SPECTRAL CHARACTERIZATION OF GRAPHS WITH SMALL SECOND LARGEST LAPLACIAN EIGENVALUE

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The family $\mathcal{G}$ of connected graphs with second largest Laplacian eigenvalue at most θ, where $\theta = 3.2470$ is the largest root of the equation $\mu^3 - 5\mu^2 + 6\mu - 1 = 0$, is characterized by Wu, Yu and Shu [Y.R. Wu, G.L. Yu and J.L. Shu, Graphs with small second largest Laplacian eigenvalue, European J. Combin. 36 (2014) 190–197]. Let $G(a, b, c, d)$ be a graph with order $n = 2a + b + 2c + 3d + 1$ that consists of a triangle(s), b pendant edge(s), c pendant path(s) of length 2 and d pendant path(s) of length 3, sharing a common vertex. In this paper, we first prove that the graph $G(a, b, c, d)$ is determined by its Laplacian spectrum. Then we conclude that except for two graphs, all the graphs in $\mathcal{G}$ are determined by their Laplacian spectra.

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Key words: Laplacian spectrum, L-cospectral graphs, second largest Laplacian eigenvalue.

1. INTRODUCTION

The graphs considered in this paper are simple and undirected. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E(G)$, where its order and size are $|V(G)| = n(G) = n$ and $|E(G)| = m(G) = m$, respectively. Denote by $d_i(G)$ the degree of a vertex v_i in G. We denote the diagonal matrix of vertex degrees by $D(G)$ and denote the adjacency matrix by $A(G)$. The maximum eigenvalue of $A(G)$ is called *the index* of G. The matrix $L(G) = D(G) - A(G)$ ($Q(G) = D(G) + A(G)$) is called the Laplacian matrix (signless Laplacian matrix) of G. We use $\Phi(G; \mu)$ to denote the Laplacian characteristic polynomial of $L(G)$. Its eigenvalues will be called the Laplacian eigenvalues of graph G. Assume that $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G) = 0$ are the Laplacian eigenvalues of the graph G. The *Laplacian spectrum* of the graph G, denoted by $Spec_L(G)$, is the multiset of its Laplacian eigenvalues. Two graphs G and H are said to be *L-cospectral*, denoted by $Spec_L(G) = Spec_L(H)$ if they share the same Laplacian spectrum (*i.e.*, equal Laplacian characteristic polynomial). A graph G is said to be *determined by the Laplacian spectrum*

(*DLS* for short) if for any graph H, $Spec_L(G) = Spec_L(H)$ implies that H is isomorphic to G. Similar terminology will be used for $A(G)$ and $Q(G)$. So we can speak of *adjacency spectrum, signless Laplacian spectrum,* $Spec_A(G)$, $Spec_Q(G)$, *A-cospectral, Q-cospectral, DAS and DQS*.

Gutman, Gineityte, Lepović and Petrović [7] discovered some connections between photoelectron spectra and the Laplacian eigenvalues of the underlying molecular graphs. Petrović, Gutman, Lepović and Milekić [15] stressed that the results of determining graphs with a small number of Laplacian eigenvalues can be of interest in the photoelectron spectroscopy of organic compounds and characterized all connected bipartite graphs with $\mu_3(G) < 2$. Recently, there has been a lot of interest in the work of determining graphs with a small number of Laplacian eigenvalues exceeding a given value or studying the bounds of the k-th Laplacian eigenvalue. For example, Merris [11] studied the relations between the structure of graphs and the number of eigenvalues greater than two. Zhang [18] studied the graphs with fourth Laplacian eigenvalue less than two. Zhang also characterized all connected bipartite graphs whose third largest Laplacian eigenvalue is less than three in [19]. The background of spectral graph theory and terminology not defined can be found in [2] for references.

van Dam and Haemers [3] asked the question *Which graphs are determined by their spectra.* This is a difficult problem in the theory of graph spectra. The exact characterization of graphs with second largest eigenvalue exceeding a given value is extensively studied, however, whether they are determined by their Laplacian spectra or not is less considered. Recently, Omidi [13] showed that graphs of index less than 2 are determined by their Laplacian spectra. Li, Guo and Shiu [9] studied extremal graphs for the extremal values of the second largest Laplacian eigenvalue. They also showed that graphs with second largest Laplacian eigenvalue at most 3 are determined by their Laplacian spectra. For the detailed background and some known results on this subject, we refer the readers to the excellent surveys [12, 5, 6, 4] and the references therein.

The family $\mathcal{G}$ of connected graphs with second largest Laplacian eigenvalue at most θ, where $\theta = 3.2470$ is the largest root of the equation $\mu^3 - 5\mu^2 + 6\mu - 1 = 0$, is characterized by Wu, Yu and Shu in [17]. Let $G(a, b, c, d)$ be a graph with order $n = 2a + b + 2c + 3d + 1$ that consists of a triangle(s), b pendant edge(s), c pendant path(s) of length 2 and d pendant path(s) of length 3, sharing a common vertex (see Figure 1). The graph $G(0, b, c, d)$ is also known as a starlike tree (see [8]), $G(a, b, c, 0)$ is also known as a firefly graph (see [9]) and $G(a, 0, 0, 0)$ is also known as a friendship graph (see [16]). It is well known that the starlike tree $G(0, b, c, d)$ is *DLS* (see [14]), the firefly graph $G(a, b, c, 0)$

is also DLS (see [9]), and the friendship graph $G(a,0,0,0)$ were shown to be DLS in [10], DQS in [16], with one exception in the case $a = 16$, DAS in [1].

In this paper, we first show that the graph $G(a,b,c,d)$ is determined by its Laplacian spectrum. Then we conclude that except for two graphs, all the graphs in $\mathcal{G}$ are determined by their Laplacian spectra.

2. PRELIMINARIES

We first present some well known results which will play an important role throughout this paper.

LEMMA 2.1 ([3]). *Let G and H be L-cospectral graphs. Then*

(i) G and H have the same number of vertices;

(ii) G and H have the same number of edges;

(iii) G and H have the same number of spanning trees;

(iv) G and H have the same number of components;

(v) $\sum_{i=1}^{n} d_i(G)^2 = \sum_{i=1}^{n} d_i(H)^2$;

(vi) $q(G) = 6n_3(G) - \sum_{i=1}^{n} d_i(G)^3 = 6n_3(H) - \sum_{i=1}^{n} d_i(H)^3 = q(H)$, denote by $n_3(G)$ the number of triangles in G.

THEOREM 2.2 ([14]). *Let G be a starlike tree. Then G is determined by its Laplacian spectrum.*

THEOREM 2.3 ([9]). *The firefly graph is determined by its Laplacian spectrum.*

Now we quote a theorem due to Wu, Yu and Shu [17] which characterizes all connected graphs with second largest Laplacian eigenvalue no more than θ.

THEOREM 2.4 ([17]). *Let G be a connected graph. Then $G \in \mathcal{G}$, i.e. $\mu_2(G) \leq \theta$ if and only if G is a subgraph of one of the graphs $G(a,b,c,d)$ $(a,b,c,d \geq 0)$, U_3^3, B_3^1, S_1, B_4^6, U_4^{13} and H shown in Figure 1.*

By Theorems 2.2, 2.3 and 2.4, $\mathcal{G}$ contains all the graphs, labelled in Tables 1–4 and H, $G(a,b,c,d)$ see Fig. 1. Denote by P_n and C_n the path and cycle on n vertices, respectively. For convenience in the following discussion, $\mathcal{G}$ can be classified as $\mathcal{G} = W_0 \cup W_1 \cup W_2 \cup W_3 \cup W_4$, where

(i) $W_0 = \{P_6, T_1, T_2\}$ (see Table 1),

(ii) $W_1 = \{C_6, U_3^i, U_4^j\}$ for $(i = 1, 2, \cdots, 13; j = 1, 2, \cdots, 13)$ (see Tables 2 and 3),

(iii) $W_2 = \{B_3^i, B_4^j\}$ for $(i = 1, 2, \cdots, 5; j = 1, 2, \cdots, 7)$ (see Table 4),

(iv) $W_3 = \{S_1, S_2, S_3\}$ (see Table 1),

(v) $W_4 = \{H, G(a,b,c,d)\}$, where $2a + b + 2c + 3d + 1 \geq 11$, $a > 0, d > 0$.

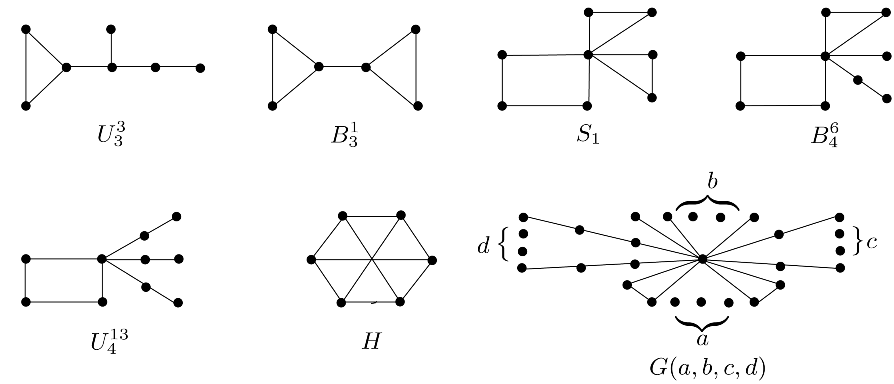
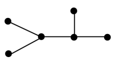
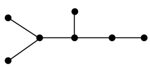
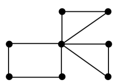
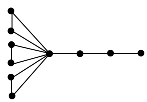
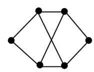


Fig. 1 – $G(a,b,c,d)$, U_3^3, B_3^1, S_1, B_4^6, U_4^{13} and H.

Table 1
$G \in W_0$ and $G \in W_3$

$n(G)$	$n=6$	$n=7$	$n=8$	$n=10$	$n=6$
G	 T_1	 T_2	 S_1	 S_2	 S_3

3. GRAPHS IN $\mathcal{G}$ ARE NOT L-COSPECTRAL

As is known, P_6 and C_6 are DLS. Thus, according to (i) and (ii) of Theorem 2.1, we have the following Lemmas 3.1–3.4.

LEMMA 3.1. *$G \in W_i$ and $H \in W_j$ are not L-cospectral if $i \neq j$.*

LEMMA 3.2. *The graphs in $W_0 = \{P_6, T_1, T_2\}$ (see Table 1) are not L-cospectral.*

LEMMA 3.3. *The graphs in $W_3 = \{S_1, S_2, S_3\}$ (see Table 1) are not L-cospectral.*

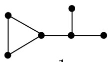
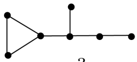
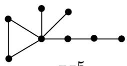
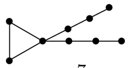
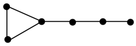
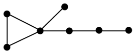
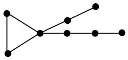
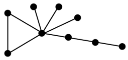
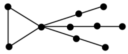
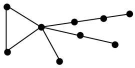
LEMMA 3.4. *The graphs in $W_4 = \{H, G(a,b,c,d)\}$, where $2a+b+2c+3d+1 \geq 11$, $a > 0, d > 0$ (see Figure 1) are not L-cospectral.*

LEMMA 3.5. *The graphs in $W_1 = \{C_6, U_3^i, U_4^j\}$ $(i = 1,2,\cdots,13; j = 1,2,\cdots,13)$ (see Tables 2 and 3) are not L-cospectral.*

Proof. Since the number of spanning trees of graph in U_3^i is 3, but the number of spanning trees of graph in U_4^j is 4, we conclude that $G \in U_3^i$ and $H \in U_4^j$ are not L-cospectral. On the other hand, by Theorem 2.1 (vi), the graphs

Table 2

$G \in U_3^i$ $(i = 1, 2, \cdots, 13)$

$n(G)$	$n = 6$	$n = 7$	$n = 8$	$n = 9$	$n = 10$
G	 U_3^1	 U_3^3	 U_3^5	 U_3^7	 U_3^{10}
$q(G)$	-66	-74	-154	-108	-170
G	 U_3^2	 U_3^4	 U_3^6	 U_3^8	 U_3^{11}
$q(G)$	-54	-92	-100	-246	-170
G				 U_3^9	 U_3^{12}
$q(G)$				-162	-374
G					 U_3^{13}
$q(G)$					-254

in U_3^i and U_4^j $(i = 1, 2, \cdots, 13; j = 1, 2, \cdots, 13)$ (see Tables 2 and 3) are not L-cospectral, except for U_3^{10} and U_3^{11}. At last, by a direct calculation, we have

$$Spec_L(U_3^{10}) = [6.1068, 3.2470, 3.0797, 3, 1.5550, 1.4469, 1, 0.3676, 0.1981, 0],$$

$$Spec_L(U_3^{11}) = [6.1504, 3.1871, 3, 2.6180, 2.3204, 1.4757, 0.6298, 0.3820, 0.2366, 0].$$

Thus, we conclude that all graphs in W_1 are not L-cospectral. It completes this proof.

By similar arguments, we have the following lemma for graphs in W_2.

LEMMA 3.6. *The graphs in* $W_2 = \{B_3^i, B_4^j\}$ $(i = 1, 2, \cdots, 5; j = 1, 2, \cdots, 7)$ *(see Table 4) are not L-cospectral except for* B_4^1 *and* B_4^2*, where* $Spec_L(B_4^1) = Spec_L(B_4^2) = [5.2361, 3, 3, 2, 0.7639, 0]$.

4. THE LAPLACIAN SPECTRAL CHARACTERIZATION OF $G(a, b, c, d)$

In this section, we will consider the DLS-graphs in $G(a, b, c, d)$. In the following, we first compute the Laplacian polynomial of $G(a, b, c, d)$ and then prove $G(a, b, c, d)$ is DLS.

Table 3

$G \in U_4^j \ (j = 1, 2, \cdots, 13)$

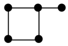
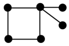
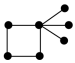
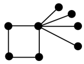
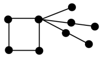
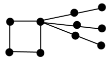
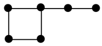
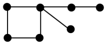
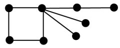
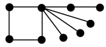
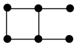
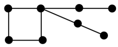
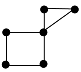
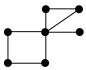
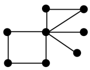
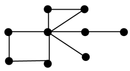
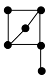
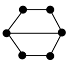
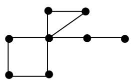
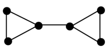
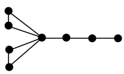
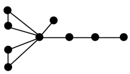
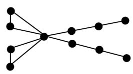
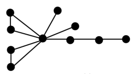
$n(G)$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$
G	 U_4^1	 U_4^2	 U_4^3	 U_4^6	 U_4^8	 U_4^{11}	 U_4^{13}
$q(G)$	-32	-52	-90	-152	-244	-168	-176
G			 U_4^4	 U_4^7	 U_4^9	 U_4^{12}	
$q(G)$			-60	-98	-160	-252	
G			 U_4^5		 U_4^{10}		
$q(G)$			-72		-106		

Table 4

$G \in W_2$

$n(G)$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$
G	 B_4^1	 B_4^4	 B_4^5	 B_4^7	
$q(G)$	-98	-160	-252	-260	
G	 B_4^2  B_4^3		 B_4^6		
$q(G)$	-98 -86		-168		
G	 B_3^1		 B_3^2	 B_3^3	 B_3^4
$q(G)$	-74		-162	-254	-262
G					 B_3^5
$q(G)$					-382

Lemma 4.1. *Let $G(a,b,c,d)$ be a graph with order $n = 2a+b+2c+3d+1$ that consists of a triangles, b pendant edges, c pendant paths of length 2 and d pendant paths of length 3, sharing a common vertex (see Figure 1). Then*

$$\Phi(G(a,b,c,d);\mu) = (\mu^2-4\mu+3)^a(\mu-1)^b(\mu^2-3\mu+1)^{c-1}(\mu^3-5\mu^2+6\mu-1)^{d-1}f(\mu),$$

where $f(\mu) = \mu^7 - (2a+b+c+d+9)\mu^6 + (16a+8b+8c+8d)\mu^5 - (44a+22b+23c+23d+46)\mu^4 + (48a+24b+29c+29d+33)\mu^3 - (18a+9b+15c+16d+10)\mu^2 + (2a+b+2c+3d+1)\mu$.

Proof. Let $G = G(a,b,c,d)$ and we label the a triangles, b pendant edges, c pendant paths of length 2 and d pendant paths of length 3 by $v_{11}v_{12}v_{21}v_{22}\cdots v_{a1}v_{a2}$; $u_1\cdots u_b$; $w_{11}w_{12}\cdots w_{c1}w_{c2}$ and $s_{11}s_{12}s_{13}\cdots s_{d1}\ s_{d2}s_{d3}$, respectively, where u is the maximum degree vertex of $G(a,b,c,d)$ (see Fig. 1). Then the vertices of $G(a,b,c,d)$ can be partitioned as $\{u\}\cup V\cup U\cup W\cup S$, where

$$V = \{v_{11}, v_{12}, v_{21}, v_{22}, \cdots, v_{a1}, v_{a2}\}, U = \{u_1, ..., u_b\},$$

$$W = \{w_{11}, w_{12}, \cdots, w_{c1}, w_{c2}\}, S = \{s_{11}, s_{12}, s_{13}, \cdots, s_{d1}, s_{d2}, s_{d3}\}.$$

The rows and columns of the matrix $\mu I - L(G(a,b,c,d))$ are arranged as the ordering in accordance with the vertices in $\{u\}$, V, U, W and S, respectively. And then by expanding the determinant of $\mu I - L(G(a,b,c,d))$ along the first row, we obtain

$$\Phi(G(a,b,c,d);\mu)=(\mu^2-4\mu+3)^a(\mu-1)^b(\mu^2-3\mu+1)^{c-1}(\mu^3-5\mu^2+6\mu-1)^{d-1}f(\mu),$$

where $f(\mu) = \mu^7 - (2a+b+c+d+9)\mu^6 + (16a+8b+8c+8d)\mu^5 - (44a+22b+23c+23d+46)\mu^4 + (48a+24b+29c+29d+33)\mu^3 - (18a+9b+15c+16d+10)\mu^2 + (2a+b+2c+3d+1)\mu$.

Theorem 4.2. *The graph $G(a,b,c,d)$ ($a>0, d>0$ and $2a+b+2c+3d+1\geq 11$) (in W_4) displayed in Figure 1 is determined by its Laplacian spectrum.*

Proof. For $G(a,b,c,d)\in\mathcal{G}$ with $n\geq 11$, if there exists a graph F which is L-cospectral with $G(a,b,c,d)$. Then from Theorem 2.4, we have $F\in\mathcal{G}$. By Lemmas 3.1 and 3.4 in section 3, we may write $F = G(a',b',c',d')$. Moreover, it is well known that the number of edges (or vertices) and spanning trees of a graph can be determined by its L-spectrum by Lemma 2.1. Then by Lemma 4.1 we have

(1) $$2a+b+2c+3d = 2a'+b'+2c'+3d',$$

(2) $$3a = 3a',$$

(3) $$\Phi(G(a,b,c,d);\mu) = \Phi(G(a',b',c',d');\mu).$$

In addition, from Eqs. (2) and (3), we have

$$(4)\quad \begin{aligned}&(\mu-1)^{b}(\mu^2-3\mu+1)^{c-1}(\mu^3-5\mu^2+6\mu-1)^{d-1}f(\mu)\\=&(\mu-1)^{b'}(\mu^2-3\mu+1)^{c'-1}(\mu^3-5\mu^2+6\mu-1)^{d'-1}g(\mu),\end{aligned}$$

where

$$(5)\quad \begin{aligned}f(\mu)=\mu^7&-(2a+b+c+d+9)\mu^6+(16a+8b+8c+8d)\mu^5\\&-(44a+22b+23c+23d+46)\mu^4\\&+(48a+24b+29c+29d+33)\mu^3\\&-(18a+9b+15c+16d+10)\mu^2+(2a+b+2c+3d+1)\mu\end{aligned}$$

and

$$(6)\quad \begin{aligned}g(\mu)=\mu^7&-(2a'+b'+c'+d'+9)\mu^6+(16a'+8b'+8c'+8d')\mu^5\\&-(44a'+22b'+23c'+23d'+46)\mu^4\\&+(48a'+24b'+29c'+29d'+33)\mu^3\\&-(18a'+9b'+15c'+16d'+10)\mu^2\\&+(2a'+b'+2c'+3d'+1)\mu.\end{aligned}$$

By Eq. (4), we get

$$(7)\quad (\mu-1)^{b-b'}(\mu^2-3\mu+1)^{c-c'}(\mu^3-5\mu^2+6\mu-1)^{d-d'}f(\mu)=g(\mu),$$

Clearly, the term in $g(\mu)$ with the largest exponent is μ^7, and similarly for $(\mu-1)^{b-b'}(\mu^2-3\mu+1)^{c-c'}(\mu^3-5\mu^2+6\mu-1)^{d-d'}f(\mu)$. So Eq. (7) implies $b=b'$, $c=c'$ and $d=d'$. That is $G(a,b,c,d)=G(a',b',c',d')$. Therefore, each $G(a,b,c,d)\in\mathcal{G}$ with $n\geq 11$ is determined by its Laplacian spectrum. We complete this proof.

Combining the Lemmas 3.1–3.6 in Section 3 with the Theorem 4.2 in Section 4, we obtain:

Theorem 4.3. *Connected graphs with $\mu_2\leq\theta$ are DLS except for B_4^1 and B_4^2 shown in Table 4, where $\theta=3.2470$ is the largest root of the equation $\mu^3-5\mu^2+6\mu-1=0$.*

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