SPECTRAL CHARACTERIZATION OF GRAPHS WITH SMALL SECOND LARGEST LAPLACIAN EIGENVALUE

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The family $G$ of connected graphs with second largest Laplacian eigenvalue at most $\theta$, where $\theta = 3.2470$ is the largest root of the equation $\mu^3 - 5\mu^2 + 6\mu - 1 = 0$, is characterized by Wu, Yu and Shu [Y.R. Wu, G.L. Yu and J.L. Shu, Graphs with small second largest Laplacian eigenvalue, European J. Combin. 36 (2014) 190–197]. Let $G(a, b, c, d)$ be a graph with order $n = 2a + b + 2c + 3d + 1$ that consists of $a$ triangle(s), $b$ pendant edge(s), $c$ pendant path(s) of length 2 and $d$ pendant path(s) of length 3, sharing a common vertex. In this paper, we first prove that the graph $G(a, b, c, d)$ is determined by its Laplacian spectrum. Then we conclude that except for two graphs, all the graphs in $G$ are determined by their Laplacian spectra.

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Key words: Laplacian spectrum, L-cospectral graphs, second largest Laplacian eigenvalue.

1. INTRODUCTION

The graphs considered in this paper are simple and undirected. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E(G)$, where its order and size are $|V(G)| = n(G) = n$ and $|E(G)| = m(G) = m$, respectively. Denote by $d_i(G)$ the degree of a vertex $v_i$ in $G$. We denote the diagonal matrix of vertex degrees by $D(G)$ and denote the adjacency matrix by $A(G)$. The maximum eigenvalue of $A(G)$ is called the index of $G$. The matrix $L(G) = D(G) - A(G)$ ($Q(G) = D(G) + A(G)$) is called the Laplacian matrix (signless Laplacian matrix) of $G$. We use $\Phi(G; \mu)$ to denote the Laplacian characteristic polynomial of $L(G)$. Its eigenvalues will be called the Laplacian eigenvalues of graph $G$. Assume that $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G) = 0$ are the Laplacian eigenvalues of the graph $G$. The Laplacian spectrum of the graph $G$, denoted by $Spec_L(G)$, is the multiset of its Laplacian eigenvalues. Two graphs $G$ and $H$ are said to be L-cospectral, denoted by $Spec_L(G) = Spec_L(H)$ if they share the same Laplacian spectrum (i.e., equal Laplacian characteristic polynomial). A graph $G$ is said to be determined by the Laplacian spectrum.
(DLS for short) if for any graph $H$, $\text{Spec}_L(G) = \text{Spec}_L(H)$ implies that $H$ is isomorphic to $G$. Similar terminology will be used for $A(G)$ and $Q(G)$. So we can speak of adjacency spectrum, signless Laplacian spectrum, $\text{Spec}_A(G)$, $\text{Spec}_Q(G)$, $A$-cospectral, $Q$-cospectral, DAS and DQS.

Gutman, Gineityte, Lepović and Petrović [7] discovered some connections between photoelectron spectra and the Laplacian eigenvalues of the underlying molecular graphs. Petrović, Gutman, Lepović and Milekić [15] stressed that the results of determining graphs with a small number of Laplacian eigenvalues can be of interest in the photoelectron spectroscopy of organic compounds and characterized all connected bipartite graphs with $\mu_3(G) < 2$. Recently, there has been a lot of interest in the work of determining graphs with a small number of Laplacian eigenvalues exceeding a given value or studying the bounds of the $k$-th Laplacian eigenvalue. For example, Merris [11] studied the relations between the structure of graphs and the number of eigenvalues greater than two. Zhang [18] studied the graphs with fourth Laplacian eigenvalue less than two. Zhang also characterized all connected bipartite graphs whose third largest Laplacian eigenvalue is less than three in [19]. The background of spectral graph theory and terminology not defined can be found in [2] for references.

van Dam and Haemers [3] asked the question Which graphs are determined by their spectra. This is a difficult problem in the theory of graph spectra. The exact characterization of graphs with second largest eigenvalue exceeding a given value is extensively studied, however, whether they are determined by their Laplacian spectra or not is less considered. Recently, Omidi [13] showed that graphs of index less than 2 are determined by their Laplacian spectra. Li, Guo and Shiu [9] studied extremal graphs for the extremal values of the second largest Laplacian eigenvalue. They also showed that graphs with second largest Laplacian eigenvalue at most 3 are determined by their Laplacian spectra. For the detailed background and some known results on this subject, we refer the readers to the excellent surveys [12, 5, 6, 4] and the references therein.

The family $\mathcal{G}$ of connected graphs with second largest Laplacian eigenvalue at most $\theta$, where $\theta = 3.2470$ is the largest root of the equation $\mu^3 - 5\mu^2 + 6\mu - 1 = 0$, is characterized by Wu, Yu and Shu in [17]. Let $G(a, b, c, d)$ be a graph with order $n = 2a + b + 2c + 3d + 1$ that consists of $a$ triangle(s), $b$ pendant edge(s), $c$ pendant path(s) of length 2 and $d$ pendant path(s) of length 3, sharing a common vertex (see Figure 1). The graph $G(0, b, c, d)$ is also known as a starlike tree (see [8]), $G(a, b, c, 0)$ is also known as a firefly graph (see [9]) and $G(a, 0, 0, 0)$ is also known as a friendship graph (see [16]). It is well known that the starlike tree $G(0, b, c, d)$ is DLS (see [14]), the firefly graph $G(a, b, c, 0)$
is also DLS (see [9]), and the friendship graph $G(a,0,0,0)$ were shown to be
DLS in [10], DQS in [16], with one exception in the case $a=16$, DAS in [1].
In this paper, we first show that the graph $G(a,b,c,d)$ is determined by
its Laplacian spectrum. Then we conclude that except for two graphs, all the
graphs in $\mathcal{G}$ are determined by their Laplacian spectra.

2. PRELIMINARIES

We first present some well known results which will play an important
role throughout this paper.

Lemma 2.1 ([3]). Let $G$ and $H$ be $L$-cospectral graphs. Then
(i) $G$ and $H$ have the same number of vertices;
(ii) $G$ and $H$ have the same number of edges;
(iii) $G$ and $H$ have the same number of spanning trees;
(iv) $G$ and $H$ have the same number of components;
(v) $\sum_{i=1}^{n} d_i(G)^2 = \sum_{i=1}^{n} d_i(H)^2$;
(vi) $q(G) = 6n_3(G) - \sum_{i=1}^{n} d_i(G)^3 = 6n_3(H) - \sum_{i=1}^{n} d_i(H)^3 = q(H)$, denote
by $n_3(G)$ the number of triangles in $G$.

Theorem 2.2 ([14]). Let $G$ be a starlike tree. Then $G$ is determined by
its Laplacian spectrum.

Theorem 2.3 ([9]). The firefly graph is determined by its Laplacian spec-
trum.

Now we quote a theorem due to Wu, Yu and Shu [17] which characterizes
all connected graphs with second largest Laplacian eigenvalue no more than $\theta$.

Theorem 2.4 ([17]). Let $G$ be a connected graph. Then $G \in \mathcal{G}$, i.e.
$\mu_2(G) \leq \theta$ if and only if $G$ is a subgraph of one of the graphs $G(a,b,c,d)$
$(a,b,c,d \geq 0)$, $U_3^3$, $B_3^1$, $S_1$, $B_4^6$, $U_4^{13}$ and $H$ shown in Figure 1.

By Theorems 2.2, 2.3 and 2.4, $\mathcal{G}$ contains all the graphs, labelled in
Tables 1–4 and $H$, $G(a,b,c,d)$ see Fig. 1. Denote by $P_n$ and $C_n$ the path and
cycle on $n$ vertices, respectively. For convenience in the following discussion,
$\mathcal{G}$ can be classified as $\mathcal{G} = W_0 \cup W_1 \cup W_2 \cup W_3 \cup W_4$, where
(i) $W_0 = \{P_6, T_1, T_2\}$ (see Table 1),
(ii) $W_1 = \{C_6, U_3^i, U_4^j\}$ for $(i = 1,2,\cdots,13; j = 1,2,\cdots,13)$ (see Tables 2
and 3 ),
(iii) $W_2 = \{B_3^i, B_4^j\}$ for $(i = 1,2,\cdots,5; j = 1,2,\cdots,7)$ (see Table 4),
(iv) $W_3 = \{S_1, S_2, S_3\}$ (see Table 1),
(v) $W_4 = \{H, G(a,b,c,d)\}$, where $2a + b + 2c + 3d + 1 \geq 11$, $a > 0, d > 0$. 
3. GRAPHS IN $\mathcal{G}$ ARE NOT L-COSPECTRAL

As is known, $P_6$ and $C_6$ are DLS. Thus, according to $(i)$ and $(ii)$ of Theorem 2.1, we have the following Lemmas 3.1–3.4.

**Lemma 3.1.** $G \in W_i$ and $H \in W_j$ are not L-cospectral if $i \neq j$.

**Lemma 3.2.** The graphs in $W_0 = \{P_6, T_1, T_2\}$ (see Table 1) are not L-cospectral.

**Lemma 3.3.** The graphs in $W_3 = \{S_1, S_2, S_3\}$ (see Table 1) are not L-cospectral.

**Lemma 3.4.** The graphs in $W_4 = \{H, G(a,b,c,d)\}$, where $2a + b + 2c + 3d + 1 \geq 11$, $a > 0$, $d > 0$ (see Figure 1) are not L-cospectral.

**Lemma 3.5.** The graphs in $W_1 = \{C_6, U_3^i, U_4^j\}$ $(i = 1, 2, \cdots, 13; j = 1, 2, \cdots, 13)$ (see Tables 2 and 3) are not L-cospectral.

**Proof.** Since the number of spanning trees of graph in $U_3^1$ is 3, but the number of spanning trees of graph in $U_4^j$ is 4, we conclude that $G \in U_3^i$ and $H \in U_4^j$ are not L-cospectral. On the other hand, by Theorem 2.1 $(vi)$, the graphs
in $U^i_3$ and $U^j_4$ ($i = 1, 2, \cdots, 13; j = 1, 2, \cdots, 13$) (see Tables 2 and 3) are not L-cospectral, except for $U^{10}_3$ and $U^{11}_3$. At last, by a direct calculation, we have

$$\text{Spec}_L(U^{10}_3) = [6.1068, 3.2470, 3.0797, 3.15550, 1.4469, 1, 0.3676, 0.1981, 0],$$
$$\text{Spec}_L(U^{11}_3) = [6.1504, 3.1871, 3, 2.6180, 2.3204, 1.4757, 0.6298, 0.3820, 0.2366, 0].$$

Thus, we conclude that all graphs in $W_1$ are not L-cospectral. It completes this proof.

By similar arguments, we have the following lemma for graphs in $W_2$.

**Lemma 3.6.** The graphs in $W_2 = \{B^i_3, B^j_4\}$ ($i = 1, 2, \cdots, 5; j = 1, 2, \cdots, 7$) (see Table 4) are not L-cospectral except for $B^1_4$ and $B^2_4$, where $\text{Spec}_L(B^1_4) = \text{Spec}_L(B^2_4) = [5.2361, 3, 3, 2, 0.7639, 0]$.

### 4. **The Laplacian Spectral Characterization of $G(a, b, c, d)$**

In this section, we will consider the $DLS$-graphs in $G(a, b, c, d)$. In the following, we first compute the Laplacian polynomial of $G(a, b, c, d)$ and then prove $G(a, b, c, d)$ is $DLS$. 

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**Table 2**

$G \in U^i_3$ ($i = 1, 2, \cdots, 13$)

<table>
<thead>
<tr>
<th>$n(G)$</th>
<th>$n = 6$</th>
<th>$n = 7$</th>
<th>$n = 8$</th>
<th>$n = 9$</th>
<th>$n = 10$</th>
</tr>
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<tbody>
<tr>
<td>$G$</td>
<td>$U^1_3$</td>
<td>$U^3_3$</td>
<td>$U^5_3$</td>
<td>$U^7_3$</td>
<td>$U^{10}_3$</td>
</tr>
<tr>
<td>$q(G)$</td>
<td>$-66$</td>
<td>$-74$</td>
<td>$-154$</td>
<td>$-108$</td>
<td>$-170$</td>
</tr>
<tr>
<td>$G$</td>
<td>$U^2_3$</td>
<td>$U^4_3$</td>
<td>$U^6_3$</td>
<td>$U^8_3$</td>
<td>$U^{11}_3$</td>
</tr>
<tr>
<td>$q(G)$</td>
<td>$-54$</td>
<td>$-92$</td>
<td>$-100$</td>
<td>$-246$</td>
<td>$-170$</td>
</tr>
<tr>
<td>$G$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$q(G)$</td>
<td></td>
<td></td>
<td></td>
<td>$-162$</td>
<td>$-374$</td>
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<tr>
<td>$G$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$q(G)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$-254$</td>
</tr>
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### Table 3
$G \in U_j^2$ ($j = 1, 2, \cdots, 13$)

<table>
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<th>$n(G)$</th>
<th>$n = 4$</th>
<th>$n = 5$</th>
<th>$n = 6$</th>
<th>$n = 7$</th>
<th>$n = 8$</th>
<th>$n = 9$</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$q(G)$</td>
<td>$-32$</td>
<td>$-52$</td>
<td>$-90$</td>
<td>$-152$</td>
<td>$-244$</td>
<td>$-168$</td>
<td>$-176$</td>
</tr>
<tr>
<td>$G$</td>
<td>$U_4^1$</td>
<td>$U_4^2$</td>
<td>$U_4^3$</td>
<td>$U_4^6$</td>
<td>$U_4^8$</td>
<td>$U_4^{11}$</td>
<td>$U_4^{13}$</td>
</tr>
<tr>
<td>$q(G)$</td>
<td>$-60$</td>
<td>$-98$</td>
<td>$-160$</td>
<td>$-252$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$G$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$q(G)$</td>
<td>$-72$</td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

### Table 4
$G \in W_2$

<table>
<thead>
<tr>
<th>$n(G)$</th>
<th>$n = 6$</th>
<th>$n = 7$</th>
<th>$n = 8$</th>
<th>$n = 9$</th>
<th>$n = 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q(G)$</td>
<td>$-98$</td>
<td>$-160$</td>
<td>$-252$</td>
<td>$-260$</td>
<td></td>
</tr>
<tr>
<td>$G$</td>
<td>$B_4^1$</td>
<td>$B_4^3$</td>
<td>$B_4^5$</td>
<td>$B_4^7$</td>
<td></td>
</tr>
<tr>
<td>$q(G)$</td>
<td>$-98$</td>
<td>$-86$</td>
<td></td>
<td>$-168$</td>
<td></td>
</tr>
<tr>
<td>$G$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$q(G)$</td>
<td>$-74$</td>
<td>$-162$</td>
<td>$-254$</td>
<td>$-262$</td>
<td></td>
</tr>
<tr>
<td>$G$</td>
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<td></td>
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</tr>
<tr>
<td>$q(G)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$-382$</td>
</tr>
</tbody>
</table>
Lemma 4.1. Let $G(a, b, c, d)$ be a graph with order $n = 2a + b + 2c + 3d + 1$ that consists of $a$ triangles, $b$ pendant edges, $c$ pendant paths of length 2 and $d$ pendant paths of length 3, sharing a common vertex (see Figure 1). Then

$\Phi(G(a, b, c, d); \mu) = (\mu^2 - 4\mu + 3)^a(\mu - 1)^b(\mu^2 - 3\mu + 1)^c - 1(\mu^3 - 5\mu^2 + 6\mu - 1)^d - 1 f(\mu),$

where $f(\mu) = \mu^7 - (2a + b + c + d + 9)\mu^6 + (16a + 8b + 8c + 8d)\mu^5 - (44a + 22b + 23c + 23d + 46)\mu^4 + (48a + 24b + 29c + 29d + 33)\mu^3 - (18a + 9b + 15c + 16d + 10)\mu^2 + (2a + b + 2c + 3d + 1)\mu.$

Proof. Let $G = G(a, b, c, d)$ and we label the $a$ triangles, $b$ pendant edges, $c$ pendant paths of length 2 and $d$ pendant paths of length 3 by $v_{11}v_{12}v_{21}v_{22} \cdots v_{a1}v_{a2}$; $u_1 \cdots u_b$; $w_{11}w_{12} \cdots w_{c1}w_{c2}$ and $s_{11}s_{12}s_{13} \cdots s_{d1} s_{d2}s_{d3}$, respectively, where $u$ is the maximum degree vertex of $G(a, b, c, d)$ (see Fig. 1). Then the vertices of $G(a, b, c, d)$ can be partitioned as $\{u\} \cup V \cup U \cup W \cup S,$ where

$V = \{v_{11}, v_{12}, v_{21}, v_{22}, \cdots, v_{a1}, v_{a2}\}, U = \{u_1, \ldots, u_b\},$

$W = \{w_{11}, w_{12}, \cdots, w_{c1}, w_{c2}\}, S = \{s_{11}, s_{12}, s_{13}, \cdots, s_{d1}, s_{d2}, s_{d3}\}.$

The rows and columns of the matrix $\mu I - L(G(a, b, c, d))$ are arranged as the ordering in accordance with the vertices in $\{u\}, V, U, W$ and $S,$ respectively. And then by expanding the determinant of $\mu I - L(G(a, b, c, d))$ along the first row, we obtain

$\Phi(G(a, b, c, d); \mu) = (\mu^2 - 4\mu + 3)^a(\mu - 1)^b(\mu^2 - 3\mu + 1)^c - 1(\mu^3 - 5\mu^2 + 6\mu - 1)^d - 1 f(\mu),$

where $f(\mu) = \mu^7 - (2a + b + c + d + 9)\mu^6 + (16a + 8b + 8c + 8d)\mu^5 - (44a + 22b + 23c + 23d + 46)\mu^4 + (48a + 24b + 29c + 29d + 33)\mu^3 - (18a + 9b + 15c + 16d + 10)\mu^2 + (2a + b + 2c + 3d + 1)\mu.$

Theorem 4.2. The graph $G(a, b, c, d)$ ($a > 0, d > 0$ and $2a + b + 2c + 3d + 1 \geq 11$) (in $W_4$) displayed in Figure 1 is determined by its Laplacian spectrum.

Proof. For $G(a, b, c, d) \in \mathcal{G}$ with $n \geq 11$, if there exists a graph $F$ which is L-cospectral with $G(a, b, c, d).$ Then from Theorem 2.4, we have $F \in \mathcal{G}.$ By Lemmas 3.1 and 3.4 in section 3, we may write $F = G(a', b', c', d').$ Moreover, it is well known that the number of edges (or vertices) and spanning trees of a graph can be determined by its L-spectrum by Lemma 2.1. Then by Lemma 4.1 we have

(1) $2a + b + 2c + 3d = 2a' + b' + 2c' + 3d',$
(2) $3a = 3a',$
(3) $\Phi(G(a, b, c, d); \mu) = \Phi(G(a', b', c', d'); \mu).$
In addition, from Eqs. (2) and (3), we have

\[ (\mu - 1)^b(\mu^2 - 3\mu + 1)^{c-1}(\mu^3 - 5\mu^2 + 6\mu - 1)^{d-1} f(\mu) \]

\[ = (\mu - 1)^{b'}(\mu^2 - 3\mu + 1)^{c'-1}(\mu^3 - 5\mu^2 + 6\mu - 1)^{d'-1} g(\mu), \]

where

\[ f(\mu) = \mu^7 - (2a + b + c + d + 9)\mu^6 + (16a + 8b + 8c + 8d)\mu^5 \]

\[ - (44a + 22b + 23c + 23d + 46)\mu^4 \]

\[ + (48a + 24b + 29c + 29d + 33)\mu^3 \]

\[ - (18a + 9b + 15c + 16d + 10)\mu^2 + (2a + b + 2c + 3d + 1)\mu \]

and

\[ g(\mu) = \mu^7 - (2a' + b' + c' + d' + 9)\mu^6 + (16a' + 8b' + 8c' + 8d')\mu^5 \]

\[ - (44a' + 22b' + 23c' + 23d' + 46)\mu^4 \]

\[ + (48a' + 24b' + 29c' + 29d' + 33)\mu^3 \]

\[ - (18a' + 9b' + 15c' + 16d' + 10)\mu^2 + (2a' + b' + 2c' + 3d' + 1)\mu. \]

By Eq. (4), we get

\[ (\mu - 1)^{b-b'}(\mu^2 - 3\mu + 1)^{c-c'}(\mu^3 - 5\mu^2 + 6\mu - 1)^{d-d'} f(\mu) = g(\mu), \]

Clearly, the term in \( g(\mu) \) with the largest exponent is \( \mu^7 \), and similarly for

\[ (\mu - 1)^{b-b'}(\mu^2 - 3\mu + 1)^{c-c'}(\mu^3 - 5\mu^2 + 6\mu - 1)^{d-d'} f(\mu). \]

So Eq. (7) implies \( b = b', c = c' \) and \( d = d' \). That is \( G(a, b, c, d) = G(a', b', c', d') \). Therefore, each \( G(a, b, c, d) \in G \) with \( n \geq 11 \) is determined by its Laplacian spectrum. We complete this proof.

Combining the Lemmas 3.1–3.6 in Section 3 with the Theorem 4.2 in Section 4, we obtain:

**THEOREM 4.3.** Connected graphs with \( \mu_2 \leq \theta \) are DLS except for \( B_4^1 \) and \( B_4^2 \) shown in Table 4, where \( \theta = 3.2470 \) is the largest root of the equation \( \mu^3 - 5\mu^2 + 6\mu - 1 = 0 \).

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