BOUNDEDNESS OF THE STRONG KKT MULTIPLIERS FOR PROPER AND ISOLATED EFFICIENCIES IN NONSMOOTH MULTIOBJECTIVE OPTIMIZATION

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In this paper, using the idea of upper semi-regular convexificators, we propose constraint qualification and study existence and boundedness of the strong Karush-Kuhn-Tucker multipliers for proper and isolated efficiencies in nonsmooth multiobjective optimization problems with inequality, equality constraints and an arbitrary set constraint. Moreover, sufficient optimality conditions are studied for a (local) properly efficient solution.

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1. INTRODUCTION

Multiobjective programming is an extension of mathematical programming where a scaler valued objective function is replaced by a vector valued function. Necessary optimality conditions for these problems were studied by several authors in the smooth and nonsmooth cases; see, for instance, [1, 2, 11, 13–15, 19, 20, 23, 28, 33]. In multiobjective optimization problems, under the same constraint qualifications used in nonlinear programming, the Karush-Kuhn-Tucker(in short, KKT) type necessary conditions for (weakly) efficient solutions guaranteed only that the Lagrange multiplier vector associated with the objective function is nonzero. Thus, some multipliers corresponding to the components of the vector objective function may be zero. This means that the components of the vector-valued objective function have no role in the necessary conditions for efficiency. The conditions which ensure us that all multipliers of the vector objective function are nonzero are called strong KKT conditions.

During the past decade, there have been a lot of papers devoted to study the strong KKT conditions for multiobjective optimization problems, see [2,

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11, 13–15, 24, 30]. Maeda [24], Preda and Chitescu [30] derived strong Kuhn-Tucker necessary conditions for a Pareto minimum of differentiable and continuously differentiable multiobjective optimization problems, respectively. On the line of their work, strong Kuhn-Tucker necessary conditions were presented in [2, 11, 13, 14] in the smooth and nonsmooth cases.

The concept of convexificators was first introduced by Demyanov [5] in 1994 as a generalization of the notion of upper convex and lower concave approximations. Convexificators can be viewed as a weaker version of the notion of subdifferentials so that it will lead in sharper results in nonsmooth analysis. In this article, we use the notion of semi-regular convexificators, which are introduced in [6], as a particular case of the convexificators. Note that, for a locally Lipschitz function the Clarke subdifferential [3], Michel Penot subdifferential [25], Mordukhovich [26] and Treiman [36] provide examples of the semi-regular convexificators. Recently, there have been many papers devoted to study of the strong Karush-Kuhn-Tucker conditions for multiobjective programming problems in terms of convexificators, we refer to [11, 14, 22, 23]. Along with (weakly) efficient solutions, many works devoted to the study of proper efficient solutions and isolated efficient solutions/or strict efficiencies of a vector optimization problem have been published. For instance, in [34, 35] optimality conditions for proper efficiencies by using the Clarke and the Mordukhovich subdifferentials are stated, while Ginchev *et al.* [10] have obtained optimality conditions for proper efficient solutions and isolated minimizers in terms of the Dini derivative.

Boundedness and nonemptiness of the Kuhn-Tucker multipliers set for an optimization problem have been studied by several researchers in the smooth and nonsmooth cases; see [7, 18, 27, 29]. Recently, using the idea of upper convexificators, existence and boundedness of the multipliers set for a nonsmooth Lipschitz multiobjective optimization problem are discussed in [21].

In this work, by using the idea of convexificators, we study the nonemptiness and boundedness of the strong KKT multiplier sets for local properly efficient solutions and local isolated minimizers of a nonsmooth multiobjective optimization problem with equality, inequality constraints and an arbitrary set constraint. With this goal, we introduce the generalized Mangasarian-Fromovitz constraint qualification in terms of convexificators. Then we show that they are necessary for the existence and boundedness of the strong KKT multipliers for local properly efficient solutions and local isolated minimizers where the objective and inequality functions are locally Lipschitz and equality constraints are continuously differentiable. In addition, sufficient conditions for properly efficient solutions are presented based on convexificator.

The outline of the paper is as follows. Section 2 is devoted to notations, basic definitions and some preliminary results to be used in the rest of the paper. In Section 3, we first introduce an extended version of the Mangasarian-Fromovitz constraint qualification for a nonsmooth optimization problem with equality, inequality and set constraints via convexificators. Then a necessary condition is presented for the set of strong KKT multipliers for local properly efficient solutions and local isolated minimizers of a multiobjective optimization problem to be nonempty and bounded. Sufficient conditions for properly efficient solutions of such a problem are also supplied.

2. PRELIMINARIES

Throughout this paper, $\mathbb{R}^n$ is the usual n-dimensional Euclidean space. The inner product and the norm of the space in question are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let S be a subset of $\mathbb{R}^n$. The convex hull of S, the closure of S and interior of S are denoted by $\operatorname{co} S, \operatorname{cl} \mathrm{S}$ and $\operatorname{int} \mathrm{S}$, respectively. The negative and strictly negative polar cones S^- and S^s are defined respectively by

$$S^- = \{u \in \mathbb{R}^n : \langle x, u \rangle \leq 0 \ \ \forall x \in S\},$$
$$S^s = \{u \in \mathbb{R}^n : \langle x, u \rangle < 0 \ \ \forall x \in S\}.$$

Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be two vectors in $\mathbb{R}^n$. Then,

$$\begin{aligned} x = y &\Longleftrightarrow x_i = y_i, \quad i = 1, \dots, n, \\ x \leq y &\Longleftrightarrow x_i \leq y_i, \quad i = 1, \dots, n, and \quad x \neq y. \\ x < y &\Longleftrightarrow x_i < y_i, \quad i = 1, \dots, n. \end{aligned}$$

The contingent cone (or the Bouligand tangent cone) $T^B(S, x)$ and the Clarke tangent cone $T^c(S, x)$ to S at $x \in \operatorname{cl} \mathrm{S}$ are defined respectively by

$$\begin{aligned} T^B(S, x) &= \{d \in \mathbb{R}^n : \exists t_k \downarrow 0, \ \exists \ d_k \to d \ \text{ such that } \ x + t_k d_k \in S, \ \ \forall k\}, \\ T^c(S, x) &= \{d \in \mathbb{R}^n : \forall x_k \in S, \ x_k \to x, \ \forall t_k \downarrow 0, \ \exists \ d_k \to d \ \text{ such that} \\ &\qquad x_k + t_k d_k \in S, \ \ \forall k\}. \end{aligned}$$

The Clarke normal cone $N^c(S, x)$ and the Fréchet normal cone $\hat{N}(S, x)$ at $x \in \operatorname{cl} \mathrm{S}$ are defined respectively by

$$N^c(S, x) = T^c(S, x)^- = \{\zeta \in \mathbb{R}^n : \ \ \langle \zeta, d \rangle \leq 0 \ \ \forall d \in T^c(S, x)\},$$
$$\hat{N}(S, x) = T^B(S, x)^- = \{\zeta \in \mathbb{R}^n : \ \langle \zeta, d \rangle \leq 0 \ \ \forall d \in T^B(S, x)\}.$$

Note that the cones $T^c(S, x)$ and $N^c(S, x)$ are nonempty, closed and convex, $N^c(S, x) = T^c(S, x)^-$, and $T^c(S, x) \subseteq T^B(S, x)$. Moreover, $T^B(S, x)$ are always closed, but not necessarily convex, and that, if S is a convex set, then $T^c(S, x)$

and $T^B(S,x)$ coincide and are convex. For more information on the Clarke tangent cone and contingent cones, we refer the reader to [3].

Now we turn our attention to the notion of convexificator and some of its important properties. As mentioned before, this notion plays a basic role in the main results of this paper. Let $f:\mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R}\cup\{+\infty\}$ be a given function and $x \in \operatorname{dom} f := \{x \in \mathbb{R}^n : f(x) < \infty\}$. The lower and upper Dini derivatives of f at x in the direction $v \in \mathbb{R}^n$ are given respectively as

$$f^-(x;v) := \liminf_{t\downarrow 0} \frac{f(x+tv)-f(x)}{t},$$
$$f^+(x;v) := \limsup_{t\downarrow 0} \frac{f(x+tv)-f(x)}{t}.$$

If $f^+(x;v) = f^-(x;v)$, their common value is called the directional derivative of f at x in the direction v, and is denoted by $f'(x;v)$. If f is Fréchet differentiable at x with Fréchet derivative $\nabla f(x)$, then for all $v \in \mathbb{R}^n$, $f'(x;v) = \langle \nabla f(x), v\rangle$. It is noteworthy that if $f:\mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz, then both the lower and upper Dini derivatives exist finitely. Now we recall the definitions of the upper and lower convexificators from [17]:

- f is said to have an upper convexificator at $x \in \mathbb{R}^n$ if there is a closed set $\partial^* f(x) \subset \mathbb{R}^n$ such that for each $u \in \mathbb{R}^n$,
$$f^-(x;u) \le \sup_{\xi \in \partial^* f(x)} \langle \xi, u\rangle.$$
- f is said to have a lower convexificator at x if there is a closed set $\partial_* f(x) \subset \mathbb{R}^n$ such that for each $u \in \mathbb{R}^n$,
$$f^+(x;u) \ge \inf_{\xi \in \partial_* f(x)} \langle \xi, u\rangle.$$
A closed set $\partial^* f(x) \subset \mathbb{R}^n$ is said to be a convexificator of f at x iff it is both upper and lower convexificator of f at x.
- f is said to have an upper regular convexificator at x if there is a closed set $\partial^* f(x) \subset \mathbb{R}^n$ such that for each $u \in \mathbb{R}^n$,
$$f^+(x;u) = \sup_{\xi \in \partial^* f(x)} \langle \xi, u\rangle.$$
- f is said to have a lower regular convexificator at x if there is a closed set $\partial_* f(x) \subset \mathbb{R}^n$ such that for each $u \in \mathbb{R}^n$,
$$f^-(x;u) = \inf_{\xi \in \partial_* f(x)} \langle \xi, u\rangle.$$

The upper convexificator is also known as the Jeyakumar-Luc subdifferential of f at x [37]. We point out that if a continuous function $f:\mathbb{R}^n \to \mathbb{R}$ admits a

locally bounded upper convexificator at x, then it is locally Lipschitz at x (see [17], Corollary 5.2).
In [16], the notion of convexificators was extended and used to unify and strengthen various results in nonsmooth analysis and optimization. Along the lines of [6], we give now the definition of upper semi-regular convexificators which will be useful in what follows:

- The function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to have an upper semi-regular convexificator at $x \in \mathbb{R}^n$ if there is a closed set $\partial^* f(x) \subset \mathbb{R}^n$ such that for each $u \in \mathbb{R}^n$,
$$f^+(x; u) \leq \sup_{\xi \in \partial^* f(x)} \langle \xi, u \rangle.$$
- f is said to have a lower semi-regular convexificator at $x \in \mathbb{R}^n$ if there is a closed set $\partial_* f(x) \subset \mathbb{R}^n$ such that for each $u \in \mathbb{R}^n$,
$$f^-(x; u) \geq \inf_{\xi \in \partial_* f(x)} \langle \xi, u \rangle.$$

Obviously, an upper (lower) regular convexificator of f is also an upper (lower) semi-regular convexificator of f and each upper (lower) semi-regular convexificator is an upper (lower) convexificator.

Remark 2.1. It is clear that every differentiable function has an upper regular convexificator given by $\partial^* f(x) = \{\nabla f(x)\}$. Since a locally Lipschitz function is differentiable almost everywhere, it admits upper regular convexificator over a dense set. If $f : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz at $\overline{x}$, then the Clarke subdifferential, Michel-Penot subdifferential, Mordukhovich subdifferential and Treiman subdifferential are examples of upper semi-regular convexificators for f at $\overline{x}$. Also, for a locally Lipschitz function from $\mathbb{R}^n$ to $\mathbb{R}$, which is regular in the sense of Clarke [3], the Clarke subdifferential is an upper regular convexificator at each $x \in \mathbb{R}^n$ (see [6]). It is worth to mention that the MichelPenot subdifferential of a locally Lipschitz function is a convexificator of f which is upper regular over a dense subset of $\mathbb{R}^n$. Moreover, the convex hull of an upper semi-regular convexificator of a locally Lipschitz function may be strictly contained in these known subdifferentials [17].

3. MAIN RESULTS

In this section, using the idea of convexificators, we propose constraint qualification and study nonemptiness and boundedness of the strong KKT multipliers set for a nonsmooth optimization problem at local properly efficient solutions and local isolated minimizers. Indeed, we establish necessary conditions for local properly efficient solutions and local isolated minimizers of a multiobjective optimization problem via convexificator. Also, by imposing assumptions of ∂^*-invexity-infineness, we supply sufficient conditions for properly efficient solution of such a problem.

Let C be a nonempty subset of $\mathbb{R}^n$, and let $K = \{1, \ldots, p\}$, $I = \{1, \ldots, m\}$ and $J = \{1, \ldots, q\}$ be index sets. Suppose that $f = (f_k), k \in K$, $g = (g_i), i \in I$ and $h = (h_j), j \in J$ are vector functions defined on $\mathbb{R}^n$. Let us consider the following constrained multiobjective optimization problem (P):

$$\min \{f(x) \mid x \in \mathbb{S}\}.$$

Here the feasible set $\mathbb{S}$ is defined by

$$\mathbb{S} = \{x \in \mathbb{R}^n : g_i(x) \leq 0, \ i \in I, \quad h_j(x) = 0, \ j \in J, \ x \in C\}.$$

Let $\overline{x}$ is a feasible point of (P). We set

$$I(\overline{x}) = \{i \in I : g_i(\overline{x}) = 0\},$$
$$H = \{x \in \mathbb{R}^n : h_j(x) = 0, \quad \forall j \in J\}.$$

The following assumption is needful to derive necessary conditions for local properly efficient solutions and local isolated efficient solutions.

Assumption 3.1. For every $k \in K$ and $i \in I$, the functions f_k and g_i are locally Lipschitz functions at $\overline{x}$, and admit bounded upper semi-regular convexificators $\partial^* f_k(\overline{x})$ and $\partial^* g_i(\overline{x})$ at $\overline{x}$, respectively. Also h_j, $j \in J$ are continuously differentiable.

Let $\overline{x}$ be a feasible point for (P). The set of all strong Karush-Kuhn-Tucker multiplier vectors at $\overline{x}$ is denoted by $\Lambda(\overline{x})$, i.e. $(\lambda, \mu, \gamma) \in \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^q$ belong to $\Lambda(\overline{x})$ if and only if

$$(3.1) \qquad 0 \in \sum_{k=1}^{p} \lambda_k \mathrm{co}\partial^* f_k(\overline{x}) + \sum_{i=1}^{m} \mu_i \mathrm{co}\partial^* g_i(\overline{x}) + \sum_{j=1}^{q} \gamma_j \nabla h_j(\overline{x}) + N^c(C, \overline{x}),$$

$$(3.2) \qquad \lambda_k > 0, \ k \in K, \ \sum_{k=1}^{p} \lambda_k = 1, \quad \mu_i \geq 0, \qquad \mu_i g_i(\overline{x}) = 0, \quad \forall \ i \in I.$$

A point $\bar{x} \in \mathbb{S}$ is said a locally efficient solution (locally weakly efficient solution) for (P) if there exists no $x \in \mathbb{S}$ near $\bar{x}$ such that

$$f(x) \leq f(\bar{x}) \ (f(x) < f(\bar{x})).$$

Now, we recall from [12, 9] several notions of solutions (efficiency concepts), among them the notion of a properly efficient point and the notion of an isolated efficient point.

Definition 3.2. A point $\bar{x} \in \mathbb{S}$ is called a local isolated efficient solution of problem (P) iff there exists a neighborhood $\mathcal{U}$ of $\bar{x}$ and a constant $M > 0$ such that

$$\max_{1\leq k\leq m} \{f_k(x) - f_k(\bar{x})\} \geq M\|x - \bar{x}\|, \qquad \forall x \in \mathcal{U} \cap \mathbb{S}.$$

Definition 3.3. A point $\bar{x} \in \mathbb{S}$ is called a local properly efficient solution of problem (P) iff there exist a neighborhood $\mathcal{U}$ of $\bar{x}$ and $\eta = (\eta_1, \ldots, \eta_n) > 0$ such that

$$\langle \eta, f(x)\rangle \geq \langle \eta, f(\bar{x})\rangle, \qquad \forall x \in \mathcal{U} \cap \mathbb{S}.$$

We denote the sets of local efficient solutions, local isolated efficient solutions and local properly efficient solutions of problem (P) by $\operatorname{loc}\mathbb{S}(\mathrm{P})$, $\operatorname{loc}\mathbb{S}^{\mathrm{i}}(\mathrm{P})$ and $\operatorname{loc}\mathbb{S}^{\mathrm{p}}(\mathrm{P})$, respectively. When $\mathcal{U} = \mathbb{R}^n$, one has the concepts of efficient solution, isolated efficient solutions and properly efficient solution for problem (P), and in this case we denote these solution sets by $\mathbb{S}(P)$, $\mathbb{S}^i(P)$ and $\mathbb{S}^p(P)$, respectively. It is known that (see *e.g.*, [8, 9]) for our framework the inclusions

$$\operatorname{loc}\mathbb{S}^{\mathrm{i}}(\mathrm{P}) \subset \operatorname{loc}\mathbb{S}(\mathrm{P}) \qquad \text{and} \qquad \operatorname{loc}\mathbb{S}^{\mathrm{p}}(\mathrm{P}) \subset \operatorname{loc}\mathbb{S}(\mathrm{P})$$

are valid, and the converse inclusions do not hold in general. Moreover, the sets $\operatorname{loc}\mathbb{S}^{\mathrm{i}}(\mathrm{P})$ and $\operatorname{loc}\mathbb{S}^{\mathrm{p}}(\mathrm{P})$ could be different.

To deduce our next results, the following Lemma are necessary.

LEMMA 3.4 ([19]). *A point $\bar{x} \in \mathbb{S}(P)$ if and only if $\bar{x}$ solves the scalar program*

$$(P_k) \qquad \begin{cases} \text{Minimize} & f_k(x) \\ \text{subject to} & f_s(x) \leq f_s(\bar{x}), \qquad \forall s \neq k \\ & g(x) \leq 0, \quad h(x) = 0,\ x \in C \end{cases}$$

for each $k \in K$.

LEMMA 3.5. *A point $\bar{x} \in \operatorname{loc}\mathbb{S}^{\mathrm{i}}(\mathrm{P})$ if and only if $\bar{x}$ is a local isolated minimizer of the scalar program (P_k) for each $k \in K$.*

Proof. Let $\bar{x} \in \operatorname{loc}\mathbb{S}^{\mathrm{i}}(\mathrm{P})$ and k is arbitrary. Then, there exists $M > 0$ and a neighborhood $\mathcal{U}$ of $\bar{x}$ such that

$$\max_{1\leq k\leq n} \{f_k(x) - f_k(\bar{x})\} \geq M\|x - \bar{x}\|, \qquad \forall x \in \mathcal{U} \cap \mathbb{S}. \tag{3.3}$$

We put $\mathcal{U}_k := \mathcal{U},\ M_k := M$ and

$$\mathbb{S}_k := \{x \in \mathbb{R}^n | f_j(x) \leq f_j(\bar{x}),\ \ j \neq k,\ g(x) \leq 0,\ h(x) = 0,\ x \in C\}.$$

Thus, from (3.3) we have

$$f_k(x) - f_k(\bar{x}) \geq M\|x - \bar{x}\|, \qquad \forall x \in \mathcal{U}_k \cap \mathbb{S}_k.$$

Therefore, $\bar{x}$ is a local isolated minimizer of problem (P_k).

Now, suppose that $\bar{x}$ is a local isolated minimizer for (P_k) for each $k \in K$. Thus, there exist $M_k > 0$ and a neighborhood $\mathcal{U}_k$ of $\bar{x}$ such that

$$f_k(x) - f_k(\bar{x}) \geq M_k \|x - \bar{x}\|, \qquad \forall x \in \mathcal{U}_k \cap \mathbb{S}_k.$$

We put $\mathcal{U} = \bigcup_{k=1}^{n} (\mathcal{U}_k \cap \mathbb{S}_k)$ and $M = \min_{1 \leq k \leq n} M_k$. Let $x \in \mathcal{U} \cap \mathbb{S}$, therefore $x \in \mathcal{U}_k \cap \mathbb{S}_k$ for some k. Thus,

$$f_k(x) - f_k(\bar{x}) \geq M_k \|x - \bar{x}\|.$$

Therefore

$$\max_{1 \leq k \leq n} \{f_k(x) - f_k(\bar{x})\} \geq f_k(x) - f_k(\bar{x}) \geq M_k \|x - \bar{x}\| \geq M \|x - \bar{x}\|.$$

Hence, $\bar{x} \in \operatorname{loc} \mathbb{S}^{\mathrm{i}}(\mathrm{P})$ and the proof is complete. $\square$

Let us now state some calculus rules for upper semi-regular convexificators under appropriate conditions.

LEMMA 3.6. *Let $\partial^* f(x)$ be an upper semi-regular convexificator and $\partial_* f(x)$ be a lower semi-regular convexificator of f at x. Then, $\lambda \partial^* f(x)$ is an upper semi-regular convexificator for λf at x for every $\lambda > 0$ and $\lambda \partial_* f(x)$ is an upper semi-regular convexificator for λf at x for every $\lambda < 0$.*

Proof. This follows from the definitions of upper and lower semi-regular convexificator. $\square$

LEMMA 3.7. *Assume that the functions $f_1, \ldots, f_l : \mathbb{R}^n \to \mathbb{R}$ admit upper semi-regular convexificators $\partial^* f_1(x), \ldots, \partial^* f_l(x)$ at x, respectively. Then $\sum_{k=1}^{l} \partial^* f_k(x)$ is an upper semi-regular convexificator of $\sum_{k=1}^{l} f_k$ at x.*

Proof. This follows from the definitions. $\square$

Now, we consider the following nonsmooth optimization problem:

$$\begin{array}{llll} (\tilde{P}) & \min & f(x) & \\ & s.t. & g_i(x) \leq 0, & i \in I, \\ & & x \in \Omega, & \end{array}$$

where $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $g_i : \mathbb{R}^n \to \overline{\mathbb{R}}$ are functions for $i \in I$, and Ω is an arbitrary subset of $\mathbb{R}^n$. Using the idea of upper semi-regular convexificators, we first introduce a constraint qualification of Mangasarian-Fromovitz type for $(\tilde{P})$.

Let $\overline{x}$ be a feasible point for problem $(\tilde{P})$, we say that constraint qualification $(CQ1)$ is satisfied at $\overline{x}$ if and only if $G^s \cap T^c(\Omega, \overline{x})$ is nonempty, where

$$G = \bigcup_{i \in I(\overline{x})} \mathrm{co}\partial^* \mathrm{g_i}(\overline{\mathrm{x}}).$$

A feasible point $\overline{x}$ of problem $(\tilde{P})$ is called a local isolated minimizer of $(\tilde{P})$ if there exist a positive real A and a neighborhood $\mathcal{N} \ni \overline{x}$ such that $f(x) \geq f(\overline{x}) + A\|x - \overline{x}\|$ for all feasible point x with $x \in \mathcal{N}$.

In order to establish our main theorems, we need the following auxiliary result. The proof of the next lemma is based on the technique used in [13]. Let us give its proof to make the paper self contained.

LEMMA 3.8. *Let $\overline{x}$ be a local isolated minimizer for $(\tilde{P})$. Suppose that f and g_i are locally Lipschitz functions at $\overline{x}$, and admit bounded upper semi-regular convexificators $\partial^* f(\overline{x})$ and $\partial^* g_i(\overline{x})$ for all $i \in I$. If (CQ1) holds at $\overline{x}$, then there exists $\mu \in \mathbb{R}^m$ such that*

$$0 \in \mathrm{co}\partial^* f(\overline{x}) + \sum_{i=1}^{m} \mu_i \mathrm{co}\partial^* g_i(\overline{x}) + N^c(\Omega, \overline{x}), \tag{3.4}$$

$$\mu_i \geq 0, \quad i \in I, \qquad \mu_i g_i(\overline{x}) = 0, \quad i \in I.$$

Proof. Suppose by contradiction that for each $\mu \in \mathbb{R}^m$ and $\mu \geq 0$:

$$0 \notin \mathrm{co}\partial^* f(\overline{x}) + \sum_{i \in I(\overline{x})} \mu_i \mathrm{co}\partial^* g_i(\overline{x}) + N^c(\Omega, \overline{x}).$$

We assert that

$$0 \notin \mathrm{co}(F \cup G) + N^c(\Omega, \overline{x}), \tag{3.5}$$

where $F = \mathrm{co}\partial^* f(\overline{x})$. If $0 \in \mathrm{co}(F \cup G) + N^c(\Omega, \overline{x})$, then there exist $\mu_i \geq 0$, $i \in I(\overline{x}) \cup \{0\}$, $\xi_0 \in co\partial^* f(\overline{x})$ and $\xi_i \in \mathrm{co}\partial^* g_i(\overline{x})(\ i \in I(\overline{x}))$ and $\eta \in N^c(\Omega, \overline{x})$, such that

$$\sum_{i \in \{0\} \cup I(\overline{x})} \mu_i \xi_i + \eta = 0, \text{ and } \sum_{i \in \{0\} \cup I(\overline{x})} \mu_i = 1. \tag{3.6}$$

If $\mu_0 > 0$, nothing remains to prove. Otherwise, using $(CQ1)$ together with (3.6), we can select $d \in G^s \cap T^c(\Omega, \overline{x})$ and arrive at

$$0 = \sum_{i \in I(\overline{x})} \mu_i \langle \xi_i, d \rangle + \langle \eta, d \rangle < 0,$$

which is a contradiction. Thus, the assertion (3.5) is true, or equivalently,

$$\mathrm{co}(F \cup G) \bigcap (-N^c(\Omega, \overline{x})) = \emptyset.$$

Now using the convex separation theorem, we can find a vector $v \in \mathbb{R}^n$ and some scalar $\alpha_1,\ \alpha_2 \in \mathbb{R}$ satisfying

$$\langle \zeta, v \rangle < \alpha_1 < \alpha_2 < \langle \gamma, v \rangle, \quad \forall \zeta \in \mathrm{co}(F \cup G), \quad \forall \gamma \in -N^c(\Omega, \overline{x}).$$

Since $0 \in N^c(\Omega, \overline{x})$, the above in turn implies that $\alpha_2 < 0$ and

$$v \in N^c(\Omega, \overline{x})^- = T^c(\Omega, \overline{x}) \subseteq T^B(\Omega, \overline{x}), \quad v \in (F \cup G)^s. \tag{3.7}$$

Since f and g_is admit upper semi-regular convexificators, it follows from (3.7) that

$$f^+(\overline{x}, v) < 0, \tag{3.8}$$

$$g_i^+(\overline{x}, v) < 0, \quad i \in I(\overline{x}). \tag{3.9}$$

By the definition of the contingent cone, it follows from (3.7) that there exist sequences $t_n \downarrow 0$ and $v_n \to v$ such that

$$\overline{x} + t_n v_n \in \Omega, \qquad \forall n \in \mathbb{N}. \tag{3.10}$$

Since f and g_is ,$i \in I(\overline{x})$, are locally Lipschitz functions at $\overline{x}$, using the relations (3.8)–(3.10) together with the continuity of the constraint functions, we deduce that $\overline{x} + t_n v_n$ is a feasible point for $(\tilde{P})$ for all sufficiently large n, and also,

$$f(\overline{x} + t_n v_n) < f(\overline{x}).$$

Therefore, we have arrived at a contradiction with the assumption that $\overline{x}$ is a local isolated minimizer. $\square$

Remark 3.9. It is worth mentioning that Lemma 3.8 is valid if the local isolated minimizer condition for $\overline{x}$ is replaced with the local optimal solution of scalar program $(\tilde{P})$. Also, Lemma 3.8 is not valid if the convex hull appearing before the convexificators in (3.4) is removed and upper semi-regular convexificator replace with an upper convexificator. See [13, Example 1,2].

By using suitable constraint qualifications we can obtain strong KKT necessary optimality conditions of (P), which guarantee that the lagrange multipliers corresponding to the vector valued objective function are positive. To study nonemptiness and boundedness of these multipliers set for local properly efficient solutions and local isolated efficient solutions of (P), the following generalization of the Mangasarian-Fromovitz constraint qualification(CQ2) is introduced.

Definition 3.10. Let $\overline{x}$ be a feasible point of problem (P). We say that the generalized Mangasarian-Fromovitz constraint qualification $(CQ2)$ is satisfied at $\overline{x}$ if the following assertions hold:

(1) The set $\{\nabla h_j(\overline{x})\}_{j \in J}$ is linearly independent,
(2) For every $k \in K$, there exists a nonzero vector $d \in int\, T^c(C, \overline{x})$ such that

$$\sup_{\zeta \in \partial^* g_i(\overline{x})} \langle \zeta, d \rangle < 0, \quad i \in I(\overline{x}),$$

$$\langle \nabla h_j(\overline{x}), d\rangle = 0, \quad \forall j \in J$$
$$\langle \eta_s, d\rangle < 0, \quad \forall \eta_s \in co\partial^* f_s(\overline{x}), \quad \forall s \in K \setminus \{k\},$$

Obviously, if the convexificators are replaced by the classical gradient or subdifferentials, we obtain the smooth or nonsmooth versions of the constraint qualifications.

In the next theorem, we show that at a local isolated efficient solution $\overline{x}$, the qualification $(CQ2)$ is a necessary condition for the strong KKT multipliers set $\Lambda(\overline{x})$ to be nonempty and bounded under certain conditions.

THEOREM 3.11. *Let* $\overline{x} \in \operatorname{loc}\mathbb{S}^{\mathrm{i}}(\mathrm{P})$. *Suppose that Assumption 3.1 are fulfilled. Assume that* $\operatorname{int}\mathrm{T^c}(\mathrm{C},\overline{\mathrm{x}})$ *is nonempty. If* $(CQ2)$ *holds at* $\overline{x}$*, then the set* $\Lambda(\overline{x})$ *is a nonempty bounded subset of* $\mathbb{R}^{p+m+q}$.

Proof. First, we show that $(CQ2)$ ensures the nonemptiness of $\Lambda(\overline{x})$. Since for each $j \in J$, h_j is continuously differentiable and $\{\nabla h_j(\overline{x})\}_{j\in J}$ are linearly independent, it can be shown that [4],

$$N^c(H,\overline{x}) = \mathrm{span}\{\nabla \mathrm{h_j}(\overline{\mathrm{x}}) : \ \mathrm{j} \in \mathrm{J}\}, \tag{3.11}$$

and

$$T^c(H,\overline{x}) = \{v \in \mathbb{R}^n : \langle \nabla h_j(\overline{x}), v\rangle = 0, \quad j \in J\}. \tag{3.12}$$

Since $(CQ2)$ holds, it follows that $\operatorname{int}\mathrm{T^c}(\mathrm{C},\overline{\mathrm{x}}) \cap \mathrm{T^c}(\mathrm{H},\overline{\mathrm{x}})$ is nonempty. Thus, using [31, Theorem 5] we get

$$T^c(H,\overline{x}) \cap T^c(C,\overline{x}) \subseteq T^c(H\cap C,\overline{x}), \tag{3.13}$$

and

$$N^c(H\cap C,\overline{x}) \subseteq N^c(H,\overline{x}) + N^c(C,\overline{x}). \tag{3.14}$$

Since $\overline{x}$ is a local isolated efficient solution for (P), then by Lemma 3.5, $\overline{x}$ is a local isolated minimizer of P_k for each $k \in K$. Therefore, by Lemma 3.8, there exist $\overline{\lambda}_i^k \geq 0,\ i \in K \setminus \{k\},\ \overline{\mu}_i^k \geq 0,\ i \in I$, such that

$$0 \in \mathrm{co}\partial^* f_k(\overline{x}) + \sum_{i\in K, i\neq k} \overline{\lambda}_i^k \mathrm{co}\partial^* f_i(\overline{x}) + \sum_{i=1}^m \overline{\mu}_i^k \mathrm{co}\partial^* g_i(\overline{x}) + N^c(H\cap C,\overline{x}),$$
$$\overline{\mu}_i^k g_i(\overline{x}) = 0, \quad i \in I. \tag{3.15}$$

Therefore,

$$0 \in \sum_{k=1}^p \mathrm{co}\partial^* f_k(\overline{x}) + \sum_{k=1}^p \sum_{i\in K, i\neq k} \overline{\lambda}_i^k \mathrm{co}\partial^* f_i(\overline{x}) + \sum_{k=1}^p \sum_{i=1}^m \overline{\mu}_i^k \mathrm{co}\partial^* g_i(\overline{x}) + N^c(H\cap C,\overline{x}),$$
$$\overline{\mu}_i^k g_i(\overline{x}) = 0, \quad i \in I.$$

Now, let

$$\overline{\lambda}_i = 1 + \sum_{k\in K, k\neq i} \overline{\lambda}_i^k, \qquad \overline{\mu}_i = \sum_{k=1}^{p} \overline{\mu}_i^k.$$

Now, set $\overline{\lambda} = \sum_{k=1}^{p} \overline{\lambda}_k$, $\lambda_k = \frac{\overline{\lambda}_k}{\overline{\lambda}}$, $k \in K$ and $\mu_i = \frac{\overline{\mu}_i}{\overline{\lambda}}$, $i \in I$. From (3.11) and (3.14) there exists a vector $\gamma = (\gamma_1, \ldots, \gamma_q) \in \mathbb{R}^q$ such that

$$0 \in \sum_{k=1}^{p} \lambda_k \mathrm{co}\partial^* f_k(\overline{x}) + \sum_{i=1}^{m} \mu_i \mathrm{co}\partial^* g_i(\overline{x}) + \sum_{j=1}^{q} \gamma_j \nabla h_j(\overline{x}) + N^c(C, \overline{x}),$$

$$\lambda_k > 0,\ k \in K, \quad \sum_{k=1}^{p} \lambda_k = 1, \quad \mu_i g_i(\overline{x}) = 0, \quad i \in I.$$

Thus, $\Lambda(\overline{x})$ is nonempty.

Now, we show that $(CQ2)$ ensures the boundedness of $\Lambda(\overline{x})$. Since $\{\nabla h_j(\overline{x}) : j \in J\}$ are linearly independent, for each subset $\hat{J} \subseteq J$, by Gordan's theorem, there exists $\hat{d} \in \mathbb{R}^n$ such that

$$\langle \nabla h_j(\overline{x}), \hat{d} \rangle < 0, \quad \forall j \in \hat{J}, \tag{3.16}$$

and

$$\langle \nabla h_j(\overline{x}), \hat{d} \rangle > 0, \quad \forall j \in J \setminus \hat{J}. \tag{3.17}$$

Let $d_0 \in \mathrm{int}\, \mathrm{T^c(C, \overline{x})}$ be the vector which is satisfied in $(CQ2)$. Thus, for all $i \in I(\overline{x})$,

$$\sup_{\zeta \in \partial^* g_i(\overline{x})} \langle \zeta, d_0 \rangle < 0.$$

Then, there exists $\hat{\delta} > 0$ such that $\hat{\epsilon} \in (0, 1)$ may be chosen small enough such that for every $i \in I(\overline{x})$,

$$(1 - \hat{\epsilon})\langle \zeta_i, d_0 \rangle + \hat{\epsilon}\langle \zeta_i, \hat{d} \rangle \leq -\hat{\delta} < 0, \quad \forall\ \zeta_i \in \mathrm{co}\partial^* g_i(\overline{x}), \tag{3.18}$$

and

$$\overline{d} = (1 - \hat{\epsilon}) d_0 + \hat{\epsilon}\hat{d} \in T^c(C, \overline{x}).$$

From $(CQ2)$, we have

$$\langle \nabla h_j(\overline{x}), d_0 \rangle = 0, \quad \forall j \in J.$$

By virtue of (3.16) and (3.17), we have

$$\langle \nabla h_j(\overline{x}), \overline{d} \rangle = \hat{\epsilon}\langle \nabla h_j(\overline{x}), \hat{d} \rangle \leq -\hat{\epsilon}\hat{\rho} \leq -\hat{\beta}, \quad \forall j \in \hat{J}, \tag{3.19}$$

$$\langle \nabla h_j(\overline{x}), \overline{d} \rangle = \hat{\epsilon}\langle \nabla h_j(\overline{x}), \hat{d} \rangle \geq \hat{\epsilon}\hat{\rho} \geq \hat{\beta}, \quad \forall j \in J \setminus \hat{J}, \tag{3.20}$$

and from (3.18) for all $i \in I(\overline{x})$,

$$\langle \zeta_i, \overline{d} \rangle \leq -\hat{\delta} \leq -\hat{\beta} < 0, \quad \forall \zeta_i \in \mathrm{co}\partial^* g_i(\overline{x}), \tag{3.21}$$

where

$$\hat{\rho} = \min_{j \in J} \left| \langle \nabla h_j(\overline{x}), \hat{d} \rangle \right|, \qquad \hat{\beta} = \min\{\hat{\delta}, \hat{\epsilon}\hat{\rho}\} > 0.$$

Now take arbitrary multiplier vector $(\lambda_1, \dots, \lambda_p, \mu_1, \dots, \mu_m, \gamma_1, \dots, \gamma_q) \in \Lambda(\overline{x})$ and suppose that $\hat{J} = \{j \in J : \gamma_j > 0\}$. Therefore there exist $\xi_k \in \mathrm{co}\partial^* f_k(\overline{x}),\ k \in K,\ \zeta_i \in \mathrm{co}\partial^* g_i(\overline{x}),\ i \in I$ and $\eta \in N^c(C, \overline{x})$ such that

$$0 = \sum_{k=1}^{p} \lambda_k \xi_k + \sum_{i=1}^{m} \mu_i \zeta_i + \sum_{j=1}^{q} \gamma_j \nabla h_j(\overline{x}) + \eta.$$

Since $\overline{d} \in T^c(C, \overline{x})$, we have

$$\sum_{k=1}^{p} \lambda_k \langle \xi_k, \overline{d} \rangle + \sum_{i=1}^{m} \mu_i \langle \zeta_i, \overline{d} \rangle + \sum_{j=1}^{q} \gamma_j \langle \nabla h_j(\overline{x}), \overline{d} \rangle \geq 0,$$

which combined with (3.19)–(3.21) we obtain

$$\sum_{k=1}^{p} \lambda_k \langle \xi_k, \overline{d} \rangle \geq \hat{\beta}\big(\sum_{i=1}^{m} \mu_i + \sum_{j=1}^{q} |\gamma_j| \big).$$

Since for each $k \in K$, $\mathrm{co}\partial^* f_k(\overline{x})$ is bounded and there are only a finite number of possible subsets $\hat{J}$, it follows that there is a finite upper bound on $\frac{\sum_{k=1}^{p} \lambda_k \langle \xi_k, \overline{d} \rangle}{\hat{\beta}}$ independent of $\hat{J}$ which is also an upper bound for

$$\sum_{i=1}^{m} \mu_i + \sum_{j=1}^{q} |\gamma_j|.$$

Therefore, since $(\lambda_1, \dots, \lambda_p, \mu_1, \dots, \mu_m, \gamma_1, \dots, \gamma_q)$ was arbitrary, $\Lambda(\overline{x})$ is bounded. $\square$

The following example illustrates that the conclusion of Theorem 3.11 may fail to hold even in the smooth case if the $(CQ2)$ is not satisfied at the point under consideration.

Example 3.12. Consider the following problem:

$$\begin{array}{ll} \min & f(x) = (f_1(x), f_2(x)) = (x, x^2) \\ s.t. & g(x) = x^4 \leq 0, \\ & h(x) = 0, \qquad C = \mathbb{R}. \end{array}$$

Observe that $\mathbb{S} = \{0\}$ and $\overline{x} \in \mathbb{S}^i(P)$ for an arbitrary $M > 0$. It is easy to see that the $(CQ2)$ is not satisfied at $\overline{x}$. Moreover, for any $\lambda_1,\ \lambda_2 > 0$ and $\mu \geq 0$,

$$0 \notin \lambda_1 \mathrm{co}\partial^* f_1(\overline{x}) + \lambda_2 \mathrm{co}\partial^* f_2(\overline{x}) + \mu \mathrm{co}\partial^* g(\overline{x}) + N^c(C, \overline{x}),$$

which implies that $\Lambda(\overline{x}) = \emptyset$.

In the following theorem, we show that at the local properly efficient solution $\overline{x}$, $(CQ2)$ ensures the nonemptiness and boundedness of $\Lambda(\overline{x})$.

Theorem 3.13. *Let* $\overline{x} \in \mathrm{loc}\,\mathbb{S}^{\mathrm{p}}(\mathrm{P})$. *Suppose that Assumption 3.1 are fulfilled. Assume that* $\mathrm{int}\,\mathrm{T^c(C, \overline{x})}$ *is nonempty. If* $(CQ2)$ *holds at* $\overline{x}$*, then the set* $\Lambda(\overline{x})$ *is a nonempty bounded subset of* $\mathbb{R}^{p+m+q}$.

Proof. Let $\overline{x}$ be a local properly efficient solution of (P). Then there exist a neighborhood $\mathcal{U}$ of $\overline{x}$ and $\rho = (\rho_1, \ldots, \rho_p) \in \mathbb{R}^p$ with $\rho > 0$ such that

$$\sum_{i=1}^{p} \rho_i [f_i(x) - f_i(\bar{x})] \geq 0, \forall x \in \mathcal{U} \cap \mathbb{S}.$$

It means that $\overline{x}$ is local minimizer of the following scalar optimization problem

$$\begin{array}{lll} \min & \varphi(x) = \displaystyle\sum_{i=1}^{p} \rho_i f_i(x) & \\ s.t. & g_i(x) \leq 0, & i \in I, \\ & x \in H \cap C. & \end{array}$$

From Lemma 3.6 and Lemma 3.7, $\varphi(x)$ admits $\sum_{k=1}^{p} \rho_k co\partial^* f_k(\overline{x})$ as an upper semi-regular convexificator at $\overline{x}$. Then from Lemma 3.8 (and Remark 3.9 and (3.13) for (CQ1)) there exist $\overline{\mu} \in \mathbb{R}^m$ with $\overline{\mu} \geq 0$ and $\overline{\mu}_i g_i(\overline{x}) = 0,\ i \in I$ such that

$$0 \in \sum_{k=1}^{p} \rho_k \mathrm{co}\partial^* f_k(\overline{x}) + \sum_{i=1}^{m} \overline{\mu}_i \mathrm{co}\partial^* g_i(\overline{x}) + N^c(H \cap C, \overline{x}),$$

Therefore, from relations (3.11) and (3.14), there exists $\overline{\gamma} \in \mathbb{R}^q$ such that

$$0 \in \sum_{k=1}^{p} \rho_k \mathrm{co}\partial^* f_k(\overline{x}) + \sum_{i=1}^{m} \overline{\mu}_i \mathrm{co}\partial^* g_i(\overline{x}) + \sum_{j=1}^{q} \overline{\gamma}_j \nabla h_j(\overline{x}) + N^c(C, \overline{x}),$$
$$\overline{\mu}_i \geq 0, \quad \overline{\mu}_i g_i(\overline{x}) = 0, \ i \in I.$$

Now, by placing

$$\lambda_k = \frac{\rho_k}{\sum_{k=1}^{p} \rho_k}, \quad k \in K, \quad \mu_i = \frac{\overline{\mu}_i}{\sum_{k=1}^{p} \rho_k}, \ i \in I, \quad \text{and} \quad \gamma_i = \frac{\overline{\gamma}_j}{\sum_{k=1}^{p} \rho_k}.$$

We obtain $\Lambda(\overline{x})$ is nonempty.

Now, similar to the second part of the proof of Theorem 3.11, since $\sum_{k=1}^{p} \partial^* f_k(\overline{x})$ is bounded, we obtain $\Lambda(\overline{x})$ is bounded. $\square$

To establish sufficient conditions for (global) properly efficient solutions of problem (P) in the next theorem, we need the concept of (generalized) invexity-infineness-type for locally Lipschitz functions. In here, we suppose that for every $k \in K$ and $i \in I$, the functions f_k and g_i admits upper semi-regular convexificators $\partial^* f_k(\overline{x})$ and $\partial^* g_i(\overline{x})$ at $\overline{x}$. Also $h_j,\ j \in J$ are continuously differentiable.

Definition 3.14. We say that $(f,\ g;\ h)$ is ∂^*-invex-infine on C at $\overline{x} \in C$ if for any $x \in C,\ \zeta_k \in \mathrm{co}\partial^* f_k(\overline{x}),\ k \in K$ and $\eta_i \in \mathrm{co}\partial^* g_i(\overline{x}),\ i \in I$ there exists $v \in N^c(C, \overline{x})^-$ such that

$$\begin{aligned} f_k(x) - f_k(\overline{x}) &\geq \langle \zeta_k, v\rangle, \quad k \in K,\\ g_i(x) - g_i(\overline{x}) &\geq \langle \eta_i, v\rangle, \quad i \in I,\\ h_j(x) - h_j(\overline{x}) &= \langle \nabla h_j(\overline{x}), v\rangle, \quad j \in J. \end{aligned}$$

Observe that if C is convex, $f_k, k \in K, g_i, i \in I$ are convex, and $h_j, j \in J$ are affine, then $(f,\ g;\ h)$ is ∂^*-invex-infine on C at any $\overline{x} \in C$ with $v = x - \overline{x}$ for each $x \in C$. Moreover, since the Clarke subdifferential is itself a convexificator that may contain some other kinds of convexificator, it is easy to see that the class of ∂^*-invex-infine functions is larger than the one of invex-infine functions introduced in [32]. The following example shows that the class of ∂^*-invex-infine functions is strictly larger than the one of invex-infine functions.

Example 3.15. Let $f_1,\ f_2,\ g,\ h : \mathbb{R} \to \mathbb{R}$ be defined by $f_1(x) = g(x) = -|x|$, $f_2(x) = x^2$ and $h(x) = 0$ for each $x \in \mathbb{R}$. Consider $C = \mathbb{R}$ and $\overline{x} = 0$. We have $N^c(C, \overline{x})^- = \mathbb{R}$, $\partial^* f_1(\overline{x}) = \partial^* g(\overline{x}) = \{-1, 1\}$, and $\partial^* f_2(\overline{x}) = \nabla h(\overline{x}) = \{0\}$. For eny $x \in C$ and $\zeta \in \{-1, 1\}$ taking $v = \frac{-|x|}{\zeta}$, we have $v \in N^c(C, \overline{x})^-$ and

$$\begin{aligned} f_1(x) - f_1(\overline{x}) &= \zeta \cdot v \geq \zeta \cdot v,\\ f_2(x) - f_2(\overline{x}) &= x^2 \geq 0 \cdot v = 0,\\ g(x) - g(\overline{x}) &= \zeta \cdot v \geq \zeta \cdot v,\\ h(x) - h(\overline{x}) &= 0 = 0 \cdot v, \end{aligned}$$

which show that $(f,\ g;\ h)$ is ∂^*-invex-infine on C at $\overline{x} \in C$ where $f = (f_1, f_2)$. However, we have $T^c(C, \overline{x}) = \mathbb{R}$, the Clark subdifferentials $\partial^c f_1(\overline{x}) = \partial^c g(\overline{x}) = [-1, 1]$ and $\partial^c f_2(\overline{x}) = \partial^c h(\overline{x}) = \{0\}$. Now let $x \in C \setminus \{0\}$ and take $\zeta \in [-1, 1]$. Then the inequality $f_1(x) - f_1(\overline{x}) \geq \zeta \cdot v$(and $g(x) - g(\overline{x}) \geq \zeta \cdot v$) fails to hold for any $v \in T^c(C, \overline{x})$. So $(f, g; h)$ is not invex-infine on C at $\overline{x}$.

In what follows we are going to derive sufficient optimality conditions for local properly efficient solutions of (P) under the generalized convexity assumptions.

THEOREM 3.16. *Let $\overline{x} \in \mathbb{S}$ and $(f,\ g;\ h)$ be ∂^*-invex-infine on C at $\overline{x}$. If $\Lambda(\overline{x})$ be a nonempty set, then $\overline{x} \in \mathbb{S}^p(P)$.*

Proof. Since $\Lambda(\overline{x})$ is nonempty, thus there exist $\lambda_k > 0,\ k \in K,\ \mu_i \geq 0,\ i \in I$ and $\gamma_j \in \mathbb{R},\ j \in J$ such that (3.1) and (3.2) hold. Then, there exist $\zeta_k \in \mathrm{co}\partial^* f_k(\overline{x}),\ k \in K$ and $\eta_i \in \mathrm{co}\partial^* g_i(\overline{x}),\ i \in I$ such that

$$-\Big(\sum_{k=1}^{p} \lambda_k \zeta_k + \sum_{i=1}^{m} \mu_i \eta_i + \sum_{j=1}^{q} \gamma_j \nabla h_j(\overline{x})\Big) \in N^c(C, \overline{x}). \tag{3.22}$$

By the ∂^*-invex-infine property of $(f,\ g;\ h)$ on C at $\overline{x}$, for each $x \in C$, there is $v \in N^c(C, \overline{x})^-$ such that

$$\begin{aligned}
&\sum_{k=1}^{p} \lambda_k \langle \zeta_k, v\rangle + \sum_{i=1}^{m} \mu_i \langle \eta_i, v\rangle + \sum_{j=1}^{q} \gamma_j \langle \nabla h_j(\overline{x}), v\rangle \\
&\leq \sum_{k=1}^{p} \lambda_k [f_k(x) - f_k(\overline{x})] + \sum_{i=1}^{m} \mu_i [g_i(x) - g_i(\overline{x})] + \sum_{j=1}^{q} \gamma_j [h_j(x) - h_j(\overline{x})]
\end{aligned}$$

Due to the definition of Clarke normal cone, it follows from (3.22) and the relation $v \in N^c(C, \overline{x})^-$ that

$$0 \leq \sum_{k=1}^{p} \lambda_k \langle \zeta_k, v\rangle + \sum_{i=1}^{m} \mu_i \langle \eta_i, v\rangle + \sum_{j=1}^{q} \gamma_j \langle \nabla h_j(\overline{x}), v\rangle.$$

Since $\mu_i g_i(\overline{x}) = 0,\ i \in I$ and $h_j(\overline{x}) = 0,\ j \in J$ for each $x \in \mathbb{S}$, it follows that

$$\begin{aligned}
\sum_{k=1}^{p} \lambda_k f_k(\overline{x}) &= \sum_{k=1}^{p} \lambda_k f_k(\overline{x}) + \sum_{i=1}^{m} \mu_i g_i(\overline{x}) + \sum_{j=1}^{q} \gamma_j h_j(\overline{x}) \\
&\leq \sum_{k=1}^{p} \lambda_k f_k(x) + \sum_{i=1}^{m} \mu_i g_i(x) + \sum_{j=1}^{q} \gamma_j h_j(x) \\
&\leq \sum_{k=1}^{p} \lambda_k f_k(x).
\end{aligned}$$

This shows that $\overline{x} \in \mathbb{S}^p(P)$ which completes the proof. $\square$

The following example shows that if $(f,\ g;\ h)$ is not ∂^*-invex-infine on C at $\overline{x} \in C$, Theorem 3.16 is not satisfied even in the smooth case. Therefore, the condition "$(f,\ g;\ h)$ be ∂^*-invex-infine on C at $\overline{x} \in C$ " is necessary.

Example 3.17. Consider the following problem:

$$\begin{array}{ll} \min & f(x) = (f_1(x), f_2(x)) = (x^5, x^5) \\ s.t. & g(x) = -x^2 \leq 0, \\ & h(x) = 0, \quad C = \mathbb{R}. \end{array}$$

We have $\mathbb{S} = \mathbb{R}$ and thus $\overline{x} = 0 \in \mathbb{S}$. It is easy to see that $\Lambda(\overline{x})$ be a nonempty set. However, $\overline{x} \notin \mathbb{S}^p(P)$.

Next let us provide an example illustrating Theorem 3.13.

Example 3.18. Consider the nonsmooth optimization problem (P), where $f_1, f_2,\ g,\ h : \mathbb{R}^3 \to \mathbb{R}$ are defined by

$$f_1(x,y,z) = |\sin z| + x, \quad f_2(x,y,z) = \frac{-x}{2},$$

$$g(x,y,z) = \begin{cases} -|y||\sin(\frac{1}{x^2+y^2+z^2})| & (x,y,z) \neq (0,0,0) \\ 0 & (x,y,z) = (0,0,0), \end{cases}$$

$$h(x,y,z) = x^2 + y^2 + y + z,$$

$$(x,y,z) \in S = [-\pi,\pi] \times [-\pi,\pi] \times [-\pi,0].$$

Choosing $\overline{x} = (0,0,0)$, and $\eta = (1,2) > 0$, we have

$$f_1(x,y,z) + 2f_2(x,y,z) = |\sin z| \geq 0 = f_1(\overline{x}) + 2f_2(\overline{x}), \quad \forall\ (x,y,z) \in S.$$

This shows that $\overline{x} \in loc\,\mathbb{S}^p(P)$. Moreover, for all $V = (v_1, v_2, v_3) \in \mathbb{R}^3$,

$$\begin{array}{ll} f_1^+(\overline{x}, V) = |v_3| + v_1, & f_2^+(\overline{x}, V) = \dfrac{-v_1}{2}, \\ g^+(\overline{x}, V) = -|v_2|, & \nabla h(\overline{x}) = (0,1,1). \end{array}$$

Hence, the objective and constraint functions admit bounded upper semi-regular convexificators as follows:

$$\partial^* f_1(\overline{x}) = \{(1,0,1),(1,0,-1)\}, \quad \partial^* f_2(\overline{x}) = \{(\frac{-1}{2},0,0)\},$$

$$\partial^* g(\overline{x}) = \{(0,-1,0),(0,-\frac{1}{2},0,)\}.$$

Obviously, $T^C(S,\overline{x}) = \mathbb{R} \times \mathbb{R} \times \mathbb{R}^-$ and (CQ2) holds with the vector $(3,1,-1)$ for $k = 1$ and vector $(-3,1,-1)$ for $k = 2$. Thus, $\Lambda(\overline{x})$ is nonempty, for example $(1,2,1,1) \in \Lambda(\overline{x})$.

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REFERENCES

[1] T.Q. Bao and B.S. Mordukhovich, *Necessary conditions for super minimizers in constrained multiobjective optimization.* J. Global Optim. **43** (2009), 533–552.

[2] R.S. Burachik and M.M. Rizvi, *On weak and strong Kuhn-Tucker conditions for smooth multiobjective optimization.* J. Optim. Theory Appl. **155** (2012), 477–491.

[3] F.H. Clarke, *Optimization and Nonsmooth Analysis.* Wiley Interscience, New York, 1983.

[4] F.H. Clarke, Yu.S. Ledyaev, R.J. Stern and P.R. Wolenski, *Nonsmooth Analysis and Control Theory.* Graduate Texts in Mathematics **178**, Springer-Verlag, New York, 1998.

[5] V.F. Demyanov, *Convexification and concavification of a positively homogenous function by the same family of linear functions.* Universita di Pisa, Report **3** (1994), *208*, 802.

[6] J. Dutta and S. Chandra, *Convexificators, generalized convexity and vector optimization.* Optimization **53** (2004), 77–94.

[7] J. Gauvin, *A necessary and sufficient regularity condition to have bounded multipliers in nonconvex programming.* Math. Program. **12** (1977), 136–138.

[8] A.M. Geoffrion, *Proper efficiency and the theory of vector maximization.* J. Math. Anal. Appl. **22** (1968), 618–630.

[9] I. Ginchev, A. Guerraggio and M. Rocca, *From scalar to vector optimization.* Appl. Math. **51** (2006), 5–36.

[10] I. Ginchev, A. Guerraggio and M. Rocca, *Isolated minimizers and proper efficiency for $C^{0,1}$ constrained vector optimization problems.* J. Math. Anal. Appl. **309** (2005), 353–368.

[11] G. Giorgi, B. Jiménez and V. Novo, *Strong Kuhn-Tucker conditions and constraint qualifications in locally Lipschitz multiobjective optimization problem.* Top. **17** (2009), 288–304.

[12] A. Gopfert, H. Riahi, C. Tammer and C. Zalinescu, *Variational Methods in Partially Ordered Spaces.* Springer, New York, 2003.

[13] M. Golestani and S. Nobakhtian, *Convexificators and strong Kuhn-Tucker conditions.* Comput. Math. Appl. **64** (2012), 550–557.

[14] M. Golestani and S. Nobakhtian, *Nonsmooth multiobjective programming: strong Kuhn-Tucker conditions.* Positivity **17** (2013), 711–732

[15] M. Golestani and S. Nobakhtian, *Nonsmooth multiobjective programming and constraint qualifications.* Optimization **62** (2013), 783–795.

[16] V. Jeyakumar and D. T. Luc, *Approximate Jacobian matrices for nonsmooth continuous maps and C^1-optimization.* SIAM J. Control Optim. **36** (1998), 1815–1832.

[17] V. Jeyakumar and D. T. Luc, *Nonsmooth calculus, minimality, and monotonicity of convexificators.* J. Optim. Theory Appl. **101** (1999), 599–621.

[18] A. Jourani, *Constraint qualifications and Lagrange multipliers in nondifferentiable programming problems.* J. Optim. Theory Appl. **81** (1994), 533–548.

[19] P. Kanniapan, *Necessary conditions for optimality of non differentiable convex multiobjective programming.* J. Optim. Theory Appl. **40** (1983), 167–174.

[20] D.S. Kim and S. Schaible, *Optimality and duality for invex nonsmooth multiobjective programming problems.* Optimization **53** (2004), 165–176.

[21] X.F. Li and J.Z. Zhang, *Existence and Boundedness of the Kuhn-Tucker Multipliers in Nonsmooth Multiobjective Optimization.* J. Optim. Theory Appl. **145** (2010), 373–386.

[22] D.V. Luu, *Necessary and sufficient conditions for efficiency via convexificators.* J. Optim. Theory Appl. **160** (2014), 510–526.

[23] D.V. Luu, *Convexificators and necessary conditions for efficiency.* Optimization **63** (2014), 321–335.

[24] T. Maeda, *Constraint qualifications in multiobjective optimization problems: differentiable case.* J. Optim. Theory Appl. **80** (1994), 483–500.

[25] P. Michel and J.P. Penot, *A generalized derivative for calm and stable functions.* Differential Integral Equations **5** (1992), 433–454.

[26] B.S. Mordukhovich and Y. H. Shao, *On nonconvex subdifferential calculus in Banach spaces.* J. Convex Anal. **2** (1995), 211–227.

[27] V.H. Nguyen, J.J. Strodiot and R. Mifflin, *On conditions to have bounded multipliers in locally Lipschitz programming.* Math. Program. **18** (1980), 100–106.

[28] S. Nobakhtian, *Infine functions and nonsmooth multiobjective optimization problems.* Comput. Math. Appl. **51** (2006), 1385–1394.

[29] M. Pappalardo, *Error bounds for generalized Lagrange multipliers in locally Lipschitz programming.* J. Optim. Theory Appl. **73** (1992), 205–210.

[30] V. Preda and I. Chitescu, *On constraint qualification in multiobjective optimization problems: semidifferentiable case.* J. Optim. Theory Appl. **100** (1999), 417–433.

[31] R.T. Rockafellar, *"Clarke's tangent cones and the boundaries of closed sets in $\mathbb{R}^n$".* Nonlinear Analysis **3** (1979), 145–154.

[32] P.H. Sach, G.M. Lee and D.S. Kim, *Infine functions, nonsmooth alternative theorems and vector optimization problems,* J. Global Optim. **27** (2003), 51–81.

[33] P.H. Sach, D.S. Kim and G.M. Lee, *Strong duality for proper efficiency in vector optimization problems.* J. Optim. Theory Appl. **130** (2006), 139–151.

[34] M. Soleimani-damaneh, *Multiple-objective programs in Banach spaces: sufficiency for (proper) optimality.* Nonlinear Anal. **67** (2007), 958–962.

[35] M. Soleimani-damaneh, *Nonsmooth optimization using Mordukhovichs subdifferential.* SIAM J. Control Optim. **48** (2010), 3403–3432.

[36] J.S. Treiman, *The linear nonconvex generalized gradient and Lagrange multipliers.* SIAM J. Optim. **5** (1995), 670–680.

[37] X. Wang and V. Jeyakumar, *A Sharp Lagrangian Multiplier Rule for Nonsmooth Mathematical Programming Problems Involving Equality Constraints.* SIAM J. Optim. **10** (2000), 1136–1148.

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