ON THE CHAIN BLOCKERS OF A POSET

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Let \( P = C_a \times C_b \) be a poset where \( C_i \) is the chain \( 1 < \cdots < i \). A chain blocker of \( P \) is an inclusionwise minimal subset \( B \subseteq P \) with the property that every maximal chain in \( P \) contains at least one element of \( B \). In [1] the chain blockers of \( P \) are being expressed in term of the Catalan numbers and \( k \) fold convolution of the Catalan numbers. In this paper we give a complete description of the chain blockers of \( C_a \times C_b \), where \( a \leq 4 \) and \( b \geq 1 \). In the end algebraic consequences of the chain blockers are also provided.

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1. INTRODUCTION

Let \( P = C_a \times C_b \) be a poset where \( C_i \) is the chain \( 1 < \cdots < i \). A chain blocker of \( P \) is defined as a subset \( B \subseteq P \) such that every maximal chain in \( P \) contains at least one element of \( B \) and \( B \) is inclusionwise minimal with this property. In [1] the chain blockers of \( P \) were studied for some special cases and provided a new combinatorial interpretation of the convoluted Catalan numbers \( C(n, k) := \frac{k}{2n-k}(2n-k \choose n) \) and Catalan numbers \( C(n) = C(n, 1) \) introduced by Catalan [2] in 1887.

In Section 2 the chain blockers of \( C_a \times C_b \) are being discussed for \( a \leq 3 \) and \( b \geq 1 \). The main result of this section states that number of all chain blockers of \( C_3 \times C_b \) is given by a polynomial in \( b \) (Theorem 2.1). In Section 3 a formula for calculating number of all chain blockers of \( C_4 \times C_b \) is derived (Theorem 3.1).

Besides its combinatorial properties the chain blockers of \( P \) have its algebraic consequences. Let \( R = k[x_1, \ldots, x_n] \) be the polynomial ring over a field \( k \) and in \( n \) variables. To each poset \( P \) of cardinality \( n \) we define an ideal \( I_P \subset R \) such that the generators of \( I_P \) correspond to the maximal chains in \( P \). The chain blockers of \( P \) have one to one correspondence with the irreducible primary components of \( I_P \) (Proposition 4.1).

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2. INITIAL CASES

In this section, we give a description of the chain blockers of \( P = C_a \times C_b \) where \( a \leq 3 \) and \( b \geq 1 \). We start with giving a basic property of a chain blocker of \( P \).

**Lemma 2.1.** Let \( P = C_a \times C_b \) be a poset. Let \( A = \{(1,1), (1,2), \ldots, (1,b), (2,b), \ldots, (a,b)\} \) and \( B \subseteq P \) be a chain blocker. Then \( A \cap B \) contains exactly one element.

*Proof.* Since \( A \) is a maximal chain and \( B \) is a chain blocker therefore \( A \cap B \neq \emptyset \). Now suppose \( |A \cap B| \neq 1 \) and \( A \cap B = \{(i_1,j_1), \ldots, (i_r,j_r)\} \) with \( r > 1 \). We ordered the elements of \( A \cap B \) in the ascending order i.e. \((i_1,j_1) < \cdots < (i_r,j_r)\). If \( A \cap B \subseteq \{(1,1), (1,2), \ldots, (1,b)\} \) then a maximal chain blocked by any of \((i_k,j_k) \in \{(i_1,j_1) > \cdots > (i_{r-1},j_{r-1})\} \) is also blocked by \((i_1,j_1)\), hence by minimality of \( B \) we have \( A \cap B = \{(i_1,j_1)\} \). Similarly, if \( \{A \cap B\} \subset \{(2,b), \ldots, (a,b)\} \) then \( A \cap B = \{(i_r,j_r)\} \).

Now if \( \{A \cap B\} \cap \{(2,b), \ldots, (a,b)\} \neq \emptyset \) then it is enough to take \( i_1 = 1 \). Since \( B \) is a chain blocker so there exist two different maximal chains with one containing \( \{(i_1,j_1)\} \) but not containing \( (i_r,j_r) \) and vice versa. These maximal chains must intersect at some points say \( (i_k,j_k) \). Then if we combined these two maximal chains to get another maximal chain in such a way that the part before the point \((i_k,j_k)\) consists of points of the maximal chain containing \((i_r,j_r)\) and the part after \((i_k,j_k)\) consists of the points of the maximal chain containing \((i_1,j_1)\). Then this maximal chain is not blocked by \( B \), a contradiction. \( \square \)

We call the maximal chain \( A \) in the previous lemma as the left maximal chain of \( P \) and denote it by \( \mathcal{L}(P) \). Similarly, we call \( \{(1,1), (2,1), \ldots, (a,1), (a,2), \ldots, (a,b)\} \) as the right maximal chain of \( P \) and denote it by \( \mathcal{R}(P) \). Hence, by previous lemma and Lemma 2.2 [1], we have:

**Corollary 2.1.** Let \( B \subseteq C_a \times C_b \) be a chain blocker with \( |B| > 1 \). Then \( B \) contains exactly one element from \( \mathcal{R}(C_a \times C_b) \) and exactly one element from \( \mathcal{L}(C_a \times C_b) \).

If \( a = 1 \) then \( C_1 \times C_b \) is given by the chain

\[(1,1) < (1,2) < \cdots < (1,b),\]

which is the only its maximal chain. Thus, each element \( C_1 \times C_b \) is a chain blocker. Hence, number of chain blockers in this case is equal to \( b \). Following lemma provides a complete description of the chain blockers of \( C_2 \times C_b \).

**Lemma 2.2.** If \( B \) is a chain-blocker for \( C_2 \times C_b \) then either \( |B| = 1 \) or \( B = \{(1,j),(2,j')\} \), where \( 1 \leq j - 1 \leq j' \leq b - 1 \).
Proof. If \(|B| = 1\) then \(B = \{(1, 1)\}\) or \(B = \{(2, b)\}\) are only the chain blockers. Now let \(|B| > 1\). By Corollary 2.1, \(B = \{(1, j), (2, j')\}\) for \(j \in \{2, \ldots, b\}\) and \(j' \in \{1, \ldots, b - 1\}\). The condition \(j - 1 \leq j'\) then follows from the fact that otherwise not all chains are blocked. \(\square\)

By above lemma the number of chain blockers \(B\) of \(C_2 \times C_b\) with \(|B| > 1\) is given by

\[(b - 1) + (b - 2) + \cdots + 2 + 1 = \frac{1}{2}b(b - 1).\]

Next, we turn our attention to count number of all chain blockers of \(C_3 \times C_b\).

**Proposition 2.1.** Let \(P = C_3 \times C_b\) be a poset. The number of chain blockers of \(P\) containing \((2, 1)\) equals the number of chain blockers of \(P\) containing \((3, 1)\). Moreover both numbers equal to \(b\).

**Proof.** Here \(\mathcal{R}(P) = \{(1, 1), (2, 1), (3, 1), \ldots, (3, b)\}\) and \(\mathcal{L}(P) = \{(1, 1), \ldots, (1, b)(2, b), (3, b)\}\). By Corollary 2.1, a chain blocker \(B\) of \(P\) containing \((2, 1)\) does not contain any element from the set \(\mathcal{R}(P) \setminus (2, 1)\) and contains exactly one element from \(\mathcal{L}(P) \setminus \{(1, 1), (3, b)\}\), since if \(B\) contains \((2, 1)\) we must exclude the choices of minimum and maximum elements of \(P\).

Now let \((1, i) \in \mathcal{L}(P) \cap B\), where \(2 \leq i \leq b\). Then \(B\) must contain the set \(\{(2, 2), \ldots, (2, i - 1)\}\). If not say \((2, j) \notin B\) for some \(j \in \{2, i - 1\}\), then \(\{(1, 1) \ldots, (1, j), (2, j), (3, j), \ldots, (3, b)\}\) is a maximal chain which is not blocked by \(B\). Moreover by the minimality of \(B\), \((2, j) \notin B\) for \(j \geq i\). Hence,

\[B = \{(1, i), (2, i - 1), \ldots, (2, 2), (2, 1)\}\]

is only the chain blocker containing \((1, i)\) and \((2, 1)\). Running over all values of \(i\) we have \(b - 1\) such chain blockers. Similarly,

\[B = \{(2, b), (2, b - 1), \ldots, (2, 2), (2, 1)\}\]

is only the chain blocker containing \((2, 1)\) and \((2, b)\). Hence, total number of chain blockers containing \((2, 1)\) is equal to \(b\). Now since \((2, 1) < (3, 1)\) and \((2, 1) < (2, 2)\) but \((3, 1)\) and \((2, 2)\) are incomparable so any chain blocker of \(P\) containing \((2, 1)\) remains a chain blocker if we replace \((2, 1)\) with \((3, 1)\). Hence, we are done. \(\square\)

For the case \(C_3 \times C_b\) following theorem provides an explicit formula to calculate number of chain blockers of \(P\).

**Theorem 2.1.** Let \(P = C_3 \times C_{b+1}\) be a poset. The number of chain blockers of \(P\) is given by

\[
\frac{1}{6} (b^2 + 2)(b + 9).
\]
Proof. For a fixed element \((i, j) \in \mathcal{R}(P)\), we run over all elements of \(\mathcal{L}(P)\) one by one to count number of chain blockers containing both elements. If \((i, j) = (1, 1)\) or \((3, b + 1)\) then \(B = \{(i, j)\}\) is itself a chain blocker. Also by Proposition 2.1 the number of chain blockers \(B\) containing \((2, 1)\) or \((3, 1)\) equals to \(2(b + 1)\). Now let \(B\) be a chain blocker containing \((3, j) \in \mathcal{R}(P)\) and \((m, n) \in \mathcal{L}(P)\). We are left with the following cases:

Case I: \(2 \leq j \leq b - 1, (m, n) = (1, 2)\):

Here \(B \cap \{(2, 2), \ldots, (2, j + 1)\} \neq \emptyset\), because if the intersection is empty then \((1, 1) < (2, 1) < \ldots < (2, j + 1) < (3, j + 1) < \ldots < (3, b)\) is a maximal chain not blocked \(B\). Moreover by minimality of \(B\), we have \(|B \cap \{(2, 2), \ldots, (2, j + 1)\}| = 1\) and \(B \cap \{(2, j + 2), \ldots, (2, b)\} = \emptyset\). Thus, there are \(j\) such chain blockers. Hence, number of chain blockers for this case is given by

\[
\sum_{j=2}^{b-1} j = \frac{1}{2} b^2 - \frac{1}{2} b - 1.
\]

Case II: \(2 \leq j \leq b - 1, m = 1, 3 \leq n \leq b + 1\):

If \(3 \leq n \leq j + 1\) then by the same arguments as in Case I, \(B \cap \{(2, n - 1), \ldots, (2, j + 1)\} \neq \emptyset\) and for a fixed \(j\) and \(n\) number of chain blockers equals to \(j - n + 3\) and hence, total number for these choices equals to \(\sum_{j=2}^{b-1} \sum_{n=3}^{j+1} (j - n + 3) = \frac{1}{6} b^3 - \frac{7}{6} b + 1\). On the other hand if \(j + 2 \leq n \leq b + 1\) then \(B = \{(1, n), (2, n - 1), \ldots, (2, j + 1), (3, j)\}\) is only chain blocker containing \((1, n)\) and \((3, j)\). Thus, we have \(\sum_{j=2}^{b-1} \sum_{n=j+2}^{b+1} 1 = \frac{1}{2} b^2 - \frac{3}{2} b + 1\). Hence, total number of chain blockers for this case is given by

\[
\frac{1}{6} b^3 + \frac{1}{2} b^2 - \frac{8}{3} b + 2.
\]

Case III: \(j = b, m = 1, 2 \leq n \leq b + 1\):

Let \(B\) be a chain blocker containing \((3, b - 1)\) and \((m, n) \in \mathcal{L}(P)\). Since \((2, b)\) and \((3, b - 1)\) are incomparable so if we replace \((3, b - 1)\) by \((3, b)\) then \(B\) will remain a chain blocker. Thus, number of chain blockers in this case is given by putting \(j = b - 1\) in the previous Cases I and II. That is

\[
b - 1 + \sum_{n=3}^{b} (b - n + 2) + 1 = \frac{1}{2} b^2 + \frac{1}{2} b - 1.
\]

Case IV: \(2 \leq j \leq b, (m, n) = (2, b + 1)\):

If \(2 \leq j \leq b - 1\), then \(B = \{(2, b + 1), \ldots, (2, j + 1), (3, j)\}\) is the chain blocker. If \(j = b\), then \(B = \{(2, b + 1), (3, b)\}\). Thus, number of chain blockers for this case is \(b - 1\).

Now summing over all above cases and \(2(b + 1) + 2\) contribution from the initial choices, we have the required formula after simplification. \(\square\)
3. THE CASE $C_4 \times C_b$

Let $P = C_4 \times C_b$ be a poset. Then the right maximal chain $\mathcal{R}(P) := (1,1) < \cdots < (4,1) < \cdots < (4,b)$ and left maximal chain $\mathcal{L}(P) := (1,1) < \cdots < (1,b) < \cdots (4,b)$.

Proposition 3.1. Let $P = C_4 \times C_b$ be a poset. The number of chain blockers of $P$ containing $(2,1)$ equals to

$$2^b - b + 1.$$ 

Proof. Let $B$ be a chain blocker of $P$ containing $(2,1)$. Then by Proposition 2.1, $B$ contains exactly one element $(m,n)$ from $\mathcal{L}(P)$. If $(m,n) = (1,2)$ then $B = \{(1,2), (2,1)\}$ is itself a chain blocker. If $m = 1$ and $3 \leq n \leq b$ then either $(2,i) \in B$ or $(3,i) \in B$ for all $i = 2,\ldots,n - 2$. If it is not true for $i \in \{2,\ldots,n-2\}$ then $(1,1) < \cdots < (1,i) < (2,i) < (3,i) < (4,i) < \cdots < (4,b)$ is a maximal chain not blocked by $B$. Number of these choices are given by $2^{n-3}$. Now it remain to block a chain containing $(1,1) < \cdots < (1,n - 1) < (2,n - 1)$. For this $B$ must contains $\{(2,k), (3,k - 1),\ldots,(3,n - 1)\}$ where $k = n - 1,\ldots,b - 1$. Since these two cases are independent so number of chain blockers are given by $2^{n-3}(b - n + 1)$.

Now if $2 \leq m \leq 3$ and $n = b$, then by the same argument as before $B$ must contains either $(2,i)$ or $(3,i)$ for all $i = 2,\ldots,b - 1$ ⇒ it contributes $2 \cdot 2^{b-2}$. Thus, total number of chain blockers are given by

$$1 + \sum_{n=3}^{b} 2^{n-3}(b - n + 1) + 2 \cdot 2^{b-2},$$

on simplification we are done. \(\square\)

Corollary 3.1. Let $P = C_4 \times C_b$ be a poset, then number of chain blockers of $P$ containing $(3,1)$ is given by

$$2^b - 2.$$ 

Proof. Let $B$ be a chain blocker containing $(3,1)$ and an element $(m,n) \in \mathcal{L}(P) \setminus \{(1,2)\}$. Since $(1,3)$, $(2,2)$ and $(3,1)$ are in comparable and $(2,1) < (3,1)$, so if we replace $(3,1)$ by $(2,1)$ then $B$ will remain a chain blocker(see Fig. 1). Thus, by Proposition 3.1, the number of chain blockers containing $(3,1)$ and $(m,n) \in \mathcal{L}(P) \setminus \{(1,2)\}$ is given by $2^b - b$. Now if $(m,n) = (1,2)$ then $B = \{(1,2), (2,k), (3,k - 1),\ldots,(3,1)\}$, where $k \in \{2,\ldots,b - 1\}$. There are $b - 2$ such chain blockers. Hence, total number of chain blockers of $P$ containing $(3,1)$ equals to $2^b - 2$. \(\square\)
Proposition 3.2. Let $P = C_4 \times C_b$ be a poset. The number of chain blockers of $P$ containing $(4, 1)$ is given by

$$2^{b-1} + b - 3.$$ 

Proof. Clearly, if $B$ is a chain blocker containing $(4, 1)$ then $(3, 2) \in B$. Moreover, $B$ must contain an element $(m, n) \in \mathcal{L}(P)$. If $(m, n) = (1, 2)$ or $(1, 3)$, then either $B = \{(m, n), (2, 2), (3, 2), (4, 1)\}$ or $B = \{(m, n), (2, k), (3, k-1), \ldots, (3, 2), (4, 1)\}$ where $k = 3, \ldots, b - 1$. Similarly if $(m, n) = (1, 4)$, then $B = \{(1, 4), (2, k), (3, k-1), \ldots, (3, 2), (4, 1)\}$ where $k = 3, \ldots, b - 1$. Thus, so far we have calculated $3b - 7$ number of chain blockers for $m = 1$ and $n \in \{2, 3, 4\}$.

Now if $m = 1$ and $5 \leq n \leq b$ then either $(2, i) \in B$ or $(3, i) \in B$ for all $i = 3, \ldots, n - 2 \Rightarrow$ there are $2^{b-4}$ possibilities. Moreover to block a maximal chain containing $(1, 1) < \cdots < (1, n-1) < (2, n-1)$ then either $(2, n-1) \in B$ or $\{(2, k), (3, k-1), \ldots, (3, n-1)\} \subset B$ for $k = n, \ldots, b-1$. There are $b-n+1$ such possibilities. Since these two choices are independent hence number of chain blockers for this case is equal to $2^{n-4}(b-n+1)$.

Lastly, if $2 \leq m \leq 3$ and $n = b$, then $B$ must contains either $(2, i)$ or $(3, i)$ for all $3 \leq i \leq b$ which contributes $2(2^{b-2})$. Thus, total number of chain blockers of $P$ containing $(4, 1)$ is given by

$$3b - 7 + \sum_{n=5}^{b} 2^{n-4}(b-n+1) + 2^{b-1} = 2^{b-1} + b - 3. \quad \Box$$

Theorem 3.1. Let $P = C_4 \times C_b$ be a poset, then number of all chain
blocks of \( P \) is given by

\[
\frac{1}{24} b^4 + \frac{7}{12} b^3 - \frac{37}{24} b^2 - \frac{85}{12} b + 4 + 2^{b+2}.
\]

**Proof.** As before we fix one element of \( \mathcal{R}(P) \) and run over all elements of \( \mathcal{L}(P) \) to count number of chain blockers containing these two elements. As there are two trivial chain blockers of cardinality 1. Let \(|B| \geq 2\) and \((i, 1) \in \mathcal{R}(P)\) for \(i \in \{2, 3, 4\}\). Then number of chain blockers containing \((i, 1)\) is given by Proposition 3.1, Corollary 3.1 and Proposition 3.2. Thus, number of chain blockers for these choices including chain blockers of cardinality 1 is given by

\[5 \cdot 2^{b-1} - 1.\]

Now let \((4, j) \in \mathcal{R}(P)\) and \((1, n) \in \mathcal{L}(P)\), where \(2 \leq j \leq b - 1\) and \(2 \leq n \leq b\). We have following two main cases:

**Case I** \((3 \leq n \leq j + 2)\): Obviously \(B\) must have at least one element \((3, k)\) where \(2 \leq k \leq b - 1\), otherwise the maximal chain \((1, 1) < \cdots < (3, 1) < \cdots < (3, b) < (3, 4)\) is not blocked by \(B\). We have three choices for \((3, k) \in B\).

- **a:** If \(k < n - 1\), then \(B = \{(1, n), (2, n - 1), \ldots, (2, k - 1), (3, k), (4, j)\}\) (see Diagram 2(a)). Note that there exist \(n - 3\) such chain blockers \(B\).
- **b:** If \(n - 1 \leq k \leq j + 1\), then \(B = \{(1, n), (2, i), (3, k), (4, j)\}\) where \(n - 1 \leq i \leq k + 1\) (see Diagram 2(b)). Note that there exist \(\sum_{k=n-1}^{j+1} \sum_{i=n-1}^{k+1} 1\) such chain blockers \(B\).
- **c:** If \(j + 2 \leq k \leq b - 2\), then \(B = \{(1, n), (2, k+1), (3, k), \ldots, (3, j+1), (4, j)\}\) (see Diagram 2(c)). We have \(b - j - 3\) such chain blockers \(B\)

![Fig. 2 – \(C_4 \times C_b\).](image)
Hence, the number of chain blockers against cases \(a, b\) and \(c\) is given by:

\[
\sum_{j=2}^{b-2} \sum_{n=3}^{j+2} \{n - 3 + \sum_{k=n-1}^{j+1} \sum_{i=n-1}^{k+1} 1 + b - j - 3\}.
\]

Now if \(2 \leq j \leq b - 2\), the number of chain blockers of \(P\) containing \((1, 3)\) is the same as number of chain blockers containing \((1, 2)\). So if we fix \(n = 3\) in the above expression, then we obtain number of chain blockers containing \((1, 2)\) and \((4, j)\) for \(j \in \{2, \ldots, b - 2\}\). Moreover if \(2 \leq n \leq b\), the number of chain blockers containing \((4, b - 2)\) is the same as number of chain blockers containing \((4, b - 1)\).

Hence, by a simple calculations number of chain blockers containing \((1, n)\) and \((4, j)\) where \(2 \leq n \leq j + 2\) and \(2 \leq j \leq b - 1\) is given by:

\[
\frac{1}{24}b^4 + \frac{3}{4}b^3 - \frac{37}{24}b^2 - \frac{25}{4}b + 4.
\]

**Case II** \((j + 2 < n \leq b)\): Again \(B\) must have an element \((3, k)\) for \(k \in \{2, \ldots, b - 1\}\). For this case we have three independent subcases, namely:

- **d:** If \(k \leq j + 1\), then \(\{(3, k), (2, k+1), \ldots, (2, j+1)\} \subset B\) (see Diagram 3(d)). Note that there exist \(j\) such possibilities.

- **e:** If \(j + 1 \leq k \leq n - 3\), then \((2, i) \in B\) or \((3, i) \in B\) where \(j + 1 \leq i \leq n - 3\) (see Diagram 3(e)). Note that there exist \(2^{n-j-3}\) such possibilities.

- **f:** If \(n - 2 \leq k \leq b - 2\), then \(\{(2, b - 1), (3, b - 2), \ldots, (3, j + 2)\} \subset B\) (see Diagram 3(f)). We have \(b - n + 1\) such possibilities.

![Diagram](image.png)

Fig. 3 – \(C_4 \times C_b\).
Hence, multiplying out above independent cases \( d, e \) and \( f \), we get
\[
\sum_{j=3}^{b-3} \sum_{n=j+3}^{b} j(b - n + 1)2^{n-j-3}.
\]

Now if \((m, n) = (2, 3)\) or \((2, 4)\) then by cases \( d \) and \( e \) above, the number of chain blockers containing \((m, n)\) is given by \( j2^{b-j-1} \). Lastly, by symmetry the number of chain blockers containing \{\((4, b - 2), (2, b)\)\}, \{\((4, b - 2), (3, b)\)\} or \{\((4, b - 1), (2, b)\)\} are the same and given by \( b - 2 \). Also there is one chain blockers \( B = \{ (4, b - 1), (3, b) \} \). Finally, summing up all above cases we have the required formula. \( \Box \)

4. APPLICATIONS

In this section, we provide algebraic consequences associated to a chain blocker \( B \) of \( P \). A simplicial complex \( \Delta \) on the vertex set \( V = [n] \) is a collection of subsets of \( 2^{[n]} \) with the property that if \( A \in \Delta \) then \( \Delta \) contains all subsets of \( A \). The inclusionwise maximum elements of \( \Delta \) are called facets. Let \( \{ F_1, \ldots, F_r \} \) be the set of facets of \( \Delta \). A minimal vertex cover of \( \Delta \) is a subset \( A \subseteq V \) with the property that for every facet \( F_i \) of \( \Delta \) there exist a vertex \( v \in A \) such that \( v \in F_i \) and \( A \) is minimal with this property.

Let \( \Delta_P \) be a simplicial complex associated to a poset \( P \) in such a way that elements of \( \Delta_P \) are exactly the chains in \( P \). The set of facets of \( \Delta_P \) are the maximal chains of \( P \) and hence each chain blockers of \( P \) is a minimal vertex cover of \( \Delta_P \).

Now we are ready to relate a poset \( P \) to its algebraic counterpart. Let \( S = k[x_1, \ldots, x_n] \) be the polynomial ring over the field \( k \) and in \( n \) variables. Recall that a monomial ideal in \( S \) is an ideal generated by monomials \( u_i \). A monomial ideal is called a squarefree monomial ideal if it is generated by the square free monomials. Let \( I = I_1 \cap \cdots \cap I_r \) be an irredundant primary decomposition of \( I \), where ideals \( I_1, \ldots, I_r \) are called irreducible primary components of \( I \). For more details about primary decomposition see [6].

To each square free monomial ideal \( I \) one can associate a simplicial complex \( \Delta \). One way of this association is facet ideals and facet complex introduced by Sara Faridi [4]. A facet ideals \( I^F(\Delta) \) of \( \Delta \) is an ideal generated by square free monomial \( x_{i_1} \cdots x_{i_t} \) where \( \{ x_{i_1}, \ldots, x_{i_t} \} \) is a facet of \( \Delta \). Let \( i = < u_1, \ldots, u_r > \) be a squarefree monomial ideal. A facet complex \( \Delta^F(I) \) of \( I \) is a simplicial complex over the vertex set \( \{ v_1, \ldots, v_n \} \) and set of facets \( \{ F_1, \ldots, F_r \} \), where \( F_i = \{ v_j \mid x_j \not\in u_i, 1 \leq j \leq n \} \).

It is well know that minimal vertex covers of \( \Delta^F(I) \) correspond to the irreducible primary components of \( I^F(\Delta) \). Let \( I_P = I^F(\Delta_P) \). Note that \( I_P \)
is also the path ideals of the directed graph of Hasse diagram of $P$. The path ideal was introduced by Conca and De Negri in [3]. Some results of $I_P$ were also studied in [5]. Since the facets of $\Delta_P$ are the maximal chains of $P$, hence by definition of a chain blocker we have the following proposition.

**Proposition 4.1.** Let $P = C_a \times C_b$ be a poset and $I_P$ be the ideal as defined above. Then there is a one to one correspondence between chain blockers of $P$ and irreducible primary components of $I_P$.

Following examples demonstrate the one to one correspondence given in above proposition.

**Example 4.1.** Let $P = C_3 \times C_4$ be a poset as shown in Figure 4 (a). Then $I_P = (x_{11}x_{21}x_{31}x_{32}x_{33}x_{34}, x_{11}x_{21}x_{22}x_{32}x_{33}x_{34}, x_{11}x_{21}x_{22}x_{23}x_{33}x_{34}, x_{11}x_{21}x_{22}x_{23}x_{24}x_{34}, x_{11}x_{12}x_{22}x_{23}x_{33}x_{34}, x_{11}x_{12}x_{22}x_{23}x_{24}x_{34}, x_{11}x_{12}x_{13}x_{23}x_{24}x_{34}, x_{11}x_{12}x_{13}x_{14}x_{24}x_{34}).$

Since each irreducible primary components of $I_P$ correspond to a chain blocker of $P$. Thus, by Theorem 2.1 number of irreducible primary components of $I_P$ is given by 22. The same number is verified by the Computer Algebra System CoCoA and Singular. In particular the irreducible decomposition of $I_F(\Delta_P)$ is given by

$I_F(\Delta_P) = (x_{34}) \cap (x_{11}) \cap (x_{33}x_{24}) \cap (x_{21}x_{12}) \cap (x_{33}x_{22}x_{13}) \cap (x_{32}x_{22}x_{13}) \cap (x_{31}x_{22}x_{13}) \cap (x_{21}x_{22}x_{13}) \cap (x_{33}x_{12}x_{22}) \cap (x_{32}x_{12}x_{22}) \cap (x_{31}x_{23}x_{14}) \cap (x_{32}x_{23}x_{14}) \cap (x_{33}x_{13}x_{23}) \cap (x_{32}x_{12}x_{23}) \cap (x_{33}x_{12}x_{23}) \cap (x_{31}x_{22}x_{23}x_{24}) \cap (x_{21}x_{22}x_{23}x_{24}) \cap (x_{31}x_{22}x_{23}x_{14}) \cap (x_{21}x_{22}x_{23}x_{14}).$

Note that each irreducible component of $I_P$ correspond to the chain blocker of $P$. 

![Fig. 4 – $C_4 \times C_b$.](image-url)
Example 4.2. Let $P = C_4 \times C_4$ be poset as shown in Fig. 4 (b). Then

$$I_P = (x_{11}x_{21}x_{31}x_{41}x_{42}x_{43}x_{44}x_{45}, \ldots , x_{11}x_{12}x_{13}x_{14}x_{15}x_{25}x_{35}x_{45}).$$

Since each irreducible primary components of $I_P$ correspond to a chain blocker of $P$, then by Theorem 3.1 number of its irreducible components is given by 157. The same number is verified by Computer Algebra System CoCoA and Singular.

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