TWO PROBLEMS FOR STOCHASTIC FLOWS ASSOCIATED WITH NONLINEAR PARABOLIC EQUATIONS

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In this paper, we solve two problems for some nonlinear SPDE driven by Fisk-Stratonovich stoachastic integral. The main assumption is the commuting property of the drift and diffusion vector fields with respect of the Lie bracket. In the first problem (P1) we construct a classical solution for some nonlinear SPDE of parabolic type by assuming the compatibilty condition concerning the mentioned vector fields. The second problem (P2) is a related filtering one for a non-markovian system of SDEs involving a backward parabolic equation of Kolmogorov type with parameter.

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1. INTRODUCTION

In the last 40 years the interest area for stochastic partial differential equations (SPDEs) has been broaden to equations driven by more general stochastic perturbations and new existence and uniqueness theorems for their solutions, making use of different approaches, for example the L_2-theory as in Krylov and Rozovskii [14], the semigroup approach as in da Prato and Zabczyk [6], the L_p-theory as in Krylov [15], or the stochastic characteristics method as in Kunita [19]. SPDEs have been proved useful in solving nonlinear filtering problems of signaling processes as in Kunita [17] and Pardoux [22], or in the stochastic control with partial information as indicated in Lions and Souganidis [20], to mention just the most important. Other applications of linear or nonlinear SPDEs in quantum mechanics, stochastic dinamical systems, optimal control, mathematical finance, etc., may be found in Da Prato and Tubaro [7].

A new direction in SPDE's study which has gone through an explosive development belongs to the class of backward stochastic differential equations

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(BSDEs), introduced in the linear case by Bismut in [2] and studied in the general case by Pardoux and Peng in [23]. BSDEs have have been pointed out as having numerous applications in mathematical finance, for example in the theory of contingent claims in a complete market by Black and Scholes [3], in the analysis of dynamic risk measures by Barrieu and El Karoui [1], in the theory of recursive utilities by Duffie an Epstein [8], or in the analysis of contingent claims with constraints by El Karoui *et al.* [9], among others.

In a new approach, Lions and Souganidis [20] have extended the notion of viscosity solution from ordinary PDEs to a general class of SPDEs using the stochastic characteristics method, in order to remove the Fisk-Stratonovich integral appearing in the SPDE so that the stochastic viscosity solution can be studied pathwisely. Along the same line, Buckdahn and Ma (see [4], [5]), considered a class of nonlinear SPDEs driven by Fisk-Stratonovich integrals with the diffusion term independent of the gradient of the solution, for which they proved the existence and the uniqueness of a stochastic viscosity solution. They used a nonlinear Doss-Sussman type transformation for which the viscosity solutions were transform invariant and such that the SPDE has been succesfully converted in an ordinary PDE with random coefficients, making use of doubly stochastic backward differential equations (BDSDE) tool introduced by Pardoux and Peng [24].

The technique of stochastic characteristics, by the way used in this paper, has been used by Tubaro in [27] along the line pionereed by Kunita in [18] for first order SPDE's to study a class of seconder order (in the drift term) SDE's. The author used a semigroup approach based on the Kato-Tanabe theory in order to transform the SPDE into a linear parabolic equation with random parameter. Same technique has been used by Iftimie and Vârsan [12] in the study of some evolution equations with stochastic perturbations of the same form as in [4]. They considered Doss-Sussman transformations given by Langevin's smooth approximations of the Brownian motion, instead of the usual ones obtained by the modification of the Brownian motion or by piecewise linear approximations.

In this paper, we shall make use of the results obtained in [21] along the line developed by Varsan *et al.* when dealing with the classical initial value problem associated to the nonlinear SPDE

$$(1.1)\qquad \begin{cases} \mathrm{d}u(t,x) = \langle \nabla u(t,x), g_0(x)\rangle\, u(t,x)\mathrm{d}t + \\ \sum_{i=1}^m \langle \nabla u(t,x), g_i(x)\rangle \circ \mathrm{d}W_i(t), \\ u(0,x) = \varphi(x), \end{cases}$$

where $t \in [0,T], x \in \mathbb{R}^n$, or alternatively in an integral representation

$$(1.2)\qquad \begin{cases} u(t,x)=\varphi(x)+\int_0^t\langle\nabla u(s,x),g_0(x)\;u(s,x)\rangle\mathrm{d}s+\\ \sum_{i=1}^m\int_0^t\langle\nabla u(s,x),g_i(x)\rangle\circ\mathrm{d}W_i(s),\end{cases}$$

where the stochastic integral "$\circ$" is understood in the Fisk-Stratonovich sense while the system of characteristics defined by (1.1) has been defined in analogy to the characteristics associated to deterministic PDE's.

More precisely we consider that $\widehat{x}_\varphi(t;\lambda)$, $t\in[0,T]$, is the unique solution of the SDE driven by the complete vector fields $f\in(\mathcal{C}_b\cap\mathcal{C}_b^1\cap\mathcal{C}^2)(\mathbb{R}^n;\mathbb{R}^n)$ and $g\in(\mathcal{C}_b^1\cap\mathcal{C}^2)(\mathbb{R}^n;\mathbb{R}^n)$,

$$(1.3)\qquad \begin{cases}\mathrm{d}_t\widehat{x}=\varphi(\lambda)f(\widehat{x})\mathrm{d}t+g(\widehat{x})\circ\mathrm{d}w(t),\\ \widehat{x}(0)=\lambda,\end{cases}$$

where $\lambda\in\mathbb{R}^n$, $\varphi\in(\mathcal{C}_b^1\cap\mathcal{C}^2)(\mathbb{R}^n)$ and $w(t)\in\mathbb{R}$ is the scalar Wiener process over the complete filtered probability space $\{\Omega,\mathcal{F}\supset\{\mathcal{F}_t\},P\}$. We recall that the Fisk-Stratonovich integral "$\circ$" in (1.3) is computed using the Ito stochastic integral "$\cdot$"

$$g(x)\circ\mathrm{d}w(t)=g(x)\cdot\mathrm{d}w(t)+\frac{1}{2}\partial_x g(x)\cdot g(x)\mathrm{d}t.$$

In our paper, the drift term of the SPDE (1.3) is not of Lipschitz type, and this makes a difference with the literature cited previously where the drift is mostly assumed (locally) Lipschitz with respect to u, ∇u (see (1.1)). In this case the attempt to reduce the SPDE to a random PDE by making use of the stochastics characteristics might be difficult especially when it comes to establish the existence of solutions. In order to circumvent these difficulties we shall consider the system of characteristics defined by (1.3) in an analogous way to the characteristics associated to deterministic SPDE's – the method has been already used previously by Varsan *et al.*, for example in [11, 21]. This approach will result in a system of SDE's and ODE's which don't create problems to prove the existence of solutions.

We are constructing the gradient representation of the stochastic flow generated by the class of SDE's as in (1.3) driven by Fisk-Stratonovich integrals. We are resting on the method of the stochastic characteristics but also on the nonsingular representation of the gradient system associated with the vector fields g_j, as developed by Vârsan in [28]. This is possible by assuming that the Lie algebra generated by g_j is finite dimensional. The standing assumption is the commuting property of the drift and diffusion vector fields with respect to the usual Lie bracket as assumed in (2.1). The commuting hypothesis of the diffusion vector fields with respect to the Lie bracket has been used by Kunita [16], [18] and is reffered by some other authors as a compatibility condition (see Buckdahn and Ma [4], Remark 3.3) concerning the mentioned vector fields.

This leads us to a gradient representation for the stochastic flow associated with the stochastic differential equation obtained by means of the stochastic system of characteristics defined by (1.3), defined in analogy to the deterministic PDEs and the corresponding fundamental solution $\psi(t,x)$ of the same SPDE. We shall make use of $\psi(t,x)$ as the composition between the fundamental solution of a deterministic nonlinear Hamilton-Jacobi equation and the fundamental solution of a reduced SPDE.

We are next interested in computing expectations of functionals involving the solution of some SDE, which is naturally related with the SDE obtained by writing the system of characteristics associated to equation (1.3).

Along this line let us consider the stochastic functionals $u(t,x):=h(\psi(t,x))$, $S(t,x) := Eh(\widehat{x}_\psi(T;t,x)), t \in [0,T], x \in \mathbb{R}^n$, for a fixed $h \in (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n)$. Here $\psi(t,x)$ is the unique solution satisfying the flow equation

$$\widehat{x}_\varphi(t;\lambda) = x, \tag{1.4}$$

with respect to the unknown $\lambda \in \mathbb{R}^n$.

The evolution of the functional $h(\psi(t,x))$, $h \in \left(C_b^1 \cap C^2\right)(\mathbb{R}^n)$, introduced in Problem (I), is described in Theorem 2.1 of section 2. In the filtering issue, introduced by Problem (II), we are computing the expectation $Eh(\widehat{x}_\psi(T;t,x))$ of some functional depending on the terminal value of a non-Markovian process. We prove that under appropriate conditions the evolution of the conditional expectaction $S(t,x)$ will be defined by a pameterized nonlinear backward parabolic equation considering that $\widehat{x}_\psi(s;t,x)$ for arbitrary $s \in [t,T]$ and $x \in R^n$ is the unique solution of the SDE

$$\begin{cases} d_s\widehat{x} = \varphi(\psi(t,x))f(\widehat{x})ds + g(\widehat{x}) \circ \mathrm{d}w(s), \\ \widehat{x}(t) = x. \end{cases} \tag{1.5}$$

2. SOME PROBLEMS AND THEIR SOLUTIONS

PROBLEM (P1). Assume that g and f commute using the Lie bracket, *i.e.*

$$[g,f](x) = 0, \tag{2.1}$$

where $[g,f](x) := [\partial_x g(x)]f(x) - [\partial_x f(x)]g(x)$,

$$TVK = \rho \in [0,1), \tag{2.2}$$

where $V := \sup\{|\partial_x \varphi(x)| : x \in \mathbb{R}^n\}$.

Under the hypotheses (2.1) and (2.2), find the nonlinear SPDE of parabolic type satisfied by $u(t,x) = h(\psi(t,x)), t \in [0,T], x \in \mathbb{R}^n$, $h \in (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n)$, where $\psi(t,x)$ is the unique continuous and $\mathcal{F}_t$-adapted solution of the flow equation (1.4).

PROBLEM (P2). Using $\psi(t,x)$ found in (P1)and $h \in \mathcal{C}_p^2(R^n)$, describe the evolution of a functional $S(t,x) := Eh(\widehat{x}_\psi(T;t,x))$ using backward parabolic equations, where $\widehat{x}_\psi(s;t,x)$ is the unique solution of the SDE (1.5). Here $\mathcal{C}_p^2(R^n)$ stands for all functions $h \in \mathcal{C}^2(R^n)$ such that h, $\partial_x h$ and $\partial_x^2 h$ are satisfying a polynomial growth condition.

2.1. SOLUTION FOR THE PROBLEM (P1)

Remark 2.1. Under the hypotheses (2.1) and (2.2) of (P1), the unique solution of the flow equation (1.4) will be found as a composition $\psi(t,x) = \widehat{\psi}(t,\widehat{z}(t,x))$, where $\widehat{z}(t,x) := G(-w(t))[x]$ and $\lambda = \widehat{\psi}(t,z)$ is the unique deterministic solution satisfying the integral equation

$$\lambda = F(-\theta(t;\lambda))[z] =: \widehat{V}(t,z;\lambda). \tag{2.3}$$

Here $F(\sigma)[z]$ and $G(\tau)[z]$, $\sigma,\tau \in \mathbb{R}$, are the global flows generated by the complete vector fields f and g respectively, and $\theta(t;\lambda) = t\varphi(\lambda)$. The unique solution of (2.3) is constructed in the following

LEMMA 2.1. *Assume that the hypothesis (2.2) is satisfied. Then there exists a unique smooth deterministic mapping $\widehat{\psi}(t,z)$ for arbitrary $t \in [0,T]$ and $z \in \mathbb{R}^n$ solving the integral equation (2.3) and satisfying the estimate*

$$\begin{cases} F(\theta(t;\widehat{\psi}(t,z)))[\widehat{\psi}(t,z)] = z, \\ |\widehat{\psi}(t,z) - z| \leq \dfrac{TK}{1-\rho}|\varphi(z)|. \end{cases}$$

Moreover $\widehat{\psi}(t,z)$ is the unique solution of the nonlinear Hamilton-Jacobi equation

$$\begin{cases} \partial_t\widehat{\psi}(t,z) + \partial_z\widehat{\psi}(t,z)f(z)\varphi(\widehat{\psi}(t,z)) = 0, \\ \widehat{\psi}(0,z) = z. \end{cases} \tag{2.4}$$

Proof. The mapping $\lambda \in \mathbb{R}^n \longrightarrow \widehat{V}(t,z;\lambda)$ is a contractive application with respect to $\lambda \in \mathbb{R}^n$, uniformly of $(t,z) \in [0,T] \times \mathbb{R}^n$, which allows us to get the unique solution of (2.3) using a standard procedure. By a direct computation, we get

$$|\partial_\lambda\widehat{V}(t,z;\lambda)| = |f(\widehat{V}(t,z;\lambda))\partial_\lambda\theta(t;\lambda)| \leq TVK = \rho \in [0,1),$$

for any $t \in [0,T]$, $z \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^n$, where $\partial_\lambda\theta(t;\lambda)$ is a row vector. The corresponding convergent sequence $\{\lambda_k(t,z) : t \in [0,T], z \in \mathbb{R}^n\}_{k\geq 0}$ is constructed satisfying

$$\lambda_0(t,z) = z,\ \lambda_{k+1}(t,z) = \widehat{V}(t,z;\lambda_k(t,z))),$$
$$\begin{cases} |\lambda_{k+1}(t,z) - \lambda_k(t,z)| \le \rho^k |\lambda_1(t,z) - \lambda_0(t,z)|, \\ |\lambda_1(t,z) - \lambda_0(t,z)| \le |\widehat{V}(t,z;z) - z| \le TK|\varphi(z)|. \end{cases}$$

By consequence we obtain that $\{\lambda_k(t,z)\}_{k\ge 0}$ is convergent and

$$\widehat{\psi}(t,z) = \lim_{k\to\infty} \lambda_k(t,z),\ |\widehat{\psi}(t,z) - z| \le \frac{TK}{1-\rho}|\varphi(z)|.$$

Letting $k \to \infty$ and using the previous estimate we get the first conclusion of the Lemma. On the other hand, notice that $\widehat{V}(t,z;\lambda)$ satisfies $\widehat{V}(t,\widehat{y}(t,\lambda);\lambda) = \lambda$ where $\widehat{y}(t,\lambda) = F(\theta(t;\lambda))[\lambda]$. This shows that all the components of $\widehat{V}(t,z;\lambda) \in \mathbb{R}^n$ are the first integrals associated with the vector field $f_\lambda(z) = \varphi(\lambda)f(z)$, for each $\lambda \in \mathbb{R}^n$, *i.e.*

$$\partial_t \widehat{V}(t,\widehat{y}(t,\lambda);\lambda) + [\partial_z \widehat{V}(t,\widehat{y}(t,\lambda);\lambda)] f(\widehat{y}(t,\lambda))\varphi(\lambda) = 0. \tag{2.5}$$

In particular, for $\lambda = \widehat{\psi}(t,z)$ we get $\widehat{y}(t,\widehat{\psi}(t,z)) = z$ and (2.5) becomes the Hamilton-Jacobi equation

$$\partial_t \widehat{V}(t,z;\widehat{\psi}(t,z)) + [\partial_z \widehat{V}(t,z;\widehat{\psi}(t,z)] f(z)\varphi(\widehat{\psi}(t,z)) = 0. \tag{2.6}$$

Combining (2.3) and (2.6), we conclude by a direct computation that $\widehat{\psi}(t,z)$ satisfies the nonlinear Hamilton-Jacobi equation (2.4) and the proof is complete. □

Remark 2.2. Under the hypothesis (2.1), the stochastic flow $\widehat{x}_\varphi(t;\lambda)$ generated by the SDE (1.3) can be represented as follows

$$\widehat{x}_\varphi(t;\lambda) = G(w(t)) \circ F(\theta(t;\lambda))[\lambda] = H(t,w(t);\lambda). \tag{2.7}$$

LEMMA 2.2. *Assume that the hypotheses* (2.1) *and* (2.2) *are satisfied and consider the smooth mapping* $\lambda = \widehat{\psi}(t,z)$ *determined in Lemma 2.1. Then the stochastic flow* $\widehat{x}_\varphi(t;\lambda)$ *generated by the SDE* (1.3) *can be represented as in* (2.7)*. In addition* $\psi(t,x) = \widehat{\psi}(t,\widehat{z}(t,x))$ *is the unique solution of the flow equation* (1.4)*.*

Proof. Using the hypothesis (2.1), we see easily that $y(\theta,\sigma)[\lambda] := G(\sigma) \circ F(\theta)[\lambda]$, $\theta,\sigma \in \mathbb{R}$, $\lambda \in \mathbb{R}^n$, is the unique solution of the gradient system

$$\begin{cases} \partial_\theta y(\theta,\sigma)[\lambda] = f(y(\theta,\sigma)[\lambda]),\ \partial_\sigma y(\theta,\sigma)[\lambda] = g(y(\theta,\sigma)[\lambda]), \\ y(0,0)[\lambda] = \lambda \end{cases}$$

Applying the standard rule of stochastic derivation with respect to the smooth mapping $\varphi(\theta,\sigma) := y(\theta,\sigma)[\lambda]$ and the continuous process $\theta = \theta(t;\lambda)$, where

$\sigma = w(t)$, we get that $\widehat{y}_\varphi(t;\lambda) = y(\theta(t;\lambda), w(t))$, satisfies the SDE (1.3), *i.e.*

$$\begin{cases} \mathrm{d}_t\widehat{y}_\varphi(t;\lambda) = \varphi(\lambda) f(\widehat{y}_\varphi(t;x))\mathrm{d}t + g(\widehat{y}_\varphi(t;\lambda)) \circ \mathrm{d}w(t), \\ \widehat{y}_\varphi(0;\lambda) = \lambda. \end{cases}$$

On the other hand, the unicity of the solution satisfying (1.3) leads us to the conclusion that $\widehat{x}_\varphi(t;\lambda) = \widehat{y}_\varphi(t;\lambda)$, and the first statement of the Lemma is proved. The second one is a direct consequence of the fact that $\widehat{\psi}(t,z)$ is the solution defined in Lemma 2.1. The proof is complete. □

Lemma 2.3. *Under the hypotheses in Lemma 2.2, consider the continuous and $\mathcal{F}_t$-adapted process $\widehat{z}(t,x) = G(-w(t))[x]$. Then the following SPDE of parabolic type is valid*

$$\begin{cases} \mathrm{d}_t\widehat{z}(t,x) + \partial_x\widehat{z}(t,x)g(x)\widehat{\circ}\mathrm{d}w(t) = 0, \\ \widehat{z}(0,x) = x \end{cases} \tag{2.8}$$

where the nonstandard Fisk-Stratonovich integral "$\widehat{\circ}$" is computed using Ito stochastic integral "$\cdot$"

$$h(t,x)\widehat{\circ}\mathrm{d}w(t) = h(t,x)\cdot \mathrm{d}w(t) - \frac{1}{2}\partial_x h(t,x)g(x)\mathrm{d}t.$$

Proof. The conclusion (2.8) is a direct consequence of applying the standard rule of stochastic derivation with respect to $\sigma = w(t)$ and the smooth deterministic mapping $H(\sigma)[x] := G(-\sigma)[x]$. Along this line, using $H(\sigma)\circ G(\sigma)[\lambda] = \lambda$, for any $x = G(\sigma)[\lambda]$, we get

$$\begin{cases} \partial_\sigma\{H(\sigma)[x]\} = -\partial_x\{H(\sigma)[x]\}\cdot g(x), \\ \partial_\sigma^2\{H(\sigma)[x]\} = \partial_\sigma\{\partial_\sigma\{H(\sigma)[x]\}\} = \partial_\sigma\{-\partial_x\{H(\sigma)[x]\}\cdot g(x)\} \\ \qquad\qquad = \partial_x\{\partial_x\{H(\sigma)[x]\}\cdot g(x)\}\cdot g(x). \end{cases}$$

The standard rule of stochastic derivation leads us to the SDE

$$\mathrm{d}_t\widehat{z}(t,x) = \partial_\sigma\{H(\sigma)[x]\}_{\sigma=w(t)}\cdot \mathrm{d}w(t) + \frac{1}{2}\partial_\sigma^2\{H(\sigma)[x]\}_{\sigma=w(t)}\mathrm{d}t,$$

and rewritting the right hand side in the last relation we get the SPDE of parabolic type required. The proof is complete. □

Lemma 2.4. *Assume that hypotheses* (2.1) *and* (2.2) *are satisfied and consider $\psi(t,x)$ defined in Lemma* (2.2)*. Then $u(t,x) := h(\psi(t,x))$, $h \in (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n)$, satisfies the following nonlinear SPDE of parabolic type*

$$\begin{cases} \mathrm{d}_t u(t,x) + \langle\partial_x u(t,x), f(x)\rangle\varphi(\psi(t,x))\mathrm{d}t \\ \quad + \langle\partial_x u(t,x), g(x)\rangle\widehat{\circ}\mathrm{d}w(t) = 0 \\ u(0,x) = h(x). \end{cases} \tag{2.9}$$

Proof. By definition $\psi(t,x)=\widehat{\psi}(t,\widehat{z}(t,x))$, where $\widehat{z}(t,x)=G(-w(t))[x]$ satisfies the SPDE (2.8) and $\widehat{\psi}(t,z)$ satisfies the nonlinear Hamilton-Jacobi equation (2.4). Applying the standard rule of stochastic derivation with respect to the smooth mapping $\widehat{\psi}(t,z)$ and the stochastic process $\widehat{z}(t,x)$ we get the following nonlinear SPDE

$$\begin{cases} \mathrm{d}_t\psi(t,x)+\partial_x\psi(t,x)f(x)\varphi(\psi(t,x))\mathrm{d}t+\partial_x\psi(t,x)g(x)\widehat{\circ}\mathrm{d}w(t)=0,\\ \psi(0,x)=x. \end{cases}$$

In addition, the functional $u(t,x)$ can be rewritten $u(t,x)=\widehat{u}(t,\widehat{z}(t,x))$, where $\widehat{u}(t,z):=h(\widehat{\psi}(t,z))$ is a smooth deterministic functional satisfying the nonlinear Hamilton-Jacobi equation

$$\begin{cases} \partial_t\widehat{u}(t,z)+\langle\partial_z\widehat{u}(t,z),f(z)\rangle\varphi(\widehat{\psi}(t,z))=0,\\ \widehat{u}(0,z)=h(z). \end{cases}$$

Using (2.8) we obtain the SDPE satisfied by $u(t,x)$,

$$\begin{cases} \mathrm{d}_tu(t,x)+\langle\partial_z\widehat{u}(t,\widehat{z}(t,x)),f(\widehat{z}(t,x))\rangle\varphi(\psi(t,x))\mathrm{d}t+\\ \langle\partial_xu(t,x),g(x)\rangle\widehat{\circ}\mathrm{d}w(t)=0,\\ u(0,x)=h(x). \end{cases}$$

The hypothesis (2.1) allows us to write

$$\begin{aligned}\langle\partial_z\widehat{u}(t,\widehat{z}(t,x)),f(\widehat{z}(t,x))\rangle&=\partial_z\widehat{u}(t,\widehat{z}(t,x))[\partial_x\widehat{z}(t,x)][\partial_x\widehat{z}(t,x)]^{-1}f(\widehat{z}(t,x))\\&=\langle\partial_xu(t,x),f(x)\rangle,\end{aligned}$$

from where we get the conclusion (2.9). The proof is complete. □

Remark 2.3. The complete solution of Problem (P1) is contained in Lemmas 2.1–2.4. We are now in position to state the main result of this section.

THEOREM 2.1. *Assume that the vector fields* $f\in(\mathcal{C}_b\cap\mathcal{C}_b^1\cap\mathcal{C}^2)(\mathbb{R}^n;\mathbb{R}^n)$, $g\in(\mathcal{C}_b^1\cap\mathcal{C}^2)(\mathbb{R}^n;\mathbb{R}^n)$, *and the scalar function* $\varphi\in(\mathcal{C}_b^1\cap\mathcal{C}^2)(\mathbb{R}^n)$ *satisfy the hypotheses* (2.1) *and* (2.2). *Consider the continuous and* $\mathcal{F}_t$*-adapted process* $\psi(t,x)$, $t\in[0,T]$, $x\in\mathbb{R}^n$ *satisfying the flow equation* (1.4). *Then* $u(t,x):=h(\psi(t,x))$, *satisfies the nonlinear SPDE of parabolic type* (2.9), *for each* $h\in(\mathcal{C}_b^1\cap\mathcal{C}^2)(\mathbb{R}^n)$.

2.2. SOLUTION FOR THE PROBLEM (P2)

Using the same notations as in subsection 2.1, we consider the unique solution $\widehat{x}_\psi(s;t,x)$, $s\in[t,T]$ satisfying the SDE (1.5) for each $0\le t\le T$ and $x\in\mathbb{R}^n$. As far as the SDE (1.5) is a non-markovian system, the evolution of the functional $S(t,x) := Eh(\widehat{x}_\psi(T;t,x))$, $h \in \mathcal{C}_p^2(R^n)$, will be described using the pathwise representation of the conditional mean values functional

$$v(t,x) = E\{h(\widehat{x}_\psi(T;t,x)) \mid \psi(t,x)\}. \tag{2.10}$$

Assuming the hypotheses (2.1) and (2.2) we may write the following integral representation

$$\widehat{x}_\psi(T;t,x) = G(w(T)-w(t)) \circ F[(T-t)\varphi(\psi(t,x))][x], \tag{2.11}$$

for a solution of the SDE (1.5). We notice that the right hand side of (2.11) is a continuous mapping of the two independent random variables, $z_1 = [w(T) - w(t)]$ and $z_2 = \psi(t,x)$ which is $\mathcal{F}_t$-measurable. A direct consequence of this remark is to use the parameterized random variable $y(t,x;\lambda) = G(w(T) - w(t)) \circ F[(T-t)\varphi(\lambda)][x]$, and to compute the conditional mean values (2.10) as

$$v(t,x) = [Eh(y(t,x;\lambda))](\lambda = \psi(t,x)). \tag{2.12}$$

Here the functional $u(t,x;\lambda) := Eh(y(t,x;\lambda))$, satisfies a backward parabolic equation (Kolmogorov's equation) for each parameter $\lambda \in \mathbb{R}^n$ and we rewrite (2.12) as $v(t,x) = u(t,x;\psi(t,x))$. In conclusion, the functional $\{S(t,x)\}$ can be written as

$$S(t,x) = E[E\{h(\widehat{x}_\psi(T;t,x)) \mid \psi(t,x)\} = Eu(t,x;\psi(t,x)), \tag{2.13}$$

where $u(t,x;\lambda)$ satisfies the next backward parabolic equations with parameter λ,

$$\begin{cases} \partial_t u(t,x;\lambda) + \langle \partial_x u(t,x;\lambda), f(x,\lambda)\rangle \\ +\frac{1}{2}\langle \partial_x^2 u(t,x;\lambda) g(x), g(x)\rangle = 0 \\ u(T,x;\lambda) = h(x), \end{cases} \tag{2.14}$$

where $f(x,\lambda) := \varphi(\lambda) f(x) + \frac{1}{2}[\partial_x g(x)]g(x)$. We conclude these remarks in the next theorem.

Theorem 2.2. *Assume that the vector fields f, g and the scalar function φ of the SDE* (1.5) *satisfy the hypotheses* (2.1) *and* (2.2)*, where the continuous and $\mathcal{F}_t$-adapted process $\psi(t,x)$ is defined in Theorem 2.1. Then the evolution of the functional $S(t,x) := Eh(\widehat{x}_\psi(T;t,x))$, $t \in [0,T]$, $x \in \mathbb{R}^n$, $h \in \mathcal{C}_p^2(R^n)$ can be described as in* (2.13)*, where $u(t,x)$ satisfies the linear backward parabolic equations* (2.14) *for each $\lambda \in \mathbb{R}^n$.*

3. MULTIPLE VECTOR FIELDS CASE

Consider the case of several vector fields defining both the drift and the diffusion parts of the SDE (1.3), *i.e.*

$$\text{(3.1)}\qquad \begin{cases} \mathrm{d}_t\widehat{x} = [\sum_{i=1}^{m}\varphi_i(\lambda)f_i(\widehat{x})]\mathrm{d}t + \sum_{i=1}^{m} g_i(\widehat{x})\circ \mathrm{d}w_i(t), \\ \widehat{x}(0)=\lambda. \end{cases}$$

In what will follow the analysis presented in Theorems 2.1 and 2.2 will be extended to this multiple vector fields case.

We are given two finite sets of vector fields $\{f_1,\dots,f_m\}\subset(\mathcal{C}_b\cap\mathcal{C}_b^1\cap\mathcal{C}^2)(\mathbb{R}^n;\mathbb{R}^n)$ and $\{g_1,\dots,g_m\}\subset(\mathcal{C}_b^1\cap\mathcal{C}^2)(\mathbb{R}^n;\mathbb{R}^n)$ and let us consider the unique solution $\widehat{x}_\varphi(t,\lambda)$ of the SDE (3.1), where $\varphi=:(\varphi_1,\dots,\varphi_m)$, $\{\varphi_1,\dots,\varphi_m\}\subset(\mathcal{C}_b^1\cap\mathcal{C}^2)(\mathbb{R}^n)$ are fixed scalar functions and $w=(w_1(t),\dots,w_m(t))\in\mathbb{R}^m$ is a standard Wiener process over the complete filtered probability space $\{\Omega,\mathcal{F}\supset\{\mathcal{F}_t\},P\}$. Each Fisk-Stratonovich integral "$\circ$" in (3.1) is computed using Ito integral "$\cdot$" by

$$\text{(3.2)}\qquad g_i(x)\circ\mathrm{d}w_i(t)=g_i(x)\cdot\mathrm{d}w_i(t)+\frac{1}{2}[\partial_x g_i(x)]g_i(x)\mathrm{d}t.$$

Assume that $\psi(t,x)$ is the unique continuous and $\mathcal{F}_t$-adapted solution with respect to λ satisfying the flow equation

$$\text{(3.3)}\qquad \widehat{x}_\varphi(t;\lambda)=x.$$

For each $h\in(\mathcal{C}_b^1\cap\mathcal{C}^2)(\mathbb{R}^n)$, consider the stochastic functional $u(t,x)=h(\psi(t,x))$ and the deterministic mapping $S(t,x)=Eh(\widehat{x}_\psi(T;t,x))$, where $\widehat{x}_\psi(s;t,x), s\in[t,T]$ satisfies the following SDE

$$\text{(3.4)}\qquad \begin{cases} \mathrm{d}_s\widehat{x} = [\sum_{i=1}^{m}\varphi_i(\psi(t,x))f_i(\widehat{x})]\mathrm{d}s + \sum_{i=1}^{m} g_i(\widehat{x})\circ \mathrm{d}w_i(t), \\ \widehat{x}(t)=x. \end{cases}$$

PROBLEM (P1). Assume that

$$\text{(3.5)}\qquad \begin{cases} M=\{f_1,\dots,f_m,g_1,\dots,g_m\} \\ \text{commute with respect to the Lie bracket,} \\ i.e.\ [X_1,X_2](x)=0 \text{ for any pair } X_1,X_2\in M, \end{cases}$$

$$\text{(3.6)}\qquad TV_iK_i=\rho_i\in[0,\frac{1}{m}),$$

where $V_i:=\sup\{|\partial_x\varphi_i(x)|:x\in\mathbb{R}^n\}$ and $K_i:=\{|f_i(x)|:x\in\mathbb{R}^n\}$, $i=1,\dots,m$.

Under the hypotheses (3.5) and (3.6), find the nonlinear SPDE of parabolic type satisfied by $u(t,x) = h(\psi(t,x)), t \in [0,T], x \in \mathbb{R}^n$, $h \in (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n)$, where $\psi(t,x)$ is the unique continuous and $\mathcal{F}_t$-adapted solution with respect to λ of the flow equation (3.3).

PROBLEM (P2). Considering $\lambda = \psi(t,x)$ found in (P1) and $h \in \mathcal{C}_p^2(R^n)$, describe the evolution of the functional $S(t,x) = Eh(\widehat{x}_\psi(T;t,x))$ making use of backward parabolic equations, where $\widehat{x}_\psi(s;t,x), s \in [t,T]$ is the unique solution of the SDE (3.4).

3.1. SOLUTION FOR (P1)

Under the hypotheses (3.5) and (3.6), the unique solution of the SPDE (3.4) can be represented by

$$\widehat{x}_\varphi(t;\lambda) = G(w(t)) \circ F(\theta(t;\lambda))[\lambda] =: H(t,w(t);\lambda), \tag{3.7}$$

where $G(\sigma)[z] = G_1(\sigma_1) \circ \cdots \circ G_m(\sigma_m)[z], \sigma = (\sigma_1,\ldots,\sigma_m), F(\sigma)[z] = F_1(\sigma_1) \circ \cdots \circ F_m(\sigma_m)[z]$, $\theta(t;\lambda) := (t\varphi_1(\lambda),\ldots,t\varphi_m(\lambda))$ and $((F_i(\sigma_i)[z], G_i(\sigma_i)[z])$ are the global flows generated by (f_i, g_i), $i \in \{1,\ldots,m\}$.

The arguments for solving (P1) in the case of one pair (f,g) of vector fields as in subsection (2.1) can be also used here to get similar results. Under the representation (3.7), the unique continuous and $\mathcal{F}_t$-adapted solution $\psi(t,x)$ with respect to λ of equation (3.3) will be found as a composition $\psi(t,x) = \widehat{\psi}(t,\widehat{z}(t,x))$, $\widehat{z}(t,x) := G(-w(t))[x]$, where $\widehat{\psi}(t,z)$ is the unique solution with respect to λ satisfying the deterministic integral equation

$$\lambda = F(-\theta(t;\lambda))[z] =: \widehat{V}(t,z;\lambda). \tag{3.8}$$

LEMMA 3.1. *Asume that hypotheses* (3.5) *and* (3.6) *are satisfied. Then there exists a unique smooth mapping* $\widehat{\psi}(t,z)$ *solving the integral equation* (3.8) *and satisfying the estimate*

$$\begin{cases} F(\theta(t;\widehat{\psi}(t,z)))[\widehat{\psi}(t,z)] = z, \\ |\widehat{\psi}(t,z) - z| \le \dfrac{r(T,z)}{1-\rho}, \end{cases} \tag{3.9}$$

where $\rho = \rho_1 + \cdots + \rho_m \in [0,1)$ *and* $r(T,z) = T\sum_{i=1}^m K_i|\varphi_i(z)|$.

Moreover $\widehat{\psi}(t,z)$ *is the unique solution of the nonlinear Hamilton-Jacobi equation*

$$\begin{cases} \partial_t\widehat{\psi}(t,z) + \partial_z\widehat{\psi}(t,z)[\sum_{i=1}^m \varphi_i(\widehat{\psi}(t,z))f_i(z)] = 0, \\ \widehat{\psi}(0,z) = z. \end{cases} \tag{3.10}$$

The proof is based on the arguments of Lemma 2.1 in subsection 2.1.

Lemma 3.2. *Assume that hypotheses* (3.5) *and* (3.6) *are satisfied and consider* $\widehat{\psi}(t,z)$ *found in Lemma* (3.1). *Then the stochastic flow* $\widehat{x}_\varphi(t;\lambda)$ *generated by the SDE* (3.1) *can be represented as in* (3.7). *In addition* $\psi(t,x) = \widehat{\psi}(t,\widehat{z}(t,x))$, *is the unique solution of the stochastic flow equation* (3.3).

The proof follows the arguments used in Lemma 2.2 of subsection 2.1.

Lemma 3.3. *Under the hypothesis* (3.5), *consider the continuous and* $\mathcal{F}_t$*-adapted process* $\widehat{z}(t,x)$. *Then the following SPDE of parabolic type is valid*

$$\begin{cases} \mathrm{d}_t\widehat{z}(t,x) + \sum_{i=1}^{m} \partial_x\widehat{z}(t,x)g_i(x)\widehat{\circ}\mathrm{d}w_i(t) = 0, \\ \widehat{z}(0,x) = x, \end{cases}$$

where the Fisk-Stratonovich integral "$\widehat{\circ}$*" is computed using the Ito stochastic integral "$\cdot$"*

$$h_i(t,x)\widehat{\circ}\mathrm{d}w_i(t) = h_i(t,x)\cdot\mathrm{d}w_i(t) - \frac{1}{2}\partial_x h_i(t,x)g_i(x)\mathrm{d}t$$

Proof. The conclusion is a direct consequence of applying the standard rule of stochastic derivation with respect to $\sigma = w(t)$ and the smooth deterministic mapping $H(\sigma)[x] = G(-\sigma)[x]$. In this respect, using that $H(\sigma)\circ G(\sigma)[\lambda] = \lambda$ for any $x = G(\sigma)[\lambda]$, we get

$$\begin{cases} \partial_{\sigma_i}H(\sigma)[x] = -\partial_x\{H(\sigma)[x]\}g_i(x), \\ \partial^2_{\sigma_i}\{H(\sigma)[x]\} = \partial_{\sigma_i}\{\partial_{\sigma_i}\{H(\sigma)[x]\}\} = \partial_{\sigma_i}\{-\partial_x\{H(\sigma)[x]\}g_i(x)\} \\ \qquad\qquad = \partial_x\{\partial_x\{H(\sigma)[x]\}g_i(x)\}g_i(x), \end{cases}$$

for $\sigma \in \mathbb{R}^m$, $x \in \mathbb{R}^n$ and each $i \in \{1,\ldots,m\}$. Recall that the standard rule of stochastic derivation leads us to the SDE

$$\mathrm{d}_t\widehat{z}(t,x) = \sum_{i=1}^{m} \partial_{\sigma_i}\{H(\sigma)[x]\}_{(\sigma=w(t))}\cdot\mathrm{d}w_i(t) + \frac{1}{2}\sum_{i=1}^{m}\partial^2_{\sigma_i}\{H(\sigma)[x]\}_{(\sigma=w(t))}\mathrm{d}t.$$

Rewritting the right hand side in the last equality we get the SPDE of parabolic type required. □

Lemma 3.4. *Assume the hypotheses* (3.5) *and* (3.6) *are satisfied and consider* $\psi(t,x)$ *defined in Lemma 3.2. Then* $u(t,x) := h(\psi(t,x))$, *for arbitrary*

$h \in (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n)$, *satisfies the following nonlinear SPDE of parabolic type*

$$(3.11)\qquad \begin{cases} \mathrm{d}_t u(t,x) + \langle \partial_x u(t,x), \sum_{i=1}^m \varphi_i(\psi(t,x) f_i(x))\rangle \mathrm{d}t \\ \qquad\qquad + \sum_{i=1}^m \langle \partial_x u(t,x), g_i(x)\rangle \widehat{\circ} \mathrm{d}w_i(t) = 0, \\ u(0,x) = h(x) \end{cases}$$

The proof uses the same arguments as in lemma 2.4 of subsection 2.1.

Remark 3.1. The complete solution of Problem (P1) is contained in lemmas 2.1–2.4. We are now in position to state the main result of this section.

THEOREM 3.3. *Assume that the vector fields* $\{f_1, \ldots, f_m\} \subset (\mathcal{C}_b \cap \mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n; \mathbb{R}^n)$, $\{g_1, \ldots, g_m\} \subset (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n; \mathbb{R}^n)$ *and scalar functions* $\{\varphi_1, \ldots, \varphi_m\} \subset (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n)$ *satisfy the hypotheses 3.5 and 3.6.*

Consider the continuous and $\mathcal{F}_t$*-adapted process* $\psi(t,x)$ *satisfying the flow equation* (3.3). *Then* $u(t,x) := h(\psi(t,x))$ *satisfies the nonlinear SPDE of parabolic type* (3.11) *for each* $h \in (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n)$.

3.2. SOLUTION FOR (P2)

Using the same notations as in subsection 2.1, we consider the unique solution $\widehat{x}_\psi(s; t, x)$, $s \in [t, T]$ satisfying the SDE (3.4) for each $0 \leq t \leq T$ and $x \in \mathbb{R}^n$. As far as SDE (3.4) is a non-markovian system, the evolution of the functional $S(t,x) := Eh(\widehat{x}_\psi(T; t, x))$ will be described for each $h \in \mathcal{C}_p^2(R^n)$ using the pathwise representation of the conditioned mean values functional $v(t,x) := E\{h(\widehat{x}_\psi(T; t, x)) \mid \psi(t,x)\}$.

Assuming the hypotheses 3.5 and 3.6 we may write the following integral representation

$$(3.12)\qquad \widehat{x}_\psi(T; t, x) = G(w(T) - w(t)) \circ F[(T-t)\varphi(\psi(t,x))](x).$$

The right hand side of (3.12) is a continuous mapping of the two independent random variables, $z_1 = w(T) - w(t)$ and $z_2 = \psi(t,x)$, $\mathcal{F}_t$-measurable. Using the parameterized random variable $y(t,x;\lambda) = G(w(T) - w(t)) \circ F[(T-t)\varphi(\lambda)](x)$ we may obtain that $v(t,x) = [Eh(y(t,x;\lambda))](\lambda = \psi(t,x))$. Here, the functional $u(t,x;\lambda) = Eh(y(t,x;\lambda))$ satisfies a backward parabolic equation (Kolmogorov‘s equation) for each parameter λ and we shall obtain the representation $v(t,x) = u(t,x;\psi(t,x))$.

In conclusion, the functional $S(t,x) = Eh(\widehat{x}_\psi(T; t, x))$ can be reprezented by

$$(3.13)\qquad S(t,x) = E[E\{h(\widehat{x}_\psi(T; t, x)) \mid \psi(t,x)\}] = Eu(t,x;\psi(t,x)),$$

where $u(t,x;\lambda)$ satisfies the corresponding backward parabolic equation with parameter λ

(3.14)
$$\begin{cases} \partial_t u(t,x;\lambda) + \langle \partial_x u(t,x;\lambda), f(x,\lambda)\rangle + \frac{1}{2}\sum_{i=1}^{m} \langle \partial_x^2 u(t,x;\lambda) g_i(x), g_i(x)\rangle = 0 \\ u(T,x;\lambda) = h(x),\ f(x,\lambda) = \sum_{i=1}^{m} \varphi_i(\lambda) f_i(x) + \frac{1}{2}\sum_{i=1}^{m} [\partial_x g_i(x)] g_i(x). \end{cases}$$

We conclude these remarks by the next theorem.

THEOREM 3.4. *Assume that the vector fields* $\{f_1,\dots,f_m\} \subset (\mathcal{C}_b \cap \mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n;\mathbb{R}^n)$, $\{g_1,\dots,g_m\} \subset (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n;\mathbb{R}^n)$, *and scalar functions* $\{\varphi_1,\dots,\varphi_m\} \subset (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n)$ *of the SDE* (3.4) *satisfy the hypotheses* (3.5) *and* (3.6). *Then the evolution of the functional* $S(t,x) := Eh(\widehat{x}_\psi(T;t,x))$, *for an arbitrary* $h \in \mathcal{C}_p^2(R^n)$, *can be described as in* (3.13), *where* $u(t,x;\lambda)$ *satisfies the linear backward parabolic equations* (3.14), *for each parameter* λ.

Final remark. One may wonder about the meaning of the martingale representation associated with the non-markovian functionals $h(\widehat{x}_\psi(T;t,x))$, $h \in \mathcal{C}_p^2(R^n)$ Along this line, we may use the parameterized functional $u(t,x;\lambda)$ satisfying the backward parabolic equation (3.14). Writing $h(\widehat{x}_\psi(T;t,x)) = u(T,\widehat{x}_\psi(T;t,x)$ for $\widehat{\lambda} = \psi(t,x))$ and applying the standard rule of stochastic derivation with respect to the smooth mapping $u(s,x;\widehat{\lambda})$ and the stochastic process $\widehat{x}_\psi(s;t,x)$ we get

$$(3.15) \qquad \begin{cases} h(\widehat{x}_\psi(T;t,x)) = u(t,x;\widehat{\lambda}) \\ +\int_t^T (\partial_s + L_{\widehat{\lambda}})(u)(s,\widehat{x}_\psi(s;t,x);\widehat{\lambda})\mathrm{d}s \\ +\sum_{i=1}^m \int_t^T \langle \partial_x u(s,\widehat{x}_\psi(s;t,x);\widehat{\lambda}), g_i(x)\rangle \mathrm{d}w_i(s), \end{cases}$$

where $L_{\widehat{\lambda}}(u)(s,x;\widehat{\lambda}) := \langle \partial_x u(s,x;\widehat{\lambda}), f(x,\widehat{\lambda})\rangle + \frac{1}{2}\sum_{i=1}^m \langle \partial_x^2 u(s,x;\widehat{\lambda}) g_i(x), g_i(x)\rangle$ coincides with the parabolic operator in the PDE (3.14). Moreover we obtain the following martingale representation

$$(3.16) \qquad \begin{cases} h(\widehat{x}_\psi(T;t,x)) = u(t,x;\psi(t,x)) \\ +\sum_{i=1}^m \int_t^T \langle \partial_x u(s,\widehat{x}_\psi(s;t,x);\widehat{\lambda}), g_i(x)\rangle \cdot \mathrm{d}w_i(s), \end{cases}$$

which shows that the standard constant in the markovian case is replaced by the $\mathcal{F}_t$-measurable random variable $u(t,x;\psi(t,x))$. In addition, the backward evolution of the stochastic functional $Q(t,x) := h(\widehat{x}_\psi(T;t,x))$ given in (3.16) depends essentially on the forward evolution process $\psi(t,x)$.

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