ON THE CHAIN BLOCKERS OF A POSET

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Let $P = C_a \times C_b$ be a poset where C_i is the chain $1 < \cdots < i$. A chain blocker of P is an inlcusionwise minimal subset $B \subseteq P$ with the property that every maximal chain in P contains at least one element of B. In [1] the chain blockers of P are being expressed in term of the Catalan numbers and k fold convolution of the Catalan numbers. In this paper we give a complete description of the chain blockers of $C_a \times C_b$, where $a \leq 4$ and $b \geq 1$. In the end algebraic consequences of the chain blockers are also provided.

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1. INTRODUCTION

Let $P = C_a \times C_b$ be a poset where C_i is the chain $1 < \cdots < i$. A chain blocker of P is defined as a subset $B \subseteq P$ such that every maximal chain in P contains at least one element of B and B is inclusionwise minimal with this property. In [1] the chain blockers of P were studied for some special cases and provided a new combinatorial interpretation of the convoluted Catalan numbers $C(n, k) := \frac{k}{2n-k}\binom{2n-k}{n}$ and Catalan numbers $C(n) = C(n, 1)$ introduced by Catalan [2] in 1887.

In Section 2 the chain blockers of $C_a \times C_b$ are being discussed for $a \leq 3$ and $b \geq 1$. The main result of this section states that number of all chain blockers of $C_3 \times C_b$ is given by a polynomial in b (Theorem 2.1). In Section 3 a formula for calculating number of all chain blockers of $C_4 \times C_b$ is derived (Theorem 3.1).

Besides its combinatorial properties the chain blockers of P have its algebraic consequences. Let $R = k[x_1, \ldots, x_n]$ be the polynomial ring over a field k and in n variables. To each poset P of cardinality n we define an ideal $I_P \subset R$ such that the generators of I_P correspond to the maximal chains in P. The chain blockers of P have one to one correspondence with the irreducible primary components of I_P (Proposition 4.1).

2. INITIAL CASES

In this section, we give a description of the chain blockers of $P = C_a \times C_b$ where $a \leq 3$ and $b \geq 1$. We start with giving a basic property of a chain blocker of P.

LEMMA 2.1. *Let $P = C_a \times C_b$ be a poset. Let $A = \{(1,1),(1,2),\ldots,(1,b),(2,b),\ldots,(a,b)\}$ and $B \subseteq P$ be a chain blocker. Then $A \cap B$ contains exactly one element.*

Proof. Since A is a maximal chain and B is a chain blocker therefore $A \cap B \neq \emptyset$. Now suppose $|A \cap B| \neq 1$ and $A \cap B = \{(i_1,j_1),\ldots,(i_r,j_r)\}$ with $r > 1$. We ordered the elements of $A \cap B$ in the ascending order i.e. $(i_1,j_1) < \cdots < (i_r,j_r)$. If $A \cap B \subseteq \{(1,1),(1,2),\ldots,(1,b)\}$ then a maximal chain blocked by any of $(i_k,j_k) \in \{(i_1,j_1) > \cdots > (i_{r-1},j_{r-1})\}$ is also blocked by (i_1,j_1), hence by minimality of B we have $A \cap B = \{(i_1,j_1)\}$. Similarly, if $\{A \cap B\} \subset \{(2,b),\ldots,(a,b)\}$ then $\{A \cap B\} = \{(i_r,j_r)\}$.

Now if $\{A \cap B\} \cap \{(2,b),\ldots,(a,b)\} \neq \emptyset$ then it is enough to take $i_1 = 1$. Since B is a chain blocker so there exist two different maximal chains with one containing $\{(i_1,j_1)\}$ but not containing (i_r,j_r) and vice versa. These maximal chains must intersect at some points say (i_k,j_k). Then if we combined these two maximal chains to get another maximal chain in such a way that the part before the point (i_k,j_k) consists of points of the maximal chain containing (i_r,j_r) and the part after (i_k,j_k) consists of the points of the maximal chain containing (i_1,j_1). Then this maximal chain is not blocked by B, a contradiction. $\square$

We call the maximal chain A in the previous lemma as the left maximal chain of P and denote it by $\mathcal{L}(P)$. Similarly we call $\{(1,1),(2,1),\ldots,(a,1),(a,2),\ldots,(a,b)\}$ as the right maximal chain of P and denote it by $\mathcal{R}(P)$. Hence, by previous lemma and Lemma 2.2 [1], we have:

COROLLARY 2.1. *Let $B \subseteq C_a \times C_b$ be a chain blocker with $|B| > 1$. Then B contains exactly one element from $\mathcal{R}(C_a \times C_b)$ and exactly one element from $\mathcal{L}(C_a \times C_b)$.*

If $a = 1$ then $C_1 \times C_b$ is given by the chain

$$(1,1) < (1,2) < \cdots < (1,b),$$

which is the only its maximal chain. Thus, each element $C_1 \times C_b$ is a chain blocker. Hence, number of chain blockers in this case is equal to b. Following lemma provides a complete description of the chain blockers of $C_2 \times C_b$.

LEMMA 2.2. *If B is a chain-blocker for $C_2 \times C_b$ then either $|B| = 1$ or $B = \{(1,j),(2,j')\}$, where $1 \leq j-1 \leq j' \leq b-1$.*

Proof. If $|B| = 1$ then $B = \{(1,1)\}$ or $B = \{(2,b)\}$ are only the chain blockers. Now let $|B| > 1$. By Corollary 2.1, $B = \{(1,j),(2,j')\}$ for $j \in \{2,\ldots,b\}$ and $j' \in \{1,\ldots,b-1\}$. The condition $j-1 \leq j'$ then follows from the fact that otherwise not all chains are blocked. $\Box$

By above lemma the number of chain blockers B of $C_2 \times C_b$ with $|B| > 1$ is given by

$$(b-1)+(b-2)+\cdots 2+1 = \frac{1}{2}b(b-1).$$

Next, we turn our attention to count number of all chain blockers of $C_3 \times C_b$.

Proposition 2.1. *Let $P = C_3 \times C_b$ be a poset. The number of chain blockers of P containing $(2,1)$ equals the number of chain blockers of P containing $(3,1)$. Moreover both numbers equal to b.*

Proof. Here $\mathcal{R}(P) = \{(1,1),(2,1),(3,1),\ldots,(3,b)\}$ and $\mathcal{L}(P) = \{(1,1), \ldots, (1,b)(2,b),(3,b)$. By Corollary 2.1, a chain blocker B of P containing $(2,1)$ does not contain any element from the set $\mathcal{R}(P) \setminus (2,1)$ and contains exactly one element from $\mathcal{L}(P) \setminus \{(1,1),(3,b)\}$, since if B contains $(2,1)$ we must exclude the choices of minimum and maximum elements of P.

Now let $(1,i) \in \mathcal{L}(P) \cap B$, where $2 \leq i \leq b$. Then B must contains the set $\{(2,2),\ldots,(2,i-1)\}$. If not say $(2,j) \notin B$ for some $j \in \{2, i-1\}$, then $\{(1,1)\ldots,(1,j),(2,j),(3,j),\ldots,(3,b)\}$ is a maximal chain which is not blocked by B. Moreover by the minimality of B, $(2,j) \notin B$ for $j \geq i$. Hence,

$$B = \{(1,i),(2,i-1),\ldots,(2,2),(2,1)\}$$

is only the chain blocker containing $(1,i)$ and $(2,1)$. Running over all values of i we have $b-1$ such chain blockers. Similarly,

$$B = \{(2,b),(2,b-1),\ldots,(2,2),(2,1)\}$$

is only the chain blocker containing $(2,1)$ and $(2,b)$. Hence, total number of chain blockers containing $(2,1)$ is equal to b. Now since $(2,1) < (3,1)$ and $(2,1) < (2,2)$ but $(3,1)$ and $(2,2)$ are incomparable so any chain blocker of P containing $(2,1)$ remains a chain blocker if we replace $(2,1)$ with $(3,1)$. Hence, we are done. $\Box$

For the case $C_3 \times C_b$ following theorem provides an explicit formula to calculate number of chain blockers of P.

Theorem 2.1. *Let $P = C_3 \times C_{b+1}$ be a poset. The number of chain blockers of P is given by*

$$\frac{1}{6}(b^2+2)(b+9).$$

Proof. For a fixed element $(i,j) \in \mathcal{R}(P)$, we run over all elements of $\mathcal{L}(P)$ one by one to count number of chain blockers containing both elements. If $(i,j) = (1,1)$ or $(3, b+1)$ then $B = \{(i,j)\}$ is itself a chain blocker. Also by Proposition 2.1 the number of chain blockers B containing $(2,1)$ or $(3,1)$ equals to $2(b+1)$. Now let B be a chain blocker containing $(3,j) \in \mathcal{R}(P)$ and $(m,n) \in \mathcal{L}(P)$. We are left with the following cases:
Case I: $2 \leq j \leq b-1$, $(m,n) = (1,2)$:
Here $B \cap \{(2,2), \ldots, (2,j+1)\} \neq \emptyset$, because if the intersection is empty then $(1,1) < (2,1) < \ldots < (2,j+1) < (3,j+1) < \ldots < (3,b)$ is a maximal chain not blocked B. Moreover by minimality of B, we have $|B \cap \{(2,2), \ldots, (2,j+1)\}| = 1$ and $B \cap \{(2,j+2), \ldots, (2,b)\} = \emptyset$. Thus, there are j such chain blockers. Hence, number of chain blockers in this case is given by

$$\sum_{j=2}^{b-1} j = \frac{1}{2}b^2 - \frac{1}{2}b - 1.$$

Case II: $2 \leq j \leq b-1$, $m = 1$, $3 \leq n \leq b+1$:
If $3 \leq n \leq j+1$ then by the same arguments as in Case I, $B \cap \{(2,n-1), \ldots, (2,j+1)\} \neq \emptyset$ and for a fixed j and n number of chain blockers equals to $j-n+3$ and Hence, total number for these choices equals to $\sum_{j=2}^{b-1}\sum_{n=3}^{j+1}(j-n+3) = \frac{1}{6}b^3 - \frac{7}{6}b + 1$. On the other hand if $j+2 \leq n \leq b+1$ then $B = \{(1,n), (2,n-1), \ldots, (2,j+1), (3,j)\}$ is only chain blocker containing $(1,n)$ and $(3,j)$. Thus, we have $\sum_{j=2}^{b-1}\sum_{n=j+2}^{b+1} 1 = \frac{1}{2}b^2 - \frac{3}{2}b + 1$. Hence, total number of chain blockers for this case is given by

$$\frac{1}{6}b^3 + \frac{1}{2}b^2 - \frac{8}{3}b + 2.$$

Case III: $j = b$, $m = 1$, $2 \leq n \leq b+1$:
Let B be a chain blocker containing $(3, b-1)$ and $(m,n) \in \mathcal{L}(P)$. Since $(2,b)$ and $(3,b-1)$ are incomparable so if we replace $(3,b-1)$ by $(3,b)$ then B will remain a chain blocker. Thus, number of chain blockers in this case is given by putting $j = b-1$ in the previous Cases I and II. That is

$$b - 1 + \sum_{n=3}^{b}(b-n+2) + 1 = \frac{1}{2}b^2 + \frac{1}{2}b - 1.$$

Case IV: $2 \leq j \leq b$, $(m,n) = (2, b+1)$:
If $2 \leq j \leq b-1$, then $B = \{(2,b+1), \ldots, (2,j+1), (3,j)\}$ is the chain blocker. If $j = b$, then $B = \{(2,b+1), (3,b)\}$. Thus, number of chain blockers for this case is $b-1$.

Now summing over all above cases and $2(b+1)+2$ contribution from the initial choices, we have the required formula after simplification. $\square$

3. THE CASE $C_4 \times C_b$

Let $P = C_4 \times C_b$ be a poset. Then the right maximal chain $\mathcal{R}(P) := (1,1) < \cdots < (4,1) < \cdots < (4,b)$ and left maximal chain $\mathcal{L}(P) := (1,1) < \cdots < (1,b) < \cdots (4,b)$.

PROPOSITION 3.1. *Let $P = C_4 \times C_b$ be a poset. The number of chain blockers of P containing $(2,1)$ equals to*

$$2^b - b + 1.$$

Proof. Let B be a chain blocker of P containing $(2,1)$. Then by Proposition 2.1, B contains exactly one element (m,n) from $\mathcal{L}(P)$. If $(m,n) = (1,2)$ then $B = \{(1,2),(2,1)\}$ is itself a chain blocker. If $m = 1$ and $3 \leq n \leq b$ then either $(2,i) \in B$ or $(3,i) \in B$ for all $i = 2,\ldots,n-2$. If it is not true for $i \in \{2,\ldots,n-2\}$ then $(1,1) < \cdots < (1,i) < (2,i) < (3,i) < (4,i) < \cdots < (4,b)$ is a maximal chain not blocked by B. Number of these choices are given by 2^{n-3}. Now it remain to block a chain containing $(1,1) < \cdots < (1,n-1) < (2,n-1)$. For this B must contains $\{(2,k),(3,k-1),\ldots,(3,n-1)\}$ where $k = n-1,\ldots,b-1$. Since these two cases are independent so number of chain blockers are given by $2^{n-3}(b-n+1)$.

Now if $2 \leq m \leq 3$ and $n = b$, then by the same argument as before B must contains either $(2,i)$ or $(3,i)$ for all $i = 2,\ldots,b-1 \Rightarrow$ it contributes $2 \cdot 2^{b-2}$. Thus, total number of chain blockers are given by

$$1 + \sum_{n=3}^{b} 2^{n-3}(b-n+1) + 2 \cdot 2^{b-2},$$

on simplification we are done. $\square$

COROLLARY 3.1. *Let $P = C_4 \times C_b$ be a poset, then number of chain blockers of P containing $(3,1)$ is given by*

$$2^b - 2.$$

Proof. Let B be a chain blocker containing $(3,1)$ and an element $(m,n) \in \mathcal{L}(P) \setminus \{(1,2)\}$. Since $(1,3)$, $(2,2)$ and $(3,1)$ are in comparable and $(2,1) < (3,1)$, so if we replace $(3,1)$ by $(2,1)$ then B will remain a chain blocker(see Fig. 1). Thus, by Proposition 3.1, the number of chain blockers containing $(3,1)$ and $(m,n) \in \mathcal{L}(P) \setminus \{(1,2)\}$ is given by $2^b - b$. Now if $(m,n) = (1,2)$ then $B = \{(1,2),(2,k),(3,k-1),\ldots,(3,1)\}$, where $k \in \{2,\ldots,b-1\}$. There are $b-2$ such chain blockers. Hence, total number of chain blockers of P containing $(3,1)$ equals to $2^b - 2$. $\square$

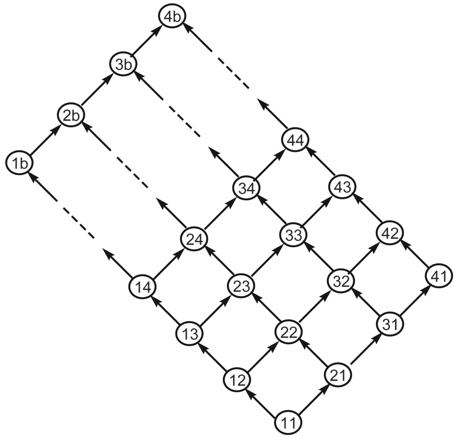


Fig. 1 – $C_4 \times C_b$.

Proposition 3.2. *Let* $P = C_4 \times C_b$ *be a poset. The number of chain blockers of* P *containing* $(4,1)$ *is given by*

$$2^{b-1} + b - 3.$$

Proof. Clearly, if B is a chain blocker containing $(4,1) \Rightarrow (3,2) \in B$. Moreover, B must contain an element $(m,n) \in \mathcal{L}(P)$. If $(m,n) = (1,2)$ or $(1,3)$, then either $B = \{(m,n),(2,2),(3,2),(4,1)\}$ or $B = \{(m,n),(2,k),(3,k-1),\ldots,(3,2),(4,1)\}$ where $k = 3,\ldots,b-1$. Similarly if $(m,n) = (1,4)$, then $B = \{(1,4),(2,k),(3,k-1),\ldots,(3,2),(4,1)\}$ where $k = 3,\ldots,b-1$. Thus, so far we have calculated $3b-7$ number of chain blockers for $m = 1$ and $n \in \{2,3,4\}$.

Now if $m = 1$ and $5 \le n \le b$ then either $(2,i) \in B$ or $(3,i) \in B$ for all $i = 3,\ldots,n-2 \Rightarrow$ there are 2^{b-4} possibilities. Moreover to block a maximal chain containing $(1,1) < \cdots < (1,n-1) < (2,n-1)$ then either $(2,n-1) \in B$ or $\{(2,k),(3,k-1),\ldots,(3,n-1)\} \subset B$ for $k = n,\ldots,b-1$. There are $b-n+1$ such possibilities. Since these two choices are independent hence number of chain blockers for this case is equal to $2^{n-4}(b-n+1)$.

Lastly, if $2 \le m \le 3$ and $n = b$, then B must contains either $(2,i)$ or $(3,i)$ for all $3 \le i \le b$ which contributes $2(2^{b-2})$. Thus, total number of chain blockers of P containing $(4,1)$ is given by

$$3b - 7 + \sum_{n=5}^{b} 2^{n-4}(b-n+1) + 2^{b-1} = 2^{b-1} + b - 3. \quad \square$$

Theorem 3.1. *Let* $P = C_4 \times C_b$ *be a poset, then number of all chain* *blockers of P is given by*

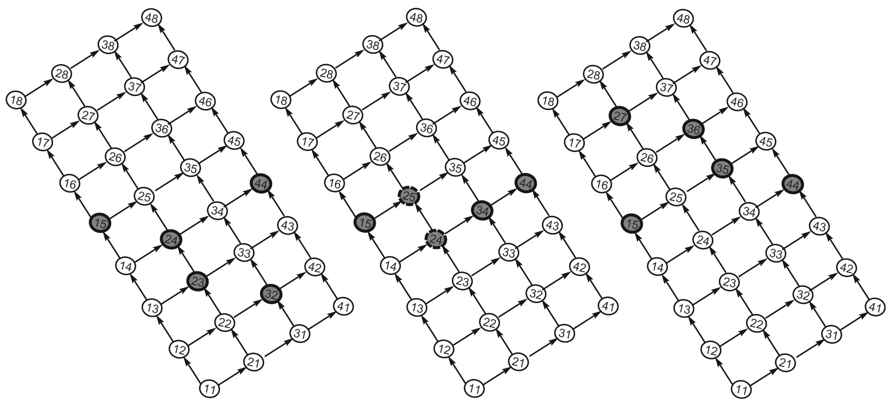
$$\frac{1}{24}b^4 + \frac{7}{12}b^3 - \frac{37}{24}b^2 - \frac{85}{12}b + 4 + 2^{b+2}.$$

Proof. As before we fix one element of $\mathcal{R}(P)$ and run over all elements of $\mathcal{L}(P)$ to count number of chain blockers containing these two elements. As there are two trivial chain blockers of cardinality 1. Let $|B| \geq 2$ and $(i,1) \in \mathcal{R}(P)$ for $i \in \{2,3,4\}$. Then number of chain blockers containing $(i,1)$ is given by Proposition 3.1, Corollary 3.1 and Proposition 3.2. Thus, number of chain blockers for these choices including chain blockers of cardinality 1 is given by

$$5 \cdot 2^{b-1} - 1.$$

Now let $(4,j) \in \mathcal{R}(P)$ and $(1,n) \in \mathcal{L}(P)$, where $2 \leq j \leq b-1$ and $2 \leq n \leq b$. We have following two main cases:
Case I ($3 \leq n \leq j+2$): Obviously B must have at least one element $(3,k)$ where $2 \leq k \leq b-1$, otherwise the maximal chain $(1,1) < \cdots < (3,1) < \cdots < (3,b) < (3,4)$ is not blocked by B. We have three choices for $(3,k) \in B$.

- **a:** If $k < n-1$, then $B = \{(1,n),(2,n-1),\ldots,(2,k-1),(3,k),(4,j)\}$ (see Diagram 2(a)). Note that there exist $n-3$ such chain blockers B.
- **b:** If $n-1 \leq k \leq j+1$, then $B = \{(1,n),(2,i),(3,k),(4,j)\}$ where $n-1 \leq i \leq k+1$ (see Diagram 2(b)). Note that there exist $\sum_{k=n-1}^{j+1}\sum_{i=n-1}^{k+1} 1$ such chain blockers B.
- **c:** If $j+2 \leq k \leq b-2$, then $B = \{(1,n),(2,k+1),(3,k),\ldots,(3,j+1),(4,j)\}$ (see Diagram 2(c)). We have $b-j-3$ such chain blockers B



(a) (b) (c)

Fig. 2 – $C_4 \times C_b$.

Hence, the number of chain blockers against cases **a**,**b** and **c** is given by:

$$\sum_{j=2}^{b-2}\sum_{n=3}^{j+2}\{n-3+\sum_{k=n-1}^{j+1}\sum_{i=n-1}^{k+1} 1+b-j-3\}.$$

Now if $2 \leq j \leq b-2$, the number of chain blockers of P containing $(1,3)$ is the same as number of chain blockers containing $(1,2)$. So if we fix $n=3$ in the above expression, then we obtain number of chain blockers containing $(1,2)$ and $(4,j)$ for $j \in \{2,\ldots,b-2\}$. Moreover if $2 \leq n \leq b$, the number of chain blockers containing $(4,b-2)$ is the same as number of chain blockers containing $(4,b-1)$.

Hence, by a simple calculations number of chain blockers containing $(1,n)$ and $(4,j)$ where $2 \leq n \leq j+2$ and $2 \leq j \leq b-1$ is given by:

$$\frac{1}{24}b^4+\frac{3}{4}b^3-\frac{37}{24}b^2-\frac{25}{4}b+4.$$

Case II ($j+2 < n \leq b$): Again B must have an element $(3,k)$ for $k \in \{2,\ldots,b-1\}$. For this case we have three independent subcases, namely:

- **d:** If $k \leq j+1$, then $\{(3,k),(2,k+1),\ldots,(2,j+1\} \subset B$ (see Diagram 3(d)). Note that there exist j such possibilities.
- **e:** If $j+1 \leq k \leq n-3$, then $(2,i) \in B$ or $(3,i) \in B$ where $j+1 \leq i \leq n-3$ (see Diagram 3(e)). Note that there exist 2^{n-j-3} such possibilities.
- **f:** If $n-2 \leq k \leq b-2$, then $\{(2,b-1),(3,b-2),\ldots,(3,j+2)\} \subset B$ (see Diagram 3(f)). We have $b-n+1$ such possibilities.

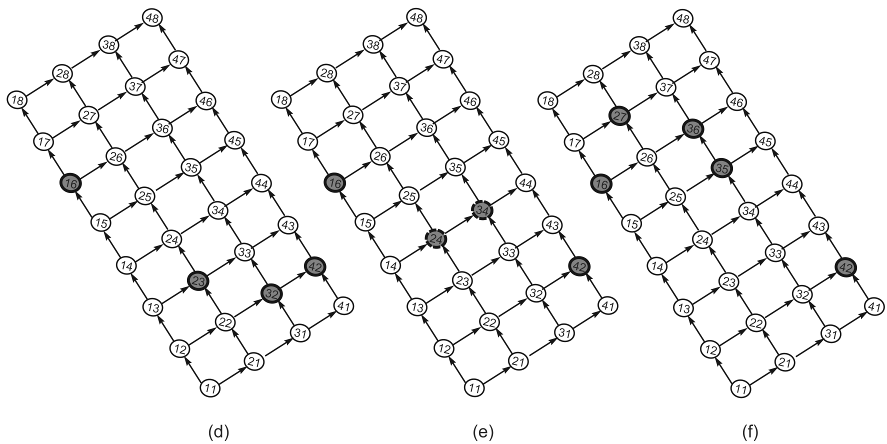


Fig. 3 – $C_4 \times C_b$.

Hence, multiplying out above independent cases **d**, **e** and **f**, we get

$$\sum_{j=3}^{b-3}\sum_{n=j+3}^{b} j(b-n+1)2^{n-j-3}.$$

Now if $(m,n)=(2,3)$ or $(2,4)$ then by cases **d** and **e** above, the number of chain blockers containing (m,n) is given by $j2^{b-j-1}$. Lastly, by symmetry the number of chain blockers containing $\{(4,b-2),(2,b)\}$, $\{(4,b-2),(3,b)\}$ or $\{(4,b-1),(2,b)\}$ are the same and given by $b-2$. Also there is one chain blockers $B=\{(4,b-1),(3,b)\}$. Finally, summing up all above cases we have the required formula. $\square$

4. APPLICATIONS

In this section, we provide algebraic consequences associated to a chain blocker B of P. A simplicial complex Δ on the vertex set $V=[n]$ is a collection of subsets of $2^{[n]}$ with the property that if $A\in\Delta$ then Δ contains all subsets of A. The inclusionwise maximum elements of Δ are called facets. Let $\{F_i,\ldots,F_r\}$ be the set of facets of Δ. A minimal vertex cover of Δ is a subset $A\subseteq V$ with the property that for every facet F_i of Δ there exist a vertex $v\in A$ such that $v\in F_i$ and A is minimal with this property.

Let Δ_P be a simplicial complex associated to a poset P in such a way that elements of Δ_P are exactly the chains in P. The set of facets of Δ_P are the maximal chains of P and hence each chain blockers of P is a minimal vertex cover of Δ_P.

Now we are ready to relate a poset P to its algebraic counterpart. Let $S=k[x_1,\ldots,x_n]$ be the polynomial ring over the field k and in n variables. Recall that a monomial ideal in S is an ideal generated by monomials u_i. A monomial ideal is called a squarefree monomial ideal if it is generated by the square free monomials. Let $I=I_1\cap\cdots\cap I_r$ be an irredundant primary decomposition of I, where ideals $I_1,\ldots,I_r$ are called irreducible primary components of I. For more details about primary decomposition see [6].

To each square free monomial ideal I one can associate a simplicial complex Δ. One way of this association is facet ideals and facet complex introduced by Sara Faridi [4]. A facet ideals $I^{\mathcal{F}}(\Delta)$ of Δ is an ideal generated by square free monomial $x_{i_1}\cdots x_{i_t}$ where $\{x_{i_1},\ldots,x_{i_t}\}$ is a facet of Δ. Let $i=<u_1,\ldots,u_r>$ be a squarefre monomial ideal. A facet complex $\Delta^{\mathcal{F}}(I)$ of I is a simplicial complex over the vertex set $\{v_1,\ldots,v_n\}$ and set of facets $\{F_1,\ldots,F_r\}$, where $F_i=\{v_j\mid x_j\backslash u_i, 1\leq j\leq n\}$.

It is well know that minimal vertex covers of $\Delta^{\mathcal{F}}(I)$ correspond to the irreducible primary components of $I^{\mathcal{F}}(\Delta)$. Let $I_P=I^{\mathcal{F}}(\Delta_P)$. Note that I_P

is also the path ideals of the directed graph of Hasse diagram of P. The path ideal was intorduced by Conca and De Negri in [3]. Some results of I_P were also studied in [5]. Since the facets of Δ_P are the maximal chains of P, hence by definition of a chain blocker we have the following proposition.

PROPOSITION 4.1. *Let $P = C_a \times C_b$ be a poset and I_P be the ideal as defined above. Then there is a one to one correspondence between chain blockers of P and irreducible primary components of I_P*

Following examples demonstrate the one to one correspondence given in above proposition.

Example 4.1. Let $P = C_3 \times C_4$ be a poset as shown in Figure 4 (a). Then $I_P = (x_{11}x_{21}x_{31}x_{32}x_{33}x_{34}, x_{11}x_{21}x_{22}x_{32}x_{33}x_{34}, x_{11}x_{21}x_{22}\ x_{23}x_{33}x_{34},\ x_{11}x_{21}x_{22}\ x_{23}x_{24}x_{34}, x_{11}x_{12}x_{22}x_{32}x_{33}x_{34}, x_{11}x_{12}x_{22}x_{23}x_{33}x_{34},\ x_{11}x_{12}x_{22}x_{23}x_{24}x_{34}, x_{11}x_{12}\ x_{13}x_{23}x_{33}x_{34},\ x_{11}x_{12}x_{13}x_{23}x_{24}\ x_{34},\ x_{11}x_{12}x_{13}x_{14}x_{24}x_{34})$.

Since each irreducible primary components of I_P correspond to a chain blocker of P. Thus, by Theorem 2.1 number of irreducible primary components of I_P is given by 22. The same number is verified by the Computer Algebra System CoCoA and Singular. In particular the irreducible decomposition of $I^{\mathcal{F}}(\Delta_P)$ is given by

$I^{\mathcal{F}}(\Delta_P) = (x_{34}) \cap (x_{11}) \cap (x_{33}x_{24}) \cap (x_{21}x_{12}) \cap (x_{33}x_{22}x_{13}) \cap (x_{32}x_{22}x_{13}) \cap (x_{31}x_{22}x_{13}) \cap (x_{21}x_{22}x_{13}) \cap (x_{33}x_{12}x_{22}) \cap (x_{32}x_{12}x_{22}) \cap (x_{31}x_{12}x_{22}) \cap (x_{32}x_{23}x_{24}) \cap (x_{33}x_{23}x_{14}) \cap (x_{32}x_{23}x_{14}) \cap (x_{33}x_{13}x_{23}) \cap (x_{32}x_{13}x_{23}) \cap (x_{32}x_{12}x_{23}) \cap (x_{33}x_{12}x_{23}) \cap (x_{31}x_{22}x_{23}x_{24}) \cap (x_{21}x_{22}x_{23}x_{24}) \cap (x_{31}x_{22}x_{23}x_{14}) \cap (x_{21}x_{22}x_{23}x_{14})$.

Note that each irreducible component of I_P correspond to the chain blocker of P.

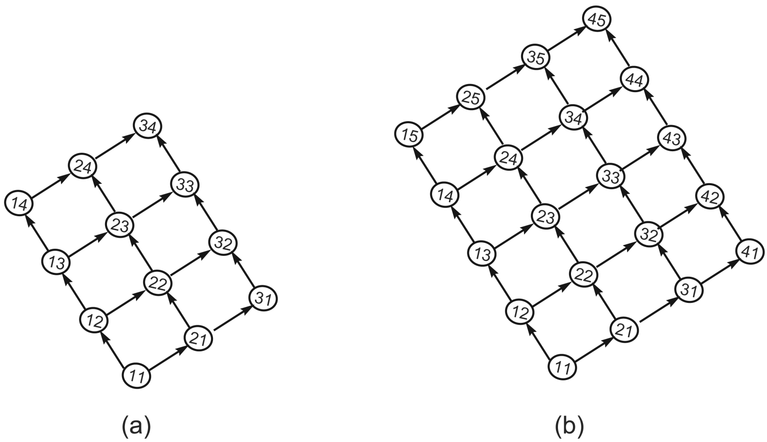


Fig. 4 – $C_4 \times C_b$.

Example 4.2. Let $P = C_4 \times C_4$ be poset as shown in Fig. 4 (b). Then

$$I_P = (x_{11}x_{21}x_{31}x_{41}x_{42}x_{43}x_{44}x_{45}, \ldots, x_{11}x_{12}x_{13}x_{14}x_{15}x_{25}x_{35}x_{45}).$$

Since each irreducible primary components of I_P correspond to a chain blocker of P, then by Theorem 3.1 number of its irreducible components is given by 157. The same number is verified by Computer Algebra System CoCoA and Singular.

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