BINOMIAL EDGE IDEALS WITH TWO ASSOCIATED PRIMES

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We study binomial edge ideals J_G with $|\operatorname{Ass}(J_G)| = 2$. We give an explicit description of the modules of deficiencies, the duals of local cohomology modules and compute the Castelnuovo-Mumford regularity. As an application, we characterize all graphs G with $|\operatorname{Ass}(J_G)| = 2$ such that S/J_G is sequentially Cohen-Macaulay.

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1. INTRODUCTION

The main purpose of this paper is to study the binomial edge ideal J_G with $|\operatorname{Ass}(J_G)| = 2$. Let G denote a connected undirected graph on n vertices. For an arbitrary field K let $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ denote the polynomial ring in $2n$ variables. To the graph G one can relate the binomial edge ideal $J_G \subset S$ generated by binomials $x_i y_j - x_j y_i, i < j$, such that $\{i, j\}$ is an edge of G. This construction was first found in [6] and independently in [8]. The algebraic properties of binomial edge ideals in terms of combinatorial properties of graphs (and vice versa) were investigated by many authors in [4–11, 14, 17–19]. The Cohen-Macaulay property of binomial edge ideals was studied in [4, 9] and [10]. As a certain generalization of the Cohen-Macaulay property the first author has studied approximately Cohen-Macaulay property as well as sequentially Cohen-Macaulay property in [17] and [18] respectively.

In the present paper, we investigate the sequentially Cohen-Macaulay property of binomial edge ideals with $|\operatorname{Ass}(J_G)| = 2$. We study the modules of deficiencies, the duals of local cohomology modules and compute the Castelnuovo-Mumford regularity.

The paper is structured as follows: in Section 2, there is preliminaries and auxiliary results that we need in the rest of the paper. In Section 3, we study binomial edge ideals with $|\operatorname{Ass}(J_G)| = 2$ and there is a proof of our main result.

2. PRELIMINARIES

First of all, we will introduce the notation used in the sequel. Moreover we summarize a few auxiliary results that we need.

Definition 2.1. For a set $T \subset [n]$ let $G_{[n]\setminus T}$ denote the graph obtained by deleting all vertices of G that belong to T.

Let $c = c(T)$ denote the number of connected components of $G_{[n]\setminus T}$. Let $G_1, \ldots, G_c$ denote the connected components of $G_{[n]\setminus T}$. Then define

$$P_T(G) = (\cup_{i\in T}\{x_i, y_i\}, J_{\tilde{G}_1}, \ldots, J_{\tilde{G}_c}),$$

where $\tilde{G}_i, i = 1, \ldots, c$, denotes the complete graph on $V(G_i), i = 1, \ldots, c$.

The following result is important for the understanding of the binomial edge ideal of G.

Lemma 2.2. *With the previous notation the following holds:*

(a) *$P_T(G) \subset S$ is a prime ideal of height $n - c + |T|$, where $|T|$ denotes the number of elements of T.*
(b) *$J_G = \cap_{T\subseteq[n]} P_T(G)$.*
(c) *$J_G \subset P_T(G)$ is a minimal prime if and only if either $T = \emptyset$ or $T \neq \emptyset$ and $c(T \setminus \{i\}) < c(T)$ each $i \in T$.*

Proof. For the proof we refer to [6]. □

Therefore J_G is an intersection of prime ideals. That is, J_G is a homogenous radical ideal with natural grading induced by the $\mathbb{N}$-grading of S.

Let M denote a finitely generated graded S-module. As an technical tool we shall use the local cohomology modules of M with respect to S_+, denoted by $H^i(M), i \in \mathbb{Z}$. For the basic properties of it we refer to the textbook of Brodmann and Sharp (see [1]). In particular the Castelnuovo-Mumford regularity $\operatorname{reg} M$ of M is defined as

$$\operatorname{reg}(M) := \max\{e(H^i(M)) + i |\operatorname{depth}(M) \leq i \leq \dim(M)\},$$

where $e(H^i(M))$ is the least integer m such that, for all $k > m$, the degree k part of the i-th local cohomology module of M is zero. For our investigations we also need the following definition.

Definition 2.3. Let M denote a finitely generated graded S-module and $d = \dim M$. For an integer $i \in \mathbb{Z}$ put

$$\omega^i(M) = \operatorname{Ext}_S^{2n-i}(M, S(-2n))$$

and call it the i-th module of deficiency. Moreover we define $\omega(M) = \omega^d(M)$ the canonical module of M. We write also $\omega_{2\times}(M) = \omega(\omega(M))$. These modules have been introduced and studied in [12].

Note that by the graded version of Local Duality (see *e.g.* [1]) there is the natural graded isomorphism $\omega^i(M) \cong \mathrm{Hom}_K(H^i(M), K)$ for all $i \in \mathbb{Z}$. If M is Cohen-Macaulay then $\omega(M)$ is also Cohen-Macaulay but the converse is not true. An S-module M is said to be canonically Cohen-Macaulay if $\omega(M)$ is Cohen-Macaulay.

Definition 2.4. An S-module M is called sequentially Cohen-Macaulay if for all $0 \leq i < d$ the module of deficiency $\omega^i(M)$ is either zero or an i-dimensional Cohen-Macaulay module. (see [13]).

3. BINOMIAL EDGE IDEALS WITH $|\mathrm{Ass}(J_G)| = 2$

In this section, we assume $|\mathrm{Ass}(J_G)| = 2$. Then $\mathrm{Ass}(J_G) = \{J_{\tilde{G}}, P_T(G)\}$ for some T because $J_{\tilde{G}}$ is always an associated prime of J_G. The following three results from the paper of Sharifan [15] are important for us.

LEMMA 3.1 ([15]). *Let G be connected graph on $[n]$ and $T = \{i \in [n] : \deg(i) = n-1\}$. Then $|\mathrm{Ass}(J_G)| = 2$ if and only if the following conditions hold.*

(1) *$T \neq \emptyset$ and $G_{[n]\setminus T}$ is disconnected.*
(2) *$G_{[n]\setminus T}$ is a disjoint union of complete graphs.*

COROLLARY 3.2 ([15]). *Let G be a connected graph on $[n]$. Then $|\mathrm{Ass}(J_G)| = 2$ if and only if G is the join of a complete graph G_1 and a graph G_2 where G_2 is a disjoint union of complete graphs.*

LEMMA 3.3 ([15]). *Let G be a connected graph on $[n]$ with $|\mathrm{Ass}(J_G)| = 2$. Then $\mathrm{depth}(S/J_G) = n - |T| + 2$ where $T = \{i \in [n] : \deg(i) = n-1\}$. In particular, J_G is Cohen-Macaulay if and only if $|T| = 1$ and $c(T) = 2$.*

The proof of all these results can be found in Section 4 of the paper [15]. The regularity of the binomial edge ideal of the disjoint union of two graphs is given by the following lemma.

LEMMA 3.4. *Let $J_{G_1} \subset S_1$ and $J_{G_2} \subset S_2$ be binomial edge ideals of graphs G_1 and G_2 where S_1 and S_2 are polynomial rings over the field K with the different set of variables, then:*

$$\mathrm{reg}(S_1/J_{G_1} \otimes S_2/J_{G_2}) = \mathrm{reg}(S_1/J_{G_1}) + \mathrm{reg}(S_2/J_{G_2})$$

Proof. Let $\mathcal{F}_\bullet$ be the minimal free graded resolution of J_{G_1} over S_1 and let $\mathcal{H}_\bullet$ be the minimal free graded resolution of J_{G_2} over S_2 then the double complex $\mathcal{F}_\bullet \otimes_K \mathcal{H}_\bullet$ is minimal free resolution of $J_{G_1} + J_{G_2}$ over S, where S

is the polynomial ring in the union of the variables of S_1 and S_2. Note that $S_1/J_{G_1} \otimes S_2/J_{G_2} = S/(J_{G_1} + J_{G_2})$ and the degree k component of $\mathcal{F}_\bullet \otimes_K \mathcal{H}_\bullet$ is

$$(\mathcal{F}_\bullet \otimes_K \mathcal{H}_\bullet)_k = \bigoplus_{i+j=k} F_i \otimes H_j.$$

Now $F_i \otimes H_j = \bigoplus_a S_1(-a) \otimes \bigoplus_b S_2(-b) = \bigoplus_{a,b} S(-a-b)$ which proves the claim. $\square$

Corollary 3.5. *Let α denotes the number of connected components of $G_{[n]/T}$ with $|V(G_i)| \geq 2$, then:*

$$\operatorname{reg}(S/P_T(G)) = \alpha$$

Proof. $\operatorname{reg}(S/J_{\tilde{G}_i}) = 1$ for all G_i with $|V(G_i)| \geq 2$. $\square$

In the following result, we will describe the modules of deficiencies $\omega^i(S/J_G)$ of the binomial edge ideal J_G with two associated primes.

Theorem 3.6. *Let G be a connected graph with $|\operatorname{Ass}(J_G)| = 2$.*

(1) *Let $c(T) > 2$ and $|T| = 1$. Then the binomial edge ideal $J_G \subset S$ has the following properties:*
   (a) $\omega^i(S/J_G) = 0$ *if and only if* $i \notin \{n+1, n-1+c(T)\}$.
   (b) $\omega^{n-1+c(T)}(S/J_G) \cong \omega^{n-1+c(T)}(S/P_T(G))$
   (c) $\omega^{n+1}(S/J_G)$ *is a $(n+1)$-dimensional Cohen-Macaulay module and there is an isomorphism* $\omega^{n+1}(\omega^{n+1}(S/J_G)) \cong (J_{\tilde{G}}, P_T(G))/J_{\tilde{G}}$.

(2) *Let $c(T) = 2$ and $|T| > 1$. Then the binomial edge ideal $J_G \subset S$ has the following properties:*
   (a) $\omega^i(S/J_G) = 0$ *if and only if* $i \notin \{n+1, n-|T|+2\}$.
   (b) $\omega^{n+1}(S/J_G) \cong \omega^{n+1}(S/J_{\tilde{G}})$
   (c) $\omega^{n-|T|+2}(S/J_G)$ *is a $(n-|T|+2)$-dimensional Cohen-Macaulay module and there is an isomorphism* $\omega^{n-|T|+2}(\omega^{n-|T|+2}(S/J_G)) \cong (J_{\tilde{G}}, P_T(G))/P_T(G)$.

(3) *Let $c(T) > 2$ and $|T| > 1$. Then the binomial edge ideal $J_G \subset S$ has the following properties:*
   (a) $\omega^i(S/J_G) = 0$ *if and only if* $i \notin \{n+1, n-|T|+2, n-|T|+c(T)\}$.
   (b) $\omega^{n+1}(S/J_G) \cong \omega^{n+1}(S/J_{\tilde{G}})$
   (c) $\omega^{n-|T|+2}(S/J_G) \cong \omega^{n-|T|+1}(S/(J_{\tilde{G}}, P_T(G)))$
   (d) $\omega^{n-|T|+c(T)}(S/J_G) \cong \omega^{n-|T|+c(T)}(S/P_T(G))$

Proof. We use the short exact sequence

$$0 \to S/J_G \to S/J_{\tilde{G}} \oplus S/P_T(G) \to S/(J_{\tilde{G}}, P_T(G)) \to 0. \tag{3.1}$$

It induces a short exact sequence

$$0 \to H^n(S/(J_{\tilde{G}}, P_T(G))) \to H^{n+1}(S/J_G) \to H^{n+1}(S/J_{\tilde{G}}) \to 0 \tag{3.2}$$

and an isomorphism

$$H^{n-1+c(T)}(S/J_G) \cong H^{n-1+c(T)}(S/P_T(G)). \tag{3.3}$$

Moreover the Cohen-Macaulayness of $S/J_{\tilde{G}}, S/P_T(G)$ and $S/(J_{\tilde{G}}, P_T(G))$ of dimensions $n+1$, $n-1+c(T)$ and n respectively imply that $H^i(S/J_G) = 0$ if $i \notin \{n+1, n-1+c(T)\}$.

The short exact sequence on local cohomology induces the following exact sequence

$$0 \to \omega^{n+1}(S/J_{\tilde{G}}) \to \omega^{n+1}(S/J_G) \to \omega^n(S/(J_{\tilde{G}}, P_T(G))) \to 0$$

by Local Duality. Taking into account that both $\omega^{n+1}(S/J_{\tilde{G}})$ and $\omega^n(S/(J_{\tilde{G}}, P_T(G)))$ are Cohen-Macaulay modules of dimension $n+1$ and n respectively, then $\operatorname{depth} \omega^{n+1}(S/J_G) \geq n$. By applying local cohomology and dualizing again it induces the following exact sequence

$$0 \to \omega^{n+1}(\omega^{n+1}(S/J_G)) \to S/J_{\tilde{G}} \xrightarrow{f} S/(J_{\tilde{G}}, P_T(G)) \to \omega^n(\omega^{n+1}(S/J_G)) \to 0.$$

The homomorphism f is induced by the commutative diagram

$$\begin{array}{ccc} S/J_{\tilde{G}} & \to & S/(J_{\tilde{G}}, P_T(G)) \\ \downarrow & & \downarrow \\ \omega_{2\times}(S/J_{\tilde{G}}) & \to & \omega_{2\times}(S/J_{\tilde{G}}, P_T(G)). \end{array}$$

Note that the vertical maps are isomorphisms. Since the upper horizontal map is surjective the lower horizontal map is surjective too. Therefore $\omega^n(\omega^{n+1}(S/J_G)) = 0$. That is $\operatorname{depth} \omega^{n+1}(S/J_G) = n+1$ and hence $\omega^{n+1}(S/J_G)$ is a Cohen-Macaulay module. Moreover $\omega^{n+1}(\omega^{n+1}(S/J_G)) \cong (J_{\tilde{G}}, P_T(G))/J_{\tilde{G}}$. This finally proves all the statements in (1). Similar arguments work also for the proofs of (2) and (3). □

LEMMA 3.7. *Let I be a graded ideal in S. Let S/I is Cohen-Macaulay with* $\dim(S/I) = d$ *and* $\operatorname{reg}(S/I) = r$ *then* $e(H^d(S/I)) = r - d$.

Proof. Note that $H^i(S/I) = 0$ for all $i \neq d$. □

COROLLARY 3.8. *Let G be a connected graph with* $|\operatorname{Ass}(J_G)| = 2$. *Then*

$$\operatorname{reg}(S/J_G) = \max\{2, \alpha\}$$

Proof. Let $c(T) > 2$ and $|T| = 1$. In this case $H^i(S/J_G) = 0$ if and only if $i \notin \{n+1, n-1+c(T)\}$. In the view of Lemma 3.7, we have $e(H^{n-1+c(T)}(S/P_T(G))) = \alpha - n + 1 - c(T)$, $e(H^{n+1}(S/J_{\tilde{G}})) = -n$ and $e(H^n(S/(J_{\tilde{G}}, P_T(G)))) = -n+1$. By using short exact sequence 3.2 and an isomorphism 3.3, we get $e(H^{n+1}(S/J_G)) = -n+1$ and $e(H^{n-1+c(T)}(S/J_G)) = \alpha - n + 1 - c(T)$. By similar arguments the remaining cases can be proved. We omit the details. $\square$

Corollary 3.9. *Let G be a connected graph with $|\operatorname{Ass}(J_G)| = 2$. Then*

(a) *S/J_G is a Cohen-Macaulay canonical ring and* $\operatorname{depth}\omega^i(S/J_G) \geq i-1$ *for all* $\operatorname{depth} S/J_G \leq i \leq \dim S/J_G$.

(b) *S/J_G is a sequentially Cohen-Macaulay ring if and only if either $c(T) = 2$ or $|T| = 1$.*

Proof. Let $c(T) > 2$ and $|T| > 1$ then by Theorem 3.6 (3), $\omega^{n-|T|+2}(S/J_G)$ is Cohen-Macaulay of dimension $n-|T|+1$. Therefore S/J_G is not sequentially Cohen-Macaulay. The converse is easily seen from the statements in (1) and (2) of Theorem 3.6. $\square$

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