# RAPID DECAY AND INVARIANT APPROXIMATION PROPERTY

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Analytic properties of invariant approximation property, studies analytic techniques from operator theory that encapsulate geometric properties of a group. We show that the following theorem holds: For a discrete group $G$ satisfying the rapid decay property with respect to a conditionally negative length function $\ell$, the reduced $C^*$-algebra $C^*_r(G)$ has the invariant approximation property. We then use this to show that some groups have invariant approximation property. We also show that if $G$ is a free product group satisfying the rapid decay property with respect to a conditionally negative length function $\ell$, then the reduced $C^*$-algebra $C^*_r(G)$ has the invariant approximation property.

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## 1. INTRODUCTION

We assume that the reader is familiar with the basic notions in operator algebras and operator spaces, see Roe [23] and Jolissaint [11], for the details on the invariant approximation property (IAP) and the Rapid Decay property (Property RD). A discrete group $\Gamma$ is said to have Rapid Decay property [11] with respect to the length function $\ell$ if there exist $C \geq 0$ and $s > 0$ such that, for all $f \in \mathbb{C}[\Gamma]$,

$$\|f\|_* \leq C \|f\|_{\ell,s} ,$$

by left convolution on $\ell^2(G)$. We give a general exposition of property RD essentially based on Jolissaint's results [11]. Who introduced the rapid decay property for groups, which generalizes Haagerup's [8] inequality for free groups. This property for groups has deep implications for the analytical, topological and geometric aspects of the groups. Jolissaint proved in his thesis that groups of polynomial growth and classical hyperbolic groups have property RD, and the only amenable discrete groups that have property RD are groups of polynomial growth. He also showed that many groups, for instance $SL_3(\mathbb{Z})$, do not

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have the Rapid Decay property [11]. De la Harpe improved Jolissaint's results and showed that the word hyperbolic groups of Gromov [7] have property RD as well, and this leads to the result of Connes and Moscovici that word hyperbolic groups satisfy the Novikov conjecture. Since then, many important works have been done on establishing the property RD, notably the works of Lafforgue [19], Chatterji [3–5] and Ruane, and Drutu and Sapir. Examples of RD groups include groups acting on CAT(0)-cube complexes [5].

The first examples of higher rank groups given by Ramagge, Robertson and Steger in [22], proved that property RD holds for discrete groups. Lafforgue [19] proved property RD for co-compact lattices in $SL_3(\mathbb{R})$ and $SL_3(\mathbb{C})$. Chatterji generalized Lafforgue's result [3] to co-compact lattices in $SL_3(\mathbb{R})$ and $E_{6(-26)}$. Chatterji, Pittet and Saloff-Coste [4] proved that the locally compact group with unimodular groups of type $PK$ have property RD. Lubotzky [5] have proved that any non uniform lattice in a higher rank contains an infinite cyclic subgroup growing exponentially with respect to the generators of any non co-compact lattice. This shows that non compact lattices in higher rank cannot have property RD. In particular, $SL_n(\mathbb{Z})$ does not have property RD for $n \geq 3$ [4]. Author had prove that the Discrete Heisenberg group and Crystallographic groups have the property RD [15].

In Section 5, we define what a coarse space is, and we study a number of ways of constructing a coarse structure on a set so as to make it into a coarse space. We also consider some of the elementary concepts associated with coarse spaces. A discrete group $G$ has natural coarse structure which allows us to define the uniform Roe algebra, $C_u^*(G)$ [23]. We say that the uniform Roe algebra, $C_u^*(G)$, is the $C^*$-algebra completion of the algebra of bounded operators on $\ell^2(X)$ which have finite propagation. The reduced $C^*$-algebra $C_r^*(G)$ is naturally contained in $C_u^*(G)$ [23]. According to Roe [23] $G$ has the invariant approximation property (IAP) if

$$C_r^*(G) = C_u^*(G)^G.$$

Section 5 makes use of the processes used in the invariant approximation property and approximation property in [1]. We prove that for a discrete group $G$ satisfying the rapid decay property with respect to a conditionally negative length function $\ell$, the reduced $C^*$-algebra $C_r^*(G)$ has the invariant approximation property (see Theorem 5.4). In section 4, we study the invariant translation approximation property (IAP). We then use this to show that the following groups have invariant approximation property (see Examples 5.6, 5.7, 5.9, and 5.11):

- The classical hyperbolic group
- Hyperbolic groups

- $CAT(0)$-cubical groups
- finitely generated coxeter group

We also show that if $G$ is a free product group satisfying the rapid decay property with respect to a conditionally negative length function $\ell$, then the reduced $C^*$-algebra $C_r^*(G)$ has the invariant approximation property (see Example 5.15).

## 2. PRELIMINARIES

Coarse geometry is the study of the large scale properties of spaces. The notion of large scale is quantified by means of a coarse structure. First we recall the following definitions:

*Definition* 2.1 ([23]). A coarse structure on a set $X$ is a collection of subsets of $X \times X$, called the *controlled* sets or *entourages* for the coarse structure, which contains the diagonal and is closed under the formation of subsets, inverses, products, and (finite) unions.

It is easy to see that the controlled sets associated to a metric space $X$ have the following properties:

(1) Any subset of a controlled set is controlled;
(2) The transpose $E^t = \{(x, y) : (y, x) \in E\}$ of a controlled set E is controlled;
(3) The composition $E_1 \circ E_2$ of controlled sets $E_1$ and $E_2$ is controlled; where

$$E_1 \circ E_2 := \{(x, z) \in X \times X : \exists\, y \in X,\ (x, y) \in E_1 \text{ and } (y, z) \in E_2\};$$

(4) A finite union of controlled sets is controlled;
(5) The diagonal $\Delta_X := \{(x, x) : x \in X\}$ is controlled.

A set equipped with a coarse structure is called a *coarse space*. Coarse geometry is the study of metric spaces (or perhaps more general objects) from a 'large scale' point of view, so that two spaces which 'look the same from a great distance' are considered equivalent. The following is an example of coarse structure.

*Example* 2.2 ([23]). Let $G$ be a finitely generated group. Then the bounded coarse structure associated to any word metric on $G$ is generated by the diagonals

$$\Delta_g = \{(h, hg) : h \in G\}$$

as $g$ runs over $G$.

Let us briefly recall basic definitions and facts concerning positive and negative type kernels and functions.

*Definition* 2.3. Let X be a set. A symmetric kernel on $X$ is a function $f : X \times X \longrightarrow \mathbb{R}$ with $f(x,y) = f(y,x)$

*Definition* 2.4 ([23]). A kernel $f$ has *conditionally positive* type if for all $m \in \mathbb{N}$, all $m$-tuples $x_1, x_2, \ldots, x_m$ of points of $X$ and for all real scalars $\lambda_1, \lambda_2, \ldots, \lambda_m$, one has

$$\sum_{i,j=1}^{m} \lambda_i \lambda_j f(x_i, x_j) \geq 0.$$

*Definition* 2.5 ([23]). A kernel $f$ has *conditionally negative* type if for all $m \in \mathbb{N}$, all $m$-tuples $x_1, x_2, \ldots, x_m$ of points of $X$, and for all real scalars $\lambda_1, \lambda_2, \ldots, \lambda_m$ such that $\sum \lambda_i = 0$, one has

$$\sum_{i,j} \lambda_i \lambda_j f(x_i, x_j) \leq 0.$$

A conditionally negative kernel on a group $G$ is a conditionally negative kernel on the set of elements of $G$ such that for any $g, h, k$, in $G$,

$$f(gh, gk) = f(h, k).$$

The following result in [23], which relates positive and negative type kernels, is known as Schoenberg's Lemma.

LEMMA 2.6 ([23]). *Let $f$ be a symmetric kernel on a space $X$. The following statements are equivalent.*

(1) *The kernel $f$ is of negative type.*
(2) *For each $t > 0$ the kernel $exp(-tf)$ is of positive type.*

*Remark* 2.7 ([23]). Let $G$ be a group; by definition the positive function on $G$ defined by $\phi : G \longrightarrow \mathbb{R}$, $(x,y) \longmapsto \phi(x^{-1}y)$, is a kernel of postive type.

We next recall some basic fact about uniform Roe algebra and metric property of a discrete group.

*Definition* 2.8 ([23]). We say that discrete metric space $X$ has *bounded geometry* if for all $R$ there exists $N$ in $\mathbb{N}$ such that for all $x \in X$ , $|B_R(x)| < N$, where $B(x,r) = \{x \in X : d(y,x) \leq r\}$.

*Definition* 2.9 ([23]). A kernel $\phi : X \times X \longrightarrow \mathbb{C}$

- is *bounded* if there, exists $M > 0$ such that $|\phi(s,t)| < M$ for all $s, t \in X$
- has *finite propagation* if there exists $R > 0$ such that $\phi(s,t) = 0$ if $d(s,t) > R$.

Let $B(X)$ be a set of bounded finite propagation kernels on $X \times X$. Each such $\phi$ defines a bounded operator on $\ell^2(X)$ via the usual formula for matrix multiplication

$$\phi * \zeta(s) = \sum_{r\in G} \phi(s,r)\zeta(r) \text{ for } \zeta \in \ell^2(X).$$

Next, we show the operator associated with a bounded kernel is bounded.

LEMMA 2.10. *Let $X$ be bounded geometry metric space. An operator associated with a bounded finite propagation kernel is bounded.*

*Proof.* Let $\phi$ and $\zeta \in \ell^2(X)$.
Consider

$$\begin{aligned} \|\phi * \zeta\|_2^2 &= \sum_{x\in X} |\phi * \zeta(x)|^2 \\ &= \sum_{x\in X} \left| \sum_{y\in X} \phi(x,y)\zeta(y) \right|^2 \end{aligned}$$

Given $x$, $\phi(x,y) \neq 0$ for $y \in B_R(x)$, where $R$ is the propagation of $\phi$. Consider

$$\begin{aligned} \left| \sum_{y\in X} \phi(x,y)\zeta(y) \right| &\leq \sum_{y\in X} |\phi(x,y)|\,|\zeta(y)| \\ &\leq \sum_{y\in X} M\,|\zeta(y)| \\ &\leq N_R M\,|\zeta(y)| \end{aligned}$$

where, by bounded geometry $N_R$ is the upper bound on the number of elements in a ball $B_R(x)$. This is independent of $x \in X$, so

$$\|\phi * \zeta\|_2^2 \leq \sum_{x\in X} N_R^2 M^2\,|\zeta(x)|^2 = N_R^2 M^2\,\|\zeta\|_2^2$$

Therefore an operator associated with a bounded kernel is bounded. □

We shall denote the finite propagation kernels on $X$ by $A^\infty(X)$.

*Definition* 2.11. The uniform Roe algebra of a metric space $X$ is the closure of $A^\infty(X)$ in the algebra $B(\ell^2(X))$ of bounded operators on $X$.

If a discrete group $G$ is equipped with its bounded coarse structure introduced in Example 2.2 then one can associated with it uniform Roe algebra $C_u^*(G)$ by repeating the above. Next we recall the left and right regular representation: We denote the group ring of $G$ by $\mathbb{C}[G]$ with the set multiplication defined by

$$\left(\sum_{s\in G} a_s s\right)\left(\sum_{t\in G} a_t t\right) = \sum_{s,t\in G} a_s a_t st$$
$$\left(\sum_{s\in G} a_s s\right)^* = \sum_{s,t\in G} \overline{a_s} s^{-1}$$

The group ring $\mathbb{C}[G]$ consists of all finitely supported complex-valued functions on $G$, that is of all finite combinations $f = \sum_{\gamma\in G} f_\gamma \delta_\gamma$ with complex coefficients. Denote $B(\ell^2(G))$ the $C^*$-algebra of all bounded linear operator on Hilbert space $\ell^2(G)$. We may distinguish between the left regular representation, which is induced by the left multiplication action, and the right regular representation, which comes from the multiplication on the right.

The left regular representation can be extended to an injective $*-$ homomorphism $\mathbb{C}[G] \longrightarrow B(\ell^2(\mathcal{H}))$, which we also denote by $\lambda$.

*Definition* 2.12 ([6]). The *left regular representation*

$$\lambda : \mathbb{C}[G] \to B(\ell^2(G))$$

is defined by

$$\lambda(s)\delta_t(r) = \delta_t(s^{-1}r) = \delta_{st}(r).$$

The *right regular representation* is given by

$$\rho(s)\delta_t(r) = \delta_t(rs) = \delta_{ts^{-1}}(r) \ for \ s, r \in G.$$

The left regular representation $\lambda$ of the group ring $\mathbb{C}[G]$ assigns to each element $f \in \mathbb{C}[G]$ a bounded operator $\lambda(f)$ which acts on any $\zeta \in \ell^2(G)$ by convolution:

$$\lambda(f)(\zeta) = f * \zeta.$$

The image $\lambda(\mathbb{C}[G])$ of the group ring under the left regular representation is a $*-$ sub-algebra of the algebra $B(\ell^2(G))$ of bounded operators on $\ell^2(G)$. The reduced $C^*$-algebra $C_r^*(G)$ of a group $G$ (which we shall assume to be discrete) arises from the study of the left regular representation $\lambda$ of the group ring $\mathbb{C}[G]$ on the Hilbert space of square-summable functions on the group.

*Definition* 2.13 ([6]). The *reduced group $C^*$-algebra $G$*, denoted by $C_r^*(G)$ is the completion of $\mathbb{C}[G]$ in the norm given, for $c \in \mathbb{C}[G]$, by

$$\|c\|_r = \|\lambda(c)\|$$

Equivalently, it is the closure of $\mathbb{C}[G]$ which is identified with its image under the left regular representation.

## 3. PROPERTY RD AND LENGTH FUNCTIONS

We will explain the basic notations related to property RD for discrete groups for further details see Jolissaint[11] and Kannan [15–17].

*Definition* 3.1. Let $G$ be a discrete group. A length function on $G$ is a map $\ell : G \longrightarrow \mathbb{R}$ taking values in the non-negative reals which satisfies the following conditions:
(1) $\ell(1) = 0$ where 1 is the identity element of the group;
(2) For every $g \in G$, $\ell(g) = \ell(g^{-1})$;
(3) For every $g, h \in G$, $\ell(gh) \leq \ell(g) + \ell(h)$.

A group equipped with a length function becomes a metric space with the left-invariant metric $d(\gamma, \mu) = \ell(\gamma^{-1}\mu)$.

*Definition* 3.2. Let $\ell$ be a length function on $G$. We define a *Sobolev norm on the group ring of G as follows:*
(1) *If $s \in \mathbb{R}$, the Sobolev space of order s is the set $H^s_\ell(G)$ of functions $\xi$ on G such that $\xi(1+\ell)^s$ belongs to $\ell^2(G)$.*
(2) *For any length function $\ell$ and positive real numbers, we define a Sobolev norm on the group ring $\mathbb{C}[G]$ by:*

$$\|f\|_{\ell,s} = \sqrt{\sum_{\gamma \in G} |f(\gamma)|^2 (1+\ell(\gamma))^{2s}}.$$

*Definition* 3.3. Let $H < \Gamma$ be a subgroup of $\Gamma$ and $\ell$ a length function on $\Gamma$. The restriction of $\ell$ to $H$ induces a length function on $H$ that we call the *induced length function.*

*Definition* 3.4. If $\ell_1$ and $\ell_2$ are length functions on $G$, we say that $\ell_2$ dominates $\ell_1$ if there exist $a, b \in \mathbb{R}$ such that $\ell_1 \leq a\ell_2 + b$. If $\ell_1$ dominates $\ell_2$ and $\ell_2$ dominates $\ell_1$, then $\ell_1$ and $\ell_2$ are said to be equivalent.

LEMMA 3.5. *If $\ell_1$ and $\ell_2$ are equivalent then $\|f\|_{\ell_1,s}$ and $\|f\|_{\ell_2,s}$ are equivalent.*

*Proof.* Since $\ell_1 \leq a\ell_2 + b$, we have

$$\begin{aligned} 1+\ell_1 &\leq 1 + a\ell_2 + b \\ &\leq 1 + b + a(1+b)\,\ell_2 \\ &\leq c(1+b)\,(1+\ell_2) \end{aligned}$$

where $C = \max\{1, a\}$ Thus,

$$\|f\|_{\ell_1,s} = \left(\sum |f(x)|^2 \{1 + l_1(x)\}^{2s}\right)^{\frac{1}{2}}$$

$$\begin{aligned} &\leq \left(\sum |f(x)|^2 \left(c(1+b)\left(1+l_2(x)\right)\right)^{2s}\right)^{\frac{1}{2}} \\ &\leq B^s \|f\|_{\ell_{2,s}} \end{aligned}$$

where $B^s = \{c(1+b)\}^s$. Similarly $\|f\|_{\ell_{2,s}} \leq \|f\|_{\ell_{1,s}}$. Therefore $\|f\|_{\ell_{2,s}}$ and $\|f\|_{\ell_{1,s}}$ are equivalent. $\square$

*Example* 3.6. Let $G$ be a discrete group with a finite generating set $S$. For convenience we will assume that $S$ is symmetric, i.e. $S^{-1} = S$. For any $g \in G$, define

$$|g|_S = \min\{k : g = s_1 \ldots s_k, s_i \in S\}.$$

This is the algebraic word length function of $G$ induced by the generating set $S$.

*Example* 3.7. Consider $\mathbb{Z}^2$ with the symmetric generating set

$$S = \{(1,0),(0,1),(0,-1),(-1,0)\}.$$

For $(m,n) \in \mathbb{Z}^2$, we have the word length function

$$|(m,n)|_S = |m| + |n|,$$

where $|m|$ and $|n|$ are the absolute values of $m$ and $n$ respectively.

Let $G$ be a countable, discrete group with symmetric finite generating sets $S$ and $S'$, yielding word-length functions $|.|_S$ and $|.|_{S'}$ respectively. As the generating sets are different, these length functions, and the metric functions they induce, are different.

*Example* 3.8. Let X be a metric space with base point $x_0 \in X$ and let $G$ be the group of isometries on X. For every $g \in G$, let $L_{x_0}(g) = d(x_0, g(x_0))$. Then $L_{x_0}$ is a length function on G.

LEMMA 3.9. *Let $(X,d)$ be a metric space, and $\ell(x) = d(x,x_o)$ be any length function where $x_o$ is a base point. If $f : (X,d_1) \longrightarrow (X,d_2)$ is a quasi-isometry then $\|f\|_{\ell_{2,s}}$ and $\|f\|_{\ell_{1,s}}$ are equivalent. If we change the base point $x_0$, we again get equivalent norms.*

*Proof.* Let $\ell$ be a length function on G. Suppose that $(X,d_1) \equiv (X,d_2)$, thus $\ell_1(x) = d_1(x,x_o) \leq d_2(x,x_o) = \ell_2(x)$. Applying Lemma 3.5, we see that $\|f\|_{\ell,x_0}$ and $\|f\|_{\ell,x_0'}$ are equivalent.

$$\begin{aligned} \ell_{x_0}(x) &= d(x,x_0) \\ &\leq d(x,x_0') + d(x_0',x_0) \\ &\leq \ell_{x_0'}(x) + b, \end{aligned}$$

where $b$ is constant. Applying Lemma 3.5, we see that $\|f\|_{\ell,x_0}$ and $\|f\|_{\ell,x_0'}$ are equivalent. $\square$

To define property RD, we first need to introduce the group ring $\mathbb{C}[G]$. This is the set of all finitely-supported functions $f : G \longrightarrow \mathbb{C}$, which forms a ring with respect to pointwise addition and the convolution product defined by

$$(f * g)(s) = \sum_{t\in\Gamma} f(t).g(t^{-1}s).$$

We are now ready to define property RD. The following definition is due to Jolissaint [11] (see also [5])

*Definition* 3.10 ([11]). Let $\ell$ be a length function on a discrete group $G$. We say that $G$ has the *Rapid Decay property* (property *RD*) with respect to the length function $\ell$ if there exist $\mathrm{C} \geq 0$ and $s > 0$ such that, for all $f \in \mathbb{C}[G]$,

$$\|f\|_* \leq C \|f\|_{\ell,s} ,$$

where $\|f\|_*$ denotes the operator norm of $f$ acting by left convolution on $\ell^2(G)$.

## 4. INVARIANT APPROXIMATION PROPERTY

In this section we will give definition of invariant approximation property Kannan [13, 14, 18] and Roe [23].

A discrete group $G$ has a natural coarse structure which allows us to define the uniform Roe algebra $C^*_u(G)$. A group $G$ can be equipped with either the left or right-invariant of the metric. A choice of one of the determines whether $C^*_\lambda(G)$ or $C^*_\rho(G)$ is a sublagebra of the uniform Roe algebra $C^*_u(G)$ of $G$ as we now explain. First we show that if the metric on $G$ is right-invariant then

$$C^*_\lambda(G) \subset C^*_u(G).$$

Let $d_1$ be the right-invariant metric on $G$

$$d_1(x, y) = d_1(xg, yg) \ \forall \ g \in G.$$

For every $g \in G$, the operator $\lambda(g)$ is given by the matrix.
Let:

$$A^\lambda_g(x, y) = \begin{cases} 1, & \text{if } x = yg, \\ 0, & \text{otherwise.} \end{cases}$$

Indeed,

$$\begin{aligned} A^\lambda_g \delta_t(s) &= \sum_{y\in G} A^\lambda_g(s, y)\delta_t(y) \\ &= \delta_t(g^{-1}s) \\ &= \delta_{gt}(s). \end{aligned}$$

Note that $A_g^\lambda$ is right-invariant

$$A_g^\lambda(xt, yt) = \begin{cases} 1, & \text{if } xt = ygt \iff x = yg, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore:

$$A_g^\lambda(x, y) = A_g^\lambda(xt, yt).$$

If the metric on $G$ is right-invariant, $A_g^\lambda$ is of finite propagation and $A_g^\lambda \in C_u^*(G)$, because $A_g^\lambda(x, y)$ is non-zero when $y^{-1}x = g$ and so

$$d_1(x, y) = d_1(y^{-1}x, e) = d_1(g, e).$$

Hence, any element of $\mathbb{C}[G]$ will give use to finite propagation and this assignment extends to an inclusion

$$C_\lambda^*(G) \hookrightarrow C_u^*(G).$$

Next we show that if the metric on $G$ is left-invariant then

$$C_\rho^*(G) \subset C_u^*(G).$$

Let $d_1$ be the left-invariant metric on $G$

$$d_1(x, y) = d_1(gx, gy) \; \forall \; g \in G.$$

For every $g \in G$, the operator $\rho(g)$ is given by the matrix.
Let:

$$A_g^\rho(x, y) = \begin{cases} 1, & \text{if } x = gy, \\ 0, & \text{otherwise..} \end{cases}$$

Indeed,

$$\begin{aligned} A_g^\rho \delta_t(s) &= \sum_{y \in G} A_g^\rho(s, y)\delta_t(y) \\ &= \delta_t(sg^{-1}) \\ &= \delta_{tg}(s). \end{aligned}$$

Note that $A_g^\rho$ is left-invariant

$$A_g^\rho(tx, ty) = \begin{cases} 1, & \text{if } tx = tgy \iff x = gy, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore:

$$A_g^\rho(x, y) = A_g^\rho(tx, ty).$$

If the metric on $G$ is right-invariant, $A_g^\rho$ is of finite propagation and $A_g^\rho \in C_u^*(G)$, because $A_g^\rho(x, y)$ is non-zero when $xy^{-1} = g$ and so

$$d_1(x, y) = d_1(xy^{-1}, e) = d_1(g, e).$$

Hence, any element of $\mathbb{C}[G]$ will give use to finite propagation and this assignment extends to an inclusion

$$C^*_\rho(G) \hookrightarrow C^*_u(G).$$

Let us now choose a right invariant metric for $G$ so that $C^*_\lambda(G) \hookrightarrow C^*_u(G)$. The right regular representation $\rho$ gives use to the adjoint action on $C^*_u(G)$ defined by

$$Ad\rho(g)T = \rho(g)T\rho(g)^* = \rho(g)T\rho(g)^{-1}$$

for all $t \in G$, $T \in C^*_u(G)$. Our remarks above show that elements of $C^*_\lambda(G)$ are invariant with respect to this action and so $C^*_\lambda(G)$ is contained in invariant subalgebra $C^*_u(G)^G$.

Lemma 4.1. *If $T \in C^*_u(G)$ has kernel $A(x,y)$, then $Ad\rho(t)T$ has kernel $A(xt,yt)$.*

*Proof.* We have that:

$$\begin{aligned}(Ad\rho(t)T\zeta)(s) &= \rho(t)(T\rho(t)^*\zeta)(s)\\ &= T\rho(t)^*\zeta(st)\\ &= \sum_{x\in G} A(st,x)(\rho(t)^{-1}\zeta)(x)\\ &= \sum_{x\in G} A(st,x)\zeta(xt^{-1}).\end{aligned}$$

Now $A(st,x)$ is non-zero whenever $x,y,t \in G$ are such that $y = xt^{-1}$, so $x = yt$ and we have

$$(Ad\rho(t)T\zeta)(s) = \sum_{x\in G} A(st,yt)\zeta(y)$$

Thus, $Ad\rho(t)T$ has kernel $A(st,yt)$. □

In general, if $T \in C^*_u(X)$ then $\forall\ x,y \in G$:

$$\begin{aligned}\langle Ad(\rho(t))T\delta_x, \delta_y\rangle &= \left\langle \rho(t)T\rho(t^{-1})\delta_x, \delta_y\right\rangle\\ &= \left\langle T\rho(t^{-1})\delta_x, \rho(t^{-1})\delta_y\right\rangle\\ &= \langle T\delta_{xt}, \delta_{yt}\rangle\,.\end{aligned}$$

So the operator $T$ is $Ad\rho-$ invariant if and only if

$$\forall\ x,y \in X\ \forall\ t \in G\ \ \langle T\delta_{xt}, \delta_{yt}\rangle = \langle T\delta_x, \delta_y\rangle\,.$$

We now define the invariant approximation property (IAP)

*Definition* 4.2 ([23]). We say that $G$ has the *invariant approximation property(IAP)* if

$$C^*_\lambda(G) = C^*_u(G)^G.$$

*Definition* 4.3 ([23]). We say that a kernel $f(x,y)$ on $X$ is *effective* if the sets

$$\{(x,y) : f(x,y) < R\}, \text{ for } R > 0,$$

generate the coarse structure on $X$.

Let $C_E(X \times X)$ denote the algebra of bounded functions $f$ on $X \times X$. which have the property that for each $\epsilon > 0$ the set

$$\{(x,y) \in X \times X : |f(x,y)| < \epsilon\}$$

is controlled. We assume that $X$ is a uniformly discrete and of bounded geometry. It can be seen that $C_E(X \times X)$ is isomorphic to $C_0(G)$, where $C_0(G)$ is the algebra of functions vanishing at $\infty$.

*Definition* 4.4. *A* $C^*$-algebra $u_n$ is an *approximate unit* in $A$, if $u_n \in A$ and for all $a \in A$; $\|u_n a - a\| \longrightarrow 0$ as $n \longrightarrow \infty$.

THEOREM 4.5 ([23]). *Let $X$ be a coarse space. The following are equivalent:*

(1) *$X$ can be coarsely embedded into a Hilbert space.*
(2) *There is an effective negative type kernel on $X$.*
(3) *The algebra $C_E(X \times X)$ has an approximate unit consisting of a sequence $\{u_n\}$ of normalized positive kernels.*

*Definition* 4.6 ([23]). We say that $f$ is a *normalized positive kernel* if $f(x,y) = 1$, for all $x, y \in X$.

LEMMA 4.7 ([23]). *Let $f$ be a normalized positive type kernel on a set $X$. Then there is a unique unital completely positive map*

$$M_f : B(\ell^2(X)) \longrightarrow B(\ell^2(X))$$

*such that*

$$\langle (M_f T)\, \delta_x, \delta_y \rangle = f(x,y) \langle T\delta_x, \delta_y \rangle,$$

*for all $T \in B(\mathcal{H})$.*

COROLLARY 4.8 ([23]). *Let $X$ be a uniformly discrete bounded geometry coarse space that is coarsely embeddable in a Hilbert space, and let $\{u_n\}$ be an approximate unit for $C_E(X \times X)$ made up of normalized positive type kernels. Then the corresponding multipliers $M_{u_n}$ define a sequence of unital completely positive maps, $C^*_u(X) \longrightarrow C^*_u(X)$, which converge pointwise to the identity.*

The following proposition shows a necessary condition to invariant approximation property.

PROPOSITION 4.9 ([23]). *Suppose that there is an approximate unit for $C_0(G)$ comprised of a sequence of functions $\phi_n$, such that*

(1) *each* $\phi_n$ *is of positive type and normalized,*

(2) *the operator* $M_{\phi_n}$ *of Schur multiplication by* $\phi_n$ *maps* $\mathbb{L}(G)$ *into* $C^*_r(G)$.

*Then* $G$ *has the invariant approximation property.*

## 5. RD AND INVARIANT APPROXIMATION PROPERTY

The following is due to Brodzki and Niblo [1].

LEMMA 5.1. *Let* $G$ *be a discrete group satisfying the rapid decay property with respect to a conditionally negative length function* $\ell$*. Let* $\phi_n$ *be any function on* $G$ *such that*

$$N = \sup_{\gamma \in G} |\phi_n(\gamma)| \, (1 + \ell(\gamma))^s \leq \infty.$$

*Then* $\phi_n$ *is a multiplier of* $C^*_r(G) \longrightarrow C^*_r(G)$ *and* $\|M_{\phi_n}\| \leq CN$.

The following is a proof of the result found in Roe [23].

PROPOSITION 5.2 ([23]). *Let* $G$ *be a discrete group satisfying the rapid decay property with respect to a length function* $\ell$*. If*

$$\sum_{g \in G} |b_g|^2 < \infty \text{ and } a_g = \phi_n(g) b_g, \;\; \phi_n(g) = exp(-\ell(g)/n)$$

*then*

$$\sum_{g \in G} |a_g|^2 \, (1 + \ell(g))^{2s} < \infty$$

*and so*

$$\sum_{g \in G} a_g \lambda(g)$$

*converges in norm to an element of* $C^*_r(G)$.

We show the following important remark, which is used for the main result of this Chapter.

*Remark* 5.3. Let $T$ in $\mathbb{L}(G)$. $T$ gives rise to a square summable sequence $\{b_g\}_{g \in G}$ as above. Indeed, in $\mathbb{L}(G)$,

$$T_n \longrightarrow T \Longleftrightarrow \forall \; x, y \in \ell^2(G) \;\; \langle T_n x, y \rangle \longrightarrow \langle Tx, y \rangle \, .$$

Take

$$\sum_{g \in G} b_g \lambda(g) \in \lambda(\mathbb{C}[G]) \subset \mathbb{B}(\ell^2(G))$$

and consider

$$\left\langle \sum_{g \in G} b_g \lambda(g) \delta_s, \delta_t \right\rangle \;\; = \;\; \left\langle \sum_{g \in G} b_g \delta_{gs}, \delta_t \right\rangle = |b_{ts^{-1}}| \, .$$

Then by Pythagoras' theorem for Hilbert space,

$$\|T\delta_s\|^2 = \sum_{g\in G} |\langle b_g\lambda(g)\delta_s, \delta_t\rangle|^2 .$$

$$\begin{aligned}\infty > \|T\delta_s\|^2 &= \sum_{t\in G} |b_{st^{-1}}|^2 \\ &= \sum_{t'\in G} |b_{t'}|^2 .\end{aligned}$$

So the Fourier coefficients $bg$ of $T$ form a square-summable sequence.

We now prove the main result of this paper.

Theorem 5.4. *Let $G$ be a discrete group satisfying the rapid decay property with respect to a conditionally negative length function $\ell$. Then the reduced $C^*$-algebra $C_r^*(G)$ has the invariant approximation property.*

*Proof.* We will use the following results from above: Proposition 4.9, Proposition 5.2, Corollary 4.8 and Remark 5.3. Let $\phi_n(\gamma) = exp(-\ell(\gamma)/n)$. Then $\phi_n(\gamma)$ is of positive type, normalized and (by Schoenberg's Lemma) $\phi_n \longrightarrow 1$ as $n \longrightarrow \infty$. It is clear that $\phi_n$ forms an approximate unit for $C_0(G)$. It remains to show that the map

$$M_{\phi_n} : \mathbb{L}(G) \longrightarrow C_r^*(G)$$

$$T \longmapsto M_{\phi_n}T$$

sends to $T$ to an element of $C_r^*(G)$. Let $\phi_n : X \longrightarrow \mathbb{C}$ be a normalized positive type kernel on $X$. We define the multiplier map

$$M_{\phi_n} : B(\ell^2(X)) \longrightarrow B(\ell^2(X))$$

such that

$$\langle (M_{\phi_n}T)\,\delta_g, \delta_h\rangle = \phi_n(g^{-1}h)\,\langle T\delta_g, \delta_h\rangle$$

for all $T$ in $\mathbb{L}(G)$. Let $\{a_g\}_{g\in G}$ be a square summable sequence function on $G$, we define

$$f : G \longrightarrow \mathbb{C}[G]$$
$$f(g) = a_g$$

such that $\|f\|_s < \infty$,

$$\begin{aligned}\|f\|_s^2 &= \sqrt{\sum_{g\in G} |f(g)|^2\,(1+\ell(g))^{2s}} \\ &= \sqrt{\sum_{g\in G} |a_g|^2\,(1+\ell(g))^{2s}}.\end{aligned}$$

Thus,

$$\sum_{g\in G} |a_g|^2 (1+\ell(g))^{2s} < \infty,$$

and by Proposition 5.2 above,

$$\sum_{g\in G} a_g\lambda(g) \in C_r^*(G).$$

If $T$ in $\mathbb{L}(G)$, then $T = \sum_{g\in G} b_g\lambda(g)$ such that $\sum_{g\in G} |b_g|^2 < \infty$ and so

$$T \longmapsto \sum_{g\in G} \phi_n(g) b_g \lambda(g).$$

Again by Proposition 5.2,

$$\sum_{g\in G} \phi_n(g) b_g \lambda(g) \in C_r^*(G).$$

Therefore

$$T \longmapsto M_{\phi_n} T$$

sends to $T$ an element of $C_r^*(G)$, and by Proposition 4.9, $G$ has the invariant approximation property. □

We now use this to show the following examples: First, we first recall the definitions of hyperbolicity for metric space.

*Definition* 5.5. A metric space $(X,d)$ is said to be *hyperbolic* if there is a constant $\delta \geq 0$ such that for any points $w,x,y,z \in X$ we have that:

$$d(w,x)+d(y,z) \leq \max\{d(w,y)+d(x,z), d(w,z)+d(x,y)\}+\delta$$

Jolissaint showed that classical hyperbolic groups have property RD [11]. Faraut and Harzallah showed that the natural metrics on these hyperbolic spaces are conditionally negative and they give rise to conditionally negative length function on these group [1]. Hence, we obtain the following example:

*Example* 5.6. Let $G$ be a classical hyperbolic group satisfying the rapid decay property with respect to a conditionally negative length function $\ell$. Then the reduced $C^*$-algebra $C_r^*(G)$ has the invariant approximation property.

We note that Ozawa has a more general result for hyperbolic groups [21].

THEOREM 5.7 ([21]). *Hyperbolic groups have the invariant translation approximation property.*

We recall the definitions of a finite-dimensional cube complex is a CAT(0) cube complex:

*Definition* 5.8. A finite-dimensional cube complex is a *CAT(0) cube complex* if the geodesic metric satisfies the CAT(0) inequality, according to which a geodesic triangle in the complex is thinner than a triangle in Euclidean space with the same side lengths.

Let $G$ be a CAT(0) cubical group, which means $G$ acts properly and cocompactly on a CAT(0) cube complex [20]. Alternatively $G$ acts properly on a CAT(0) cube complex and there is a bound on the size of point stabilizers [20]. Now according to Niblo and Reeves [20] given a group acting on a CAT(0) cube complex, they obtain a conditionally negative length kernel on the group which gives rise to a conditionally negative length function. Chatterji and Ruane [5] proved that CAT(0) cube complexes have property RD with respect to this length function provided that the action is properly discontinuous, stabilizers are uniformly bounded and the cube complexes have finite dimension. We deduce that CAT(0) cubical groups have the invariant translation approximation property.

*Example* 5.9. Let $G$ be a CAT(0) cubical group satisfying the rapid decay property with respect to a conditionally negative length function $\ell$. Then the reduced $C^*$-algebra $C^*_r(G)$ has the invariant approximation property.

We recall the definitions of coxeter group:

*Definition* 5.10 ([10]). A *coxeter* group is a discrete group $G$ given by the presentation with a finite set of generators $W = \{w_i, \dots, w_n\}$ and a finite set of relations defined as follows:

$$w_i^2 = 1 = (w_i w_j)^{m_{i,j}},$$

where $m_{i,j}$ is either $\infty$ or an integer $\geq 2$.

Chatterji proved that coxeter groups have property RD [3]. Jolissaint showed that finitely generated coxter groups have conditionally negative length function [12].

*Example* 5.11. Let $G$ be a finitely generated coxeter group satisfying the rapid decay property with respect to a conditionally negative length function $\ell$. Then the reduced $C^*$-algebra $C^*_r(G)$ has the invariant approximation property.

First we recall the definition of the free product $G_1 * G_2$ of two groups, $G_1$ and $G_2$.

*Definition* 5.12. We say that the free product of $G_1 * G_2$ of two groups $G_1$ and $G_2$ is the set consisting of the empty word (denoted by $e$) together with all reduced words $w = a_1, a_2, \dots a_n$, where the $a_j$'s are elements of either $G_1$

or $G_2$ different from the identity and satisfy the condition:

$$a_j \in G_i, \text{ implies } a_{j+1} \in G_{3-i}\,(1 \leq j \leq n-1,\, i = 1, 2).$$

*Definition* 5.13. We say that the free product of $A * B$ of two unital $C^*$-algebras $A, B$ which is a unital $*$-algebras. $A * B \longrightarrow B(\mathcal{H})$ is the $*$-representations of $A * B$

$$\|c\|_r = \sup\{\|\pi(c)\| * -\text{representation } of\ A * B\}$$

THEOREM 5.14 ([12]). *If $G_1$ and $G_2$ have property RD then so does their free product $G = G_1 * G_2$.*

*Example* 5.15. Let $G$ be a free product two groups $G_1$ and $G_2$, which satisfy the rapid decay property with respect to a conditionally negative length function $\ell$. By using Theorem 5.14, $G = G_1 * G_2$ have RD. And also Jolissaint showed that, if $G_1$ and $G_2$ have conditionally negative length function then their free product $G_1 * G_2$ also has conditionally negative length function [12] by using Theorem 5.4. Then reduced $C^*$-algebra $C_r^*(G)$ has the invariant approximation property.

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