ON THE TOTAL IRREGULARITY STRENGTH OF GENERALIZED PETERSEN GRAPH

ALI AHMAD, MUHAMMAD IBRAHIM and MUHAMMAD KAMRAN SIDDIQUI

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Let $G = (V, E)$ be a graph. A total labeling $\phi : V \cup E \to \{1, 2, \dots, k\}$ is called totally irregular total k-labeling of G if every two distinct vertices x and y in $V(G)$ satisfy $wt(x) \neq wt(y)$, and every two distinct edges x_1x_2 and y_1y_2 in $E(G)$ satisfy $wt(x_1x_2) \neq wt(y_1y_2)$, where

$$wt(x) = \phi(x) + \sum_{xz \in E(G)} \phi(xz)$$

and

$$wt(x_1x_2) = \phi(x_1) + \phi(x_1x_2) + \phi(x_2)$$

The minimum k for which a graph G has a totally irregular total k-labeling is called the total irregularity strength of G, denoted by $ts(G)$. In this paper, we determined the total irregularity strength of generalized Petersen graph.

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Key words: irregularity strength, total edge irregularity strength, total vertex irregularity strength, generalized Petersen graph, totally irregular total labeling.

1. INTRODUCTION

As a standard notation, assume that $G = G(V, E)$ is a finite, simple and undirected graph with p vertices and q edges. A labeling of a graph is any mapping that sends some set of graph elements to a set of numbers (usually positive integers). If the domain is the vertex-set or the edge-set, the labelings are called respectively vertex-labelings or edge-labelings. If the domain is $V \cup E$ then we call the labeling a total labeling. In many cases it is interesting to consider the sum of all labels associated with a graph element. This will be called the *weight* of element.

For a graph G we define a labeling $\phi : V \cup E \to \{1, 2, \dots, k\}$ to be a *total k-labeling*. A total k-labeling ϕ is defined to be an *edge irregular total k-labeling* of the graph G if for every two different edges xy and $x'y'$ their weights $\phi(x) + \phi(xy) + \phi(y)$ and $\phi(x') + \phi(x'y') + \phi(y')$ are distinct. Similarly, a total

k-labeling ϕ is defined to be a *vertex irregular total k-labeling* of G if for every two distinct vertices x and y of G their weights $\mathrm{wt}(x)$ and $\mathrm{wt}(y)$ are distinct. Here, the weight of a vertex x in G is the sum of the label of x and the labels of all edges incident with the vertex x. The minimum k for which the graph G has an edge irregular total k-labeling is called the *total edge irregularity strength* of G, $\mathrm{tes}(G)$. Analogously, the minimum k for which the graph G has a vertex irregular total k-labeling is called the *total vertex irregularity strength* of G, $\mathrm{tvs}(G)$.

The total edge irregularity strength and total vertex irregularity strength are invariants analogous to irregular assignments and irregularity strength of a graph G introduced by Chartrand et al. [10] and studied by numerous authors, see [9, 13, 14, 16, 23]. The irregular assignment is a $k-$labeling of the edges $\phi : E \to \{1, 2, \dots, k\}$ such that the sum of the labels of edges incident with a vertex is different for all the vertices of G, and the smallest k for which there is an irregular assignment is the irregularity strength, $s(G)$.

A simple lower bounds for $\mathrm{tes}(G)$ and $\mathrm{tvs}(G)$ of a (p, q)-graph G in terms of maximum degree $\triangle(G)$ and minimum degree $\delta(G)$, determined in [7], are given by the following theorems.

THEOREM 1.1 ([7]). *Let G be a (p, q)-graph with maximum degree $\Delta = \Delta(G)$. Then*

$$\mathrm{tes}(G) \geq \max\left\{\left\lceil\frac{q+2}{3}\right\rceil, \left\lceil\frac{\Delta+1}{2}\right\rceil\right\}.$$

THEOREM 1.2 ([7]). *Let G be a (p, q)-graph with minimum degree $\delta = \delta(G)$ and maximum degree $\Delta = \Delta(G)$. Then*

$$\left\lceil\frac{p+\delta}{\Delta+1}\right\rceil \leq \mathrm{tvs}(G) \leq p + \Delta - 2\delta + 1.$$

Ivančo and Jendroľ [15] posed the following conjecture:

CONJECTURE 1.1 ([15]). *Let G be an arbitrary graph different from K_5. Then*

$$\mathrm{tes}(G) = \max\left\{\left\lceil\frac{q+2}{3}\right\rceil, \left\lceil\frac{\Delta+1}{2}\right\rceil\right\}.$$

In [24] Nurdin *et. al* posed the following conjecture:

CONJECTURE 1.2 ([24]). *Let G be a connected graph having n_i vertices of degree $i (i = \delta, \delta+1, \delta+2, \dots, \Delta)$, where δ and Δ are the minimum and the maximum degree of G, respectively. Then*

$$tvs(G) = max\left\{\left\lceil\frac{\delta+n_\delta}{\delta+1}\right\rceil, \left\lceil\frac{\delta+n_\delta+n_{\delta+1}}{\delta+2}\right\rceil, \dots, \left\lceil\frac{\delta+\sum_{i=\delta}^{\Delta} n_i}{\Delta+1}\right\rceil\right\}.$$

Conjecture 1.1 has been verified for trees [15], for complete graphs and complete bipartite graphs [17, 18], for the grid [21], for hexagonal grid graphs [3] , for toroidal grid [11], for generalized prism [8], for categorical product of two cycles [1], for strong product of cycles and paths [4], for zigzag graphs [5] and for strong product of two paths [2].

Conjecture 1.2 has been verified for trees [24], for circulant graphs [6].

Combining both total edge irregularity strength and total vertex irregularity strength notions, Marzuki *et al.* [20] introduced a new irregular total k-labeling of a graph G, which is required to be at the same time both vertex and edge irregular. The minimum value of k for which such labeling exist is called total irregularity strength of graph and is denoted by $ts(G)$. Besides that, they determined the total irregularity strength of cycles and paths. Marzuki, *et al.* [20] given a lower bond of $ts(G)$ as follows.

(1) For every graph G, $ts(G) \geq \max\{tes(G), tvs(G)\}$

Ramdani and Salman [25] showed that the lower bound in (1) for some cartesian product graphs is tight. In the present paper, we investigate the total irregularity strength of the generalized Petersen graph. Let n and m be positive integers, $n \geq 3$ and $1 \leq m \leq \frac{n}{2}$. The generalized Petersen graph $P(n, m)$ is a graph with vertex set $\{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}$ and edge set consisting of all edges of the form $x_i x_{i+1}, x_i y_i$ and $y_i y_{i+m}$, where $1 \leq i \leq n$, the subscripts are reduced modulo n.

The generalized Petersen graph $P(n, m)$ has been studied extensively in recent years. Generalized Petersen graphs were first defined by Watkins [26]. Mominul Haque [22] determined the irregular total labelings of generalized Petersen graphs, Jendrol and Žoldák [19] determined the irregularity strength of generalized Petersen graphs and Chunling *et. al* [12] determined the total edge irregularity strength of generalized Petersen graphs.

In this paper, we determine the exact value of $ts(P(n, m))$.

2. TOTAL IRREGULARITY STRENGTH OF GENERALIZED PETERSEN GRAPH

In the next theorem, we determine the total irregularity strength of generalized Petersen graph.

THEOREM 2.1. *Let $P(n, m)$ be a generalized Petersen graph with $n \geq 3$ and $1 \leq m < \frac{n}{2}$. Then $ts(P(n, m)) = n + 1$.*

Proof. $P(n, m)$ has $2n$ vertices and $3n$ edges. $P(n, m)$ is 3-regular graph. From Theorem 1.1 and Theorem 1.2, we get $tes(P(n, m)) \geq \lceil \frac{3n+2}{3} \rceil = n+1$ and

$tvs(P(n,m)) \geq \lceil \frac{2n+3}{4} \rceil$. Therefore, From equation (1), we get $ts(P(n,m)) \geq n+1$. Next, we will show that $ts(P(n,m)) \leq n+1$.
Define a total labeling ϕ of $P(n,m)$ from $V(P(n,m)) \cup E(P(n,m))$ into $\{1, 2, \ldots, n+1\}$ as follow:
For $1 \leq i \leq n$, $\phi(x_i) = n+1, \phi(y_i) = 1, \phi(x_i x_{i+1}) = i+1, \phi(y_i y_{i+m}) = i$

$$\phi(x_i y_i) = \begin{cases} 1, & \text{for } \ i = 1 \\ n+2-i, & \text{for } \ 2 \leq i \leq n \end{cases}$$

Since,
$wt(y_i y_{i+m}) = \phi(y_i) + \phi(y_i y_{i+m}) + \phi(y_{i+m}) = i+2$, for $1 \leq i \leq n$.
$wt(x_i x_{i+1}) = \phi(x_i) + \phi(x_i x_{i+1}) + \phi(x_{i+1}) = 2n+3+i$, for $1 \leq i \leq n$.

$$wt(x_i y_i) = \phi(x_i) + \phi(x_i y_i) + \phi(y_i)$$

$$wt(x_i y_i) = \begin{cases} n+3, & \text{for } \ i = 1 \\ 2n+4-i, & \text{for } \ 2 \leq i \leq n \end{cases}$$

$$wt(x_i) = 2n+4+i, \quad for \quad 1 \leq i \leq n$$

$$wt(y_i) = \begin{cases} n+2, & \text{for } \ i = 1 \\ 2n+3, & \text{for } \ i = 2 \\ n+1+i, & \text{for } \ 3 \leq i \leq n \end{cases}$$

the weights of the edges and vertices of $P(n,m)$ under the labeling ϕ are distinct. It is easy to check that there are no two edges of the same weight and there are no two vertices of the same weight. So, ϕ is a totally irregular total k-labeling. We conclude that $ts(P(n,m)) = n+1$. Which completes the proof. □

For illustration, we give a totally irregular total 10-labeling for $P(9,2)$ in Fig. 1.

The weights for all vertices and the weights for all edges under the totally irregular total 10-labeling are given in Fig. 2.

Lemma 2.1. *If* $n = 4, 6, 8$ *and* $m = \frac{n}{2}$, *then* $ts(P(n,m)) = \lceil \frac{5n+4}{6} \rceil$.

Proof. For $m = \frac{n}{2}$, $P(n,m)$ has $2n$ vertices and $\frac{5n}{2}$ edges. From Theorem 1.1 and Theorem 1.2, we get $tes(P(n,m)) \geq \lceil \frac{5n+4}{6} \rceil$ and $tvs(P(n,m)) \geq \frac{n}{2} + 1$. Therefore, from equation (1), it follows that $ts(P(n,m)) \geq \lceil \frac{5n+4}{6} \rceil$. Now the existence of the optimal labeling ϕ gives the converse inequality. Let $k = \lceil \frac{5n+4}{6} \rceil$. For $1 \leq i \leq n$, $\phi(y_i) = 1$ and for $1 \leq i \leq \frac{n}{2}, \phi(y_i y_{i+m}) = i$,

$$\phi(x_i y_i) = \begin{cases} 1, & \text{for } \ 1 \leq i \leq \frac{n}{2} \\ \frac{n}{2}+1, & \text{for } \ \frac{n}{2}+1 \leq i \leq n \end{cases}$$

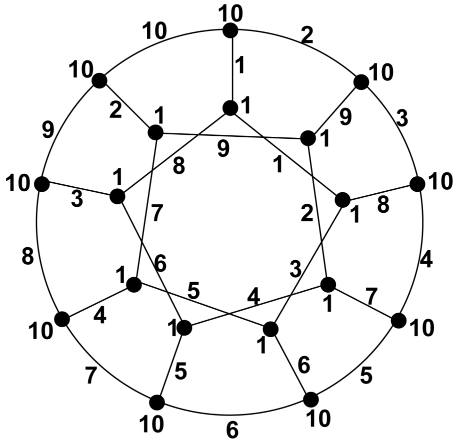

Fig. 1 – A totally irregular total 10-labeling for $P(9, 2)$.

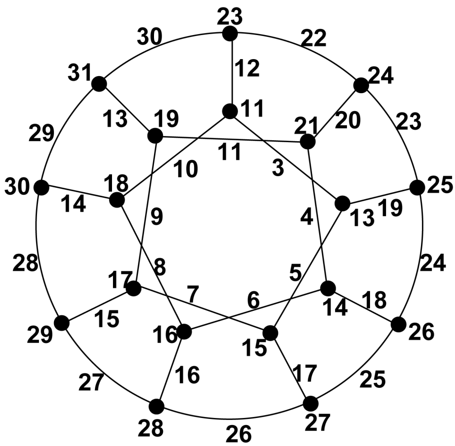

Fig. 2 – The weights of vertices and edges for $P(9, 2)$.

$$\phi(x_i x_{i+1}) = \begin{cases} \frac{n}{2} + 1 - i, & \text{for } 1 \leq i \leq \frac{n}{2} - 1 \\ k, & \text{for } \frac{n}{2} \leq i \leq n \end{cases}$$

$$\phi(x_i) = \begin{cases} \frac{n}{2} + i, & \text{for } 1 \leq i \leq \frac{n}{2} \\ \frac{3n}{2} + 1 - i, & \text{for } \frac{n}{2} + 1 \leq i \leq n \end{cases}$$

Since, $wt(x_i y_i) = \phi(x_i) + \phi(x_i y_i) + \phi(y_i)$

$$wt(x_i y_i) = \begin{cases} \frac{n}{2} + 2 + i, & \text{for } 1 \leq i \leq \frac{n}{2} \\ 2n + 3 - i, & \text{for } \frac{n}{2} + 1 \leq i \leq n \end{cases}$$

$wt(y_i y_{i+m}) = \phi(y_i) + \phi(y_i y_{i+m}) + \phi(y_{i+m}) = i + 2$, for $1 \leq i \leq \frac{n}{2}$.

For $n=4$, $wt(x_1x_2)=9, wt(x_2x_3)=12, wt(x_3x_4)=11, wt(x_4x_1)=10$.
For $n=6$, $wt(x_1x_2)=12, wt(x_2x_3)=13, wt(x_3x_4)=18, wt(x_4x_5)=17$, $wt(x_5x_6)=15, wt(x_6x_1)=14$.
For $n=8$, $wt(x_1x_2)=15, wt(x_2x_3)=16, wt(x_3x_4)=17, wt(x_4x_5)=24$, $wt(x_5x_6)=23, wt(x_6x_7)=21, wt(x_7x_8)=19, wt(x_8x_1)=18$.
$wt(y_i)=i+2$, for$1\leq i\leq n$.
For $n=4$, $wt(x_1)=10, wt(x_2)=11, wt(x_3)=15, wt(x_4)=14$.
For $n=6$, $wt(x_1)=14, wt(x_2)=11, wt(x_3)=15, wt(x_4)=22, wt(x_5)=21$, $wt(x_6)=20$.
For $n=8$, $wt(x_1)=18, wt(x_2)=14, wt(x_3)=13, wt(x_4)=19, wt(x_5)=29$, $wt(x_6)=28, wt(x_7)=27, wt(x_8)=26$.
the weights of the edges and vertices of $P(n,m)$ under the labeling ϕ are distinct, the function ϕ is a map from $V(P(n,m))\cup E(P(n,m))$ into $\{1,2,\ldots,\lceil\frac{5n+4}{6}\rceil\}$. It is easy to check that there are no two edges of the same weight and there are no two vertices of the same weight. So, ϕ is a totally irregular total k-labeling. We conclude that $ts(P(n,m))=\lceil\frac{5n+4}{6}\rceil$, for $n=4,6,8$ and $m=\frac{n}{2}$. Which completes the proof. □

THEOREM 2.2. *Let $P(n,\frac{n}{2})$ be a generalized Petersen graph with n even, $n\geq 10$. Then $ts(P(n,\frac{n}{2}))=\lceil\frac{5n+4}{6}\rceil$.*

Proof. For $m=\frac{n}{2}$, $P(n,m)$ has $2n$ vertices and $\frac{5n}{2}$ edges. From Theorem 1.1 and Theorem 1.2, we get $tes(P(n,m))\geq\lceil\frac{5n+4}{6}\rceil$ and $tvs(P(n,m))\geq\frac{n}{2}+1$. Therefore, from equation (1), we get $ts(P(n,m))\geq\lceil\frac{5n+4}{6}\rceil$. Next, we will show that $ts(P(n,m))=\lceil\frac{5n+4}{6}\rceil$. Let $k=\lceil\frac{5n+4}{6}\rceil$.

Define a total labeling ϕ of $P(n,m)$ from $V(P(n,m))\cup E(P(n,m))$ into $\{1,2,\ldots,\lceil\frac{5n+4}{6}\rceil\}$ as follow:
For $1\leq i\leq n$, $\phi(y_i)=1$ and for $1\leq i\leq\frac{n}{2},\phi(y_iy_{i+m})=i$,

$$\phi(x_i)=\begin{cases}k-\frac{n}{2}+i, & \text{for } 1\leq i\leq\frac{n}{2}\\ k+\frac{n}{2}+1-i, & \text{for } \frac{n}{2}+1\leq i\leq n\end{cases}$$

$$\phi(x_iy_i)=\begin{cases}n-k+1, & \text{for } 1\leq i\leq\frac{n}{2}\\ \frac{3n}{2}+1-k, & \text{for } \frac{n}{2}+1\leq i\leq n\end{cases}$$

$$\phi(x_ix_{i+1})=\begin{cases}\frac{5n}{2}-2k+1, & \text{for } 1\leq i\leq\frac{n}{2}-1\ \&\ i=n\\ \frac{5n}{2}-2k+2, & \text{for } \frac{n}{2}\leq i\leq n-1\end{cases}$$

Since,
$wt(y_iy_{i+m})=\phi(y_i)+\phi(y_iy_{i+m})+\phi(y_{i+m})=i+2$, for $1\leq i\leq\frac{n}{2}$.
$wt(x_iy_i)=\phi(x_i)+\phi(x_iy_i)+\phi(y_i)$

$$wt(x_iy_i) = \begin{cases} \frac{n}{2}+2+i, & \text{for } 1 \le i \le \frac{n}{2} \\ 2n+3-i, & \text{for } \frac{n}{2}+1 \le i \le n \end{cases}$$

$$wt(x_ix_{i+1}) = \phi(x_i) + \phi(x_ix_{i+1}) + \phi(x_{i+1})$$

$$wt(x_ix_{i+1}) = \begin{cases} \frac{3n}{2}+2(i+1), & \text{for } 1 \le i \le \frac{n}{2} \\ \frac{7n}{2}+3-2i, & \text{for } \frac{n}{2}+1 \le i \le n-1 \\ \frac{3n}{2}+3, & \text{for } i = n \end{cases}$$

$$wt(y_i) = 1+n-k+1+i = n-k+2+i, \quad \text{for} \quad 1 \le i \le n.$$

$$wt(x_i) = \begin{cases} \frac{11n}{2}-4k+3+i, & \text{for } 1 \le i \le \frac{n}{2}-1 \\ 6n-4k+4, & \text{for } i = \frac{n}{2} \\ 7n-4k+6-i, & \text{for } \frac{n}{2}+1 \le i \le n-1 \\ 6n-4k+5, & \text{for } i = n \end{cases}$$

Clearly the weights of the edges and vertices of $P(n, m)$ under the labeling ϕ are distinct. It is easy to check that there are no two edges of the same weight and there are no two vertices of the same weight. So, ϕ is a totally irregular total k-labeling. We conclude that $ts(P(n, m)) = \lceil \frac{5n+4}{6} \rceil$. Which completes the proof. □

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Jazan Universty,
Faculty of Computer Science
and Information Systems,
Jazan, KSA
ahmadsms@gmail.com

Bahauddin Zakariya University Multan,
Center for Advanced Studies in Pure
and Applied Mathematics,
Pakistan
mibtufail@gmail.com

Comsats Institute of Information Technology,
Department of Mathematics,
Sahiwal, Pakistan
kamransiddiqui75@gmail.com