FINITELY SUPPORTED SUBGROUPS OF A NOMINAL GROUP

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We define the notion of finitely supported subgroup of a nominal group, and present some algebraic properties of these subgroups. We prove that the family of all finitely supported subgroups of a nominal group forms a nominal complete lattice and a nominal algebraic domain.

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1. INTRODUCTION

Nominal sets represent an alternative set theory which allows a more relaxed interpretation for the notion of finiteness. They offer an elegant formalism for describing λ-terms modulo α-conversion [15], or automata on data words [12]. The theory of nominal sets has its origins in an approach developed initially by Fraenkel and Mostowski (FM) in the 1930s [14, 17], in order to prove the independence of the axiom of choice and other axioms in classical Zermelo-Fraenkel (ZF) set theory. In the last dozen years, the FM permutation model of Zermelo-Fraenkel set theory with atoms (ZFA) was axiomatized and presented as an independent set theory with atoms, named FM set theory [15]. The axioms of FM set theory are the ZFA axioms over an infinite set of atoms [15], together with the special axiom of finite support which claims that for each element x in an arbitrary set we can find a finite set supporting x. Rather than using a non-standard set theory, one could alternatively work with nominal sets [21], which are defined within ZF as usual sets endowed with some group actions satisfying a finite support requirement. Informally, whenever we consider the elements of a nominal set having a finite set of free names, the action of a permutation on such an element actually represents the renaming of these free names. The approach based on nominal sets provides a right balance between rigorous formalism and informal reasoning. This is well explained in [20] where the principles of structural recursion and induction are presented in the framework of nominal sets. There exists also an alternative definition for nominal sets in the FM framework (when the set of names is related to the set of atoms in FM). They can be defined as sets constructed according to the FM axioms with the additional property of being empty supported (invariant under all permutations). The two ways of considering nominal sets, namely in the ZF framework and in FM framework, lead to similar properties. More details are presented in Section 3.

Nominal groups [6] and nominal ordered sets [21] have already been studied in the literature. This paper is a continuation of the research effort described in [6], and provides an algebraic study on the family of finitely supported subgroups of a nominal group. Because of the finite support requirement, in the universe of nominal sets only finitely supported objects are allowed. Our goal is to present some order properties for nominal groups, in terms of finitely supported objects. More precisely, we present some properties of the family of all finitely supported subgroups of a nominal group in terms of nominal lattices and nominal domains. Actually, in this paper we present some properties of the family of all finitely (possibly non-empty) supported subgroups of a nominal group, whilst in [6] we studied only the empty-supported subgroups of a nominal group. Thus, this paper generalizes the approach presented in [6].

2. PRELIMINARIES

Let $(P, \sqsubseteq)$ be a poset. A subset U of P is directed if it is non-empty and each pair of elements in U has an upper bound in U. A poset $(D, \sqsubseteq)$ in which every directed subset has a supremum is called a directed-complete partial order, or dcpo for short. Let x and y be elements of a dcpo $(D, \sqsubseteq)$. We say that x approximates y, and denote this by $x \ll y$, if for all directed subsets U of $(D, \sqsubseteq)$ we have that $y \sqsubseteq sup(U)$ implies $x \sqsubseteq u$ for some $u \in U$. We say that x is compact if it approximates itself; the set of all compact elements in a dcpo D is denoted by $K(D)$. We note that $x \sqsubseteq y$ whenever $x \ll y$, and $x' \ll y'$ whenever $x' \sqsubseteq x \ll y \sqsubseteq y'$. We say that a subset B of a dcpo $(D, \sqsubseteq)$ is a basis for $(D, \sqsubseteq)$, if for every element x of $(D, \sqsubseteq)$ there exists a directed subset U of elements in B approximating x, with $sup(U) = x$. The directness of U shows that whenever B is a basis for $(D, \sqsubseteq)$, for each element x in D we can say that the set of elements in B approximating x is directed, and x is the supremum of the directed set of elements in B approximating it. Using the definition of approximation and the previous result we conclude that for each dcpo $(D, \sqsubseteq)$ with a basis B we have that $K(D) \subseteq B$. A dcpo is called a continuous domain if it has a basis. It is called an algebraic domain if it has a basis of compact elements. More details are in [2].

Let $(G, \cdot)$ be a group. If H is a subgroup of G we denote this by $H \leq G$.

If $S \subseteq G$, we denote by $[S]$ the subgroup of G generated by S, i.e. the smallest subgroup of G which contains S. Every element of $[S]$ can be expressed as a finite product of elements of S and inverses of elements of S. If $S \subseteq G$ is finite and $H = [S]$ we call H a finitely generated subgroup of G. The set $\mathcal{L}(G)$ of all subgroups of G ordered by inclusion forms a complete lattice. If $(H_i)_{i\in I}$ is a family of subgroups of G, the infimum of this family is $\underset{i\in I}{\cap} H_i$ and the supremum is $[\underset{i\in I}{\cup} H_i]$. Moreover, according to [2] we have that $(\mathcal{L}(G), \subseteq)$ is an algebraic domain and the compact elements in $(\mathcal{L}(G), \subseteq)$ are precisely those in $F(\mathcal{L}(G))$, where $F(\mathcal{L}(G))$ is the set of all finitely generated subgroups of a group G.

3. NOMINAL SETS AND NOMINAL POSETS

Our goal is to rephrase the previous definitions and results (in the framework of nominal sets) in terms of finitely supported objects. In order to reach this goal, we follow [21] and recall the basics of nominal sets. A complete presentation of such sets is given in Section 2 of [6].

Let A be a fixed infinite ZF-set. The following results make also sense if A is considered to be the set of atoms in the ZFA framework (characterized by the axiom "$y \in x \Rightarrow x \notin A$") and if 'ZF' is replaced by 'ZFA' in their statement. Thus, we mention that the theory of nominal sets makes sense in both ZF and ZFA.

A *permutation* of A is defined as a finitary bijection of A, i.e. a bijection of A which interchanges only finitely many elements. Let $(S_A, \circ)$ be the group of all permutations of A, and X a ZF-set.

Definition 3.1.

- An S_A*-action* on X is a function $\cdot : S_A \times X \to X$ having the properties that $Id \cdot x = x$ and $\pi \cdot (\pi' \cdot x) = (\pi \circ \pi') \cdot x$ for all $\pi, \pi' \in S_A$ and $x \in X$;
- An S_A*-set* is a pair $(X, \cdot)$ where X is a ZF-set, and $\cdot : S_A \times X \to X$ is an S_A-action on X; we simply use X whenever no confusion arises.
- Let $(X, \cdot)$ be an S_A-set. We say that $S \subset A$ *supports* x whenever for each $\pi \in Fix(S)$ we have $\pi \cdot x = x$, where $Fix(S) = \{\pi \,|\, \pi(a) = a, \forall a \in S\}$.
- Let $(X, \cdot)$ be an S_A-set. We say that X is a *nominal set* if for each $x \in X$ there exists a finite set $S_x \subset A$ which supports x.

PROPOSITION 3.2. *Let X be an S_A-set, and for each $x \in X$ let us consider $\mathcal{F}_x = \{S \subset A \,|\, S$ finite, S supports $x\}$. If $\mathcal{F}_x$ is nonempty (particularly if X is a nominal set), then it has a least element which also supports x; this element is called the* support *of x, and it is denoted by $supp(x)$.*

Definition 3.3. Let $(X, \cdot)$ be a nominal set. An element $x \in X$ is called *equivariant* if it has an empty support, i.e. $\pi \cdot x = x$ for each $\pi \in S_A$.

PROPOSITION 3.4. *Let $(X, \cdot)$ be an S_A-set, and $\pi \in S_A$ an arbitrary permutation. Then for each $x \in X$ which is finitely supported we have that $\pi \cdot x$ is finitely supported, and $supp(\pi \cdot x) = \pi(supp(x))$.*

Example 3.5.

(1) The set A of atoms is an S_A-set with the S_A-action $\cdot : S_A \times A \to A$ defined by $\pi \cdot a := \pi(a)$ for all $\pi \in S_A$ and $a \in A$. $(A, \cdot)$ is a nominal set because for each $a \in A$ we have that $\{a\}$ supports a. Moreover, $supp(a) = \{a\}$ for each $a \in A$.

(2) Any usual set X (even the set A of atoms) is an S_A-set with the S_A-action $\cdot : S_A \times X \to X$ defined by $\pi \cdot x := x$ for all $\pi \in S_A$ and $x \in X$. $(X, \cdot)$ is a nominal set because $\emptyset$ supports x for each $x \in X$. Moreover, $supp(x) = \emptyset$ for each $x \in X$.

(3) The set S_A is an S_A-set with the S_A-action $\cdot : S_A \times S_A \to S_A$ defined by $\pi \cdot \sigma := \pi \circ \sigma \circ \pi^{-1}$ for all $\pi, \sigma \in S_A$. $(S_A, \cdot)$ is a nominal set because for each $\sigma \in S_A$ we have that the finite set $\{a \in A \,|\, \sigma(a) \neq a\}$ supports σ. Moreover, $supp(\sigma) = \{a \in A \,|\, \sigma(a) \neq a\}$ for each $\sigma \in S_A$.

(4) If $(X, \cdot)$ is an S_A-set, then $\wp(X) = \{Y \,|\, Y \subseteq X\}$ is also an S_A-set with the S_A-action $\star : S_A \times \wp(X) \to \wp(X)$ defined by $\pi \star Y := \{\pi \cdot y \,|\, y \in Y\}$ for all permutations π of A, and all subsets Y of X. Note that $\wp(X)$ is not necessarily a nominal set even if X is. For example, A is a nominal set, but $\wp(A)$ is not a nominal set because the subsets of A which are at the same time infinite and coinfinite do not have the finite support property. For each nominal set $(X, \cdot)$ we denote by $\wp_{fs}(X)$ the set formed from those subsets of X which are finitely supported according to the action $\star$. We have that $(\wp_{fs}(X), \star|_{\wp_{fs}(X)})$ is a nominal set, where $\star|_{\wp_{fs}(X)} : S_A \times \wp_{fs}(X) \to \wp_{fs}(X)$ is defined by $\pi \star|_{\wp_{fs}(X)} Y := \pi \star Y$ for all $\pi \in S_A$ and $Y \in \wp_{fs}(X)$; the codomain of the action $\star|_{\wp_{fs}(X)}$ (which is in fact the action $\star$ restricted to $\wp_{fs}(X)$) is indeed included in $\wp_{fs}(X)$ according to Proposition 3.4.

(5) Let $(X, \cdot)$ and $(Y, \diamond)$ be S_A-sets. The Cartesian product $X \times Y$ is also an S_A-set with the S_A-action $\star : S_A \times (X \times Y) \to (X \times Y)$ defined by $\pi \star (x, y) = (\pi \cdot x, \pi \diamond y)$ for all $\pi \in S_A$, $x \in X$ and $y \in Y$. If $(X, \cdot)$ and $(Y, \diamond)$ are nominal sets, then $(X \times Y, \star)$ is also a nominal set.

Definition 3.6. Let $(X, \cdot)$ be a nominal set. Using the notations of Example 3.5(4), a subset Z of X is called *finitely supported* if and only if $Z \in \wp_{fs}(X)$.

Definition 3.6 is valid for functions between nominal sets just because functions are particular relations, *i.e.* particular subsets of the Cartesian product of two nominal sets.

Nominal partially ordered sets have already been considered in order to develop the domain theory over nominal sets. Nominal partially ordered sets and lattices have been considered in [22] in order to solve the Scott recursive domain equation $D \cong (D \to D)$ in the framework of nominal sets, as well as in [18] in order to analyze the Stone duality within the nominal sets theory, and in [3] in order to present some fixpoint results for nominal event structures.

Definition 3.7. A *nominal partially ordered set (nominal poset)* is a nominal set $(P, \cdot)$ together with an equivariant partial order relation $\sqsubseteq$ on P. A nominal poset is denoted by $(P, \sqsubseteq, \cdot)$ or simply by P.

A partial order relation $\sqsubseteq$ on P is a subset of the Cartesian product $P \times P$; this relation is reflexive, anti-symmetric and transitive. According to Definition 3.3, $\sqsubseteq$ is equivariant if it is finitely supported as a subset of the Cartesian product $P \times P$ in the sense of Definition 3.6, and its support is empty. This means that $\sqsubseteq$ is equivariant iff for each pair $(e, e') \in \sqsubseteq$ and each $\pi \in S_A$ we have that $\pi \star (e, e') \in \sqsubseteq$ (where $\star$ represents the action of S_A on the Cartesian product $P \times P$ constructed as in Example 3.5(5)). Denoting "$(e, e') \in \sqsubseteq$" by "$e \sqsubseteq e'$ ", the equivariance property of $\sqsubseteq$ can be expressed by

$$e \sqsubseteq e' \text{ implies } \pi \cdot e \sqsubseteq \pi \cdot e', \text{ whenever } \pi \in S_A.$$

Definition 3.8. A *nominal lattice* is a nominal set $(L, \cdot)$ together with an equivariant lattice order relation $\sqsubseteq$ on L.

Definition 3.9. A *nominal complete lattice* is a nominal poset $(L, \sqsubseteq, \cdot)$ such that every finitely supported subset $X \subseteq L$ has a least upper bound with respect to the order relation $\sqsubseteq$. The least upper bound of X is denoted by $\sqcup X$.

According to Theorem 3.1 of [3] rephrased in terms of nominal lattices, for any nominal complete lattice $(L, \sqsubseteq, \cdot)$ we have that every finitely supported subset $X \subseteq L$ has a greatest lower bound with respect to the order relation $\sqsubseteq$. We can also reformulate other definitions of the usual ZF domain theory into the framework of nominal sets, while noting that in the framework of nominal sets only finitely supported objects are allowed.

Definition 3.10.

- A nominal poset $(D, \sqsubseteq, \cdot)$ in which every finitely supported directed subset has a supremum is called a *nominal directed-complete partial order*, or shortly *nominal dcpo*.
- Let x and y be elements of a nominal dcpo $(D, \sqsubseteq, \cdot)$. We say that x *nominally approximates* y, and denote this by $x \ll_{nom} y$, if for all finitely

supported directed subsets U of $(D, \sqsubseteq, \cdot)$ we have that $y \sqsubseteq \sqcup U$ implies $x \sqsubseteq u$ for some $u \in U$.
- x is *nominal compact* if it nominally approximates itself; the set of all nominal compact elements in a nominal dcpo D is denoted by $K(D)_{nom}$.

Definition 3.11. Let $(D, \sqsubseteq, \cdot)$ be a nominal dcpo.
- We say that a nominal set $B \subseteq D$ is a *nominal basis* for $(D, \sqsubseteq, \cdot)$ if for every element x of $(D, \sqsubseteq, \cdot)$ there exists a finitely supported directed subset U of elements in B approximating x with $\sqcup U = x$.
- $(D, \sqsubseteq, \cdot)$ is called a *nominal continuous domain* if it has a nominal basis.
- $(D, \sqsubseteq, \cdot)$ is called a *nominal algebraic domain* if it has a nominal basis of nominal compact elements.

4. THE SUBGROUPS LATTICE OF A NOMINAL GROUP

Nominal groups were introduced and studied in [6]. According to [6], a nominal group is a nominal set equipped with an equivariant internal group law.

Definition 4.1. A *nominal group* is a triple $(G, \cdot, \diamond)$ such that
- $(G, \cdot)$ is a group;
- $(G, \diamond)$ is a non-trivial nominal set;
- for each $\pi \in S_A$ and each $x, y \in G$, we have $\pi \diamond (x \cdot y) = (\pi \diamond x) \cdot (\pi \diamond y)$.

Example 4.2.
(1) $(S_A, \circ, \cdot)$ is a nominal group, where $\circ$ is the usual composition of permutations and $\cdot$ is the S_A-action on S_A defined as in Example 3.5(3). Since the composition law on S_A is associative, it is easily to verify that $\pi \cdot (\sigma \circ \tau) = (\pi \cdot \sigma) \circ (\pi \cdot \tau)$ for all $\pi, \sigma, \tau \in S_A$.
(2) If $(\Sigma, \cdot)$ is a nominal set, then $(\mathbb{Z}_{ext}(\Sigma), +, \star)$ is also a nominal group. Here $\mathbb{Z}_{ext}(\Sigma)$ is the set of all extended generalized multisets over Σ defined as in [5] (i.e. the set of all functions from Σ to $\mathbb{Z}$ which have a finite algebraic support), "+" is the usual pointwise sum of extended generalized multisets, and "$\star$" is the usual S_A-action on $\mathbb{Z}^{\Sigma}$ defined as in Example 3.5(4).
(3) According to Proposition 3.6 from [6], if $(\Sigma, \cdot)$ is a nominal set, then the free group over Σ is also a nominal group.
(4) If $(X, \cdot)$ is a nominal set such that all its elements are supported by the same finite set, then, according to Proposition 5.3 from [6], $(S_X, \circ, \star)$ is also a nominal group, where $S_X = \{f : X \to X \mid f\, bijective\}$, $\circ$ is the

usual composition of functions, and $\star$ is the S_A-action on X^X defined as in Example 3.5(4).

The following definition generalizes the nominal subgroup of a nominal group introduced in [6].

Definition 4.3. Let $(G, \cdot, \diamond)$ be a nominal group. A finitely supported subgroup of G is a subgroup of G which is finitely supported as an element of $\wp(G)$.

According to Definition 3.8 in [6], any nominal subgroup of a nominal group G is a finitely supported subgroup of G with empty support. Obviously, there may exist finitely supported subgroups of G which are not nominal subgroups of G in the sense of Definition 3.8 from [6].

If $(G, \cdot, \diamond)$ is a nominal group, we denote by $\mathcal{L}(G)_{nom}$ the family of all finitely supported subgroups of G ordered by inclusion.

LEMMA 4.4. *Let $(G, \cdot, \diamond)$ be a nominal group, and F a finitely supported subset of G. Then $[F]$ is a finitely supported subgroup of G.*

Proof. We claim that $[F]$ is supported by $supp(F)$. Indeed, let us consider $\pi \in Fix(supp(F))$, and $x_1^{\varepsilon_1} \cdot x_2^{\varepsilon_2} \cdot \ldots \cdot x_n^{\varepsilon_n}$, $x_i \in F$, $\varepsilon_i = \pm 1$, $i = 1, \ldots, n$ an arbitrary element of $[F]$. Since $\pi \in Fix(supp(F))$, we have $\pi \diamond x_i \in F$ for all $i \in \{1, \ldots, n\}$. Since the internal law on G is equivariant, we have $\pi \diamond (x_1^{\varepsilon_1} \cdot x_2^{\varepsilon_2} \cdot \ldots \cdot x_n^{\varepsilon_n}) = (\pi \diamond x_1^{\varepsilon_1}) \cdot (\pi \diamond x_2^{\varepsilon_2}) \cdot \ldots \cdot (\pi \diamond x_n^{\varepsilon_n}) = (\pi \diamond x_1)^{\varepsilon_1} \cdot (\pi \diamond x_2)^{\varepsilon_2} \cdot \ldots \cdot (\pi \diamond x_n)^{\varepsilon_n} \in [F]$. Thus, $\pi \star [F] = [F]$, where $\star$ is the S_A-action on $\wp(G)$ defined as in Example 3.5(4), and so $supp(F)$ supports $[F]$. □

COROLLARY 4.5. *Let $(G, \cdot, \diamond)$ be a nominal group, and F a finite subset of G. Then $[F]$ is a finitely supported subgroup of G.*

Proof. Let $F = \{x_1, \ldots x_n\}$ be a finite subset of G. Then $supp(x_1) \cup \ldots \cup supp(x_n)$ supports F. Thus, F is finitely supported, and the result follows from Lemma 4.4. □

THEOREM 4.6. *Let $(G, \cdot, \diamond)$ be a nominal group. Then $(\mathcal{L}(G)_{nom}, \subseteq, \star)$ is a nominal complete lattice, where $\subseteq$ represents the usual inclusion relation on $\wp(G)$, and $\star$ is the S_A-action on $\wp(G)$ defined as in Example 3.5(4).*

Proof. We know that $\star$ is the S_A-action on $\wp(G)$ defined as in Example 3.5(4). We claim that the restriction of $\star$ to $\mathcal{L}(G)_{nom}$ is an S_A-action on $\mathcal{L}(G)_{nom}$, that is the codomain of the restriction function $\star|_{\mathcal{L}(G)_{nom}}$ is also $\mathcal{L}(G)_{nom}$. We should prove that for any $\pi \in S_A$ we have that $\pi \star H$ is a finitely supported subgroup of G whenever H is a finitely supported subgroup of G. Fix some $\pi \in S_A$ and $H \leq G$, H finitely supported as a subset of G. Let

$\pi \diamond h_1$ and $\pi \diamond h_2$, $h_1, h_2 \in H$ be two arbitrary elements from $\pi \star H$. Since G is a nominal group (and so, $\cdot$ is equivariant) and because H is a subgroup of G we have $(\pi \diamond h_1) \cdot (\pi \diamond h_2)^{-1} = (\pi \diamond h_1) \cdot (\pi \diamond h_2^{-1}) = \pi \diamond (h_1 \cdot h_2^{-1}) \in \pi \star H$. Since H is finitely supported as an element of the S_A-set $\wp(G)$, according to Proposition 3.4 we have that $\pi \star H$ is a finitely supported element in $\wp(G)$. Therefore, because $\pi \star H$ is also a subgroup of G, we have that $\pi \star H$ is a finitely supported subgroup of G. Thus, $(\mathcal{L}(G)_{nom}, \subseteq, \star)$ is a nominal set. The order relation $\subseteq$ on $\wp(G)$ is obviously an equivariant lattice order according to the definition of $\star$, and so $(\mathcal{L}(G)_{nom}, \subseteq, \star)$ is a nominal lattice.

Let $\mathcal{F} = (H_i)_{i \in I}$ be a finitely supported family of finitely supported subgroups of G. We know that $\cup \mathcal{F} = \underset{i \in I}{\cup} H_i$ exists in G. We have to prove that $\underset{i \in I}{\cup} H_i \in \wp_{fs}(G)$. We claim that $supp(\mathcal{F})$ supports $\underset{i \in I}{\cup} H_i$. Let $\pi \in Fix(supp(\mathcal{F}))$, and $x \in \underset{i \in I}{\cup} H_i$. There exists $j \in I$ such that $x \in H_j$. Since $\pi \in Fix(supp(\mathcal{F}))$, we have $\pi \star H_j \in \mathcal{F}$, namely there exists $k \in I$ such that $\pi \star H_j = H_k$. Therefore, $\pi \diamond x \in \pi \star H_j = H_k$, and so $\pi \diamond x \in \underset{i \in I}{\cup} H_i$. We obtain $\pi \star \underset{i \in I}{\cup} H_i = \underset{i \in I}{\cup} H_i$, and so $\underset{i \in I}{\cup} H_i$ is finitely supported. According to Lemma 4.4, we get that $[\underset{i \in I}{\cup} H_i]$ (which is the least upper bound of $\mathcal{F}$) is a finitely supported subgroup of G.

We also know that $\cap \mathcal{F} = \underset{i \in I}{\cap} H_i$ exists in G, and have to prove that $\underset{i \in I}{\cap} H_i \in \wp_{fs}(G)$. We claim that $supp(\mathcal{F})$ supports $\underset{i \in I}{\cap} H_i$. Let $\pi \in Fix(supp(\mathcal{F}))$, and $x \in \underset{i \in I}{\cap} H_i$. Then $x \in H_i$ for all $i \in I$. We have to prove that $\pi \diamond x \in H_i$ for all $i \in I$. We consider an arbitrary $j \in I$. Since $\pi \in Fix(supp(\mathcal{F}))$, there exists $k \in I$ such that $H_j = \pi \star H_k$. However, because $k \in I$, we have $x \in H_k$. Therefore, $\pi \diamond x \in \pi \star H_k = H_j$. Since j has been arbitrary chosen from I, we obtain $\pi \diamond x \in H_i$ for all $i \in I$, and so $\pi \diamond x \in \underset{i \in I}{\cap} H_i$. We obtain $\pi \star \underset{i \in I}{\cap} H_i = \underset{i \in I}{\cap} H_i$. Thus, $\underset{i \in I}{\cap} H_i$ is finitely supported. □

Corollary 4.7. *If $(G, \cdot, \diamond)$ is a nominal group, then $(\mathcal{L}(G)_{nom}, \subseteq, \star)$ is a nominal dcpo.*

Considering Theorem 3.3 in [3] translated in terms of nominal lattices, we obtain the following Tarski-type result.

Corollary 4.8. *Let $(G, \cdot, \diamond)$ be a nominal group, and $f : \mathcal{L}(G)_{nom} \to \mathcal{L}(G)_{nom}$ be an equivariant, order-preserving function over $\mathcal{L}(G)_{nom}$. Then the set of all fixed points of the function f is a nominal complete lattice.*

Lemma 4.9. *If $(H_i)_{i \in I}$ is a finitely supported directed family of finitely supported subgroups of G, then $\underset{i \in I}{\cup} H_i$ is a finitely supported subgroup of G, and* *so* $[\bigcup_{i\in I} H_i] = \bigcup_{i\in I} H_i$.

Proof. According to the proof of Theorem 4.6, because $(H_i)_{i\in I}$ is a finitely supported family of finitely supported subgroups of G, we have that $\bigcup_{i\in I} H_i$ is finitely supported in $\wp(G)$. Now, let $x, y \in \bigcup_{i\in I} H_i$. There are $i, j \in I$ such that $x \in H_i$ and $y \in H_j$. Because of the directness of $(H_i)_{i\in I}$, there exists $k \in I$ such that H_k is an upper bound both of H_i and H_j. This means that $x, y \in H_k$, and so $xy^{-1} \in H_k \subseteq \bigcup_{i\in I} H_i$. It follows that $\bigcup_{i\in I} H_i$ is a finitely supported subgroup of G. □

Proposition 4.10. *Let $(G, \cdot, \diamond)$ be a nominal group. Each finitely generated subgroup of G is nominal compact in $(\mathcal{L}(G)_{nom}, \subseteq, \star)$.*

Proof. Let $H \leq G$ be a finitely generated subgroup of G. Then $H = [F]$, where F is a finite subset of G. According to Corollary 4.5, we have that H is a finitely supported subgroup of G; this means $H \in \mathcal{L}(G)_{nom}$. Let $(H_i)_{i\in I}$ be a finitely supported directed family of finitely supported subgroups of G with $H \subseteq [\bigcup_{i\in I} H_i]$. According to Lemma 4.9, we have $H \subseteq \bigcup_{i\in I} H_i$, and so $F \subseteq H \subseteq \bigcup_{i\in I} H_i$. However, if a finite set X is covered by a (finitely supported) directed collection $(X_i)_{i\in I}$ of sets, then X is always contained in some X_i. Therefore, there exists $j \in I$ such that $F \subseteq H_j$. However, $[F] = \bigcap_{\substack{H'\leq G\\ F\subseteq H'}} H'$, and so $[F] \subseteq H_j$. This means $H \ll_{nom} H$. □

Lemma 4.11. *Let $(G, \cdot, \diamond)$ be a nominal group, $\pi \in S_A$ and F a finite subset of G. Then $\pi \star [F] = [\pi \star F]$, where $\star$ is the S_A-action on $\wp(G)$ defined as in Example 3.5(4).*

Proof. According to Corollary 4.5, $[F]$ is a finitely supported subgroup of G, and so the statement of this lemma makes sense in the framework of nominal sets. Let $x \in [F]$. Then $x = x_1^{\varepsilon_1} \cdot x_2^{\varepsilon_2} \cdot \ldots \cdot x_n^{\varepsilon_n}$, $x_i \in F$, $\varepsilon_i = \pm 1$, $i = 1, \ldots, n$. Since the internal law on G is equivariant, we get $\pi \diamond x = \pi \diamond (x_1^{\varepsilon_1} \cdot x_2^{\varepsilon_2} \cdot \ldots \cdot x_n^{\varepsilon_n}) = (\pi \diamond x_1^{\varepsilon_1}) \cdot (\pi \diamond x_2^{\varepsilon_2}) \cdot \ldots \cdot (\pi \diamond x_n^{\varepsilon_n}) = (\pi \diamond x_1)^{\varepsilon_1} \cdot (\pi \diamond x_2)^{\varepsilon_2} \cdot \ldots \cdot (\pi \diamond x_n)^{\varepsilon_n} \in [\pi \star F]$. Thus, $\pi \star [F] \subseteq [\pi \star F]$. The reverse inclusion follows analogously. Therefore, $\pi \star [F] = [\pi \star F]$. □

Corollary 4.12. *Let $(G, \cdot, \diamond)$ be a nominal group, and $F(\mathcal{L}(G))$ be the set of all finitely generated subgroups of a group G. Then $F(\mathcal{L}(G)) \subseteq \mathcal{L}(G)_{nom}$, and $F(\mathcal{L}(G))$ is a nominal set.*

Proof. According to Corollary 4.5, every finitely generated subgroup of G is a finitely supported subgroup of G. Thus, $F(\mathcal{L}(G)) \subseteq \mathcal{L}(G)_{nom}$. In order to prove that $F(\mathcal{L}(G))$ is a nominal set, it remains to prove that $\pi \star [F]$ is finitely generated whenever $\pi \in S_A$ and $[F]$ is a finitely generated subgroup of G. According to Lemma 4.11, we have $\pi \star [F] = [\pi \star F]$. Since F is finite, then $\pi \star F$ is also finite, and so $\pi \star [F]$ is finitely generated. □

Proposition 4.13. *Let $(G, \cdot, \diamond)$ be a nominal group, and H be a finitely supported subgroup of G. Then the subgroups generated by the finite subsets of H form a finitely supported directed family. Moreover, H is the union of the subgroups generated by the finite subsets of H.*

Proof. Let $A_H = \{ [F] \mid F \subseteq H \text{ and } F \text{ is } finite\}$. We have to prove that A_H is a finitely supported directed family, and $H = \underset{H' \in A_H}{\cup} H'$. First we prove that A_H is finitely supported by claiming that $supp(H)$ supports A_H. Indeed, let us consider $\pi \in Fix(supp(H))$. Let F' be an arbitrary finite subset of H; according to Lemma 4.11, we have $\pi \star [F'] = [\pi \star F']$ where $\star$ is the S_A-action on $\wp(G)$ defined as in Example 3.5(4). However, $\pi \star H = H$ because $\pi \in Fix(supp(H))$. Since $F' \subseteq H$, from the definition of $\star$, we get $\pi \star F' \subseteq \pi \star H = H$. Obviously, $\pi \star F'$ is finite, and so $[\pi \star F'] \in A_H$. Thus, $\pi \star [F'] \in A_H$, namely $supp(H)$ supports A_H.

Let $[F_1] \in A_H$ and $[F_2] \in A_H$. Then $[F_1 \cup F_2] \in A_H$, and $[F_1] = \underset{\substack{H' \leq G \\ F_1 \subseteq H'}}{\cap} H'$. We know that $F_1 \subseteq [F_1 \cup F_2]$, and so $[F_1] \subseteq [F_1 \cup F_2]$. In a similar way, $[F_2] \subseteq [F_1 \cup F_2]$. Therefore, A_H is directed.

Now let $h \in H$. Then $h \in [\{h\}]$ and $[\{h\}] \in A_H$. Therefore, $H \subseteq \underset{H' \in A_H}{\cup} H'$. For the reverse inclusion, let F be a finite subset of H. Since $[F] = \underset{\substack{H' \leq G \\ F \subseteq H'}}{\cap} H'$, we get $[F] \subseteq H$ and $\underset{H' \in A_H}{\cup} H' \subseteq H$ □

Theorem 4.14. *Let $(G, \cdot, \diamond)$ be a nominal group. Then $(\mathcal{L}(G)_{nom}, \subseteq, \star)$ is a nominal continuous domain, and a nominal basis in $(\mathcal{L}(G)_{nom}, \subseteq, \star)$ is given by $F(\mathcal{L}(G))$.*

Proof. By Corollary 4.12, we get that $F(\mathcal{L}(G))$ is a nominal set. For every finitely supported H of G, we consider $A_H = \{ [F] \mid F \subseteq H \text{ and } F \text{ is } finite\}$. Clearly $A_H \subseteq F(\mathcal{L}(G))$. According to Proposition 4.13, we know that A_H is finitely supported and directed, and $H = \underset{H' \in A_H}{\cup} H'$. According to Proposition 4.10, we know that whenever $[F] \in A_H$ we have $[F] \ll_{nom} [F] \subseteq H$, and so (after a trivial calculation) $[F] \ll_{nom} H$. Using the definition of a nominal basis in a nominal dcpo (Definition 3.11), we get that $F(\mathcal{L}(G))$ is a nominal basis in $(\mathcal{L}(G)_{nom}, \subseteq, \star)$. □

Theorem 4.15. *Let $(G, \cdot, \diamond)$ be a nominal group. Then $(\mathcal{L}(G)_{nom}, \subseteq, \star)$ is a nominal algebraic domain. Moreover, the family of all nominal compact elements in $(\mathcal{L}(G)_{nom}, \subseteq, \star)$ is precisely $F(\mathcal{L}(G))$.*

Proof. Let H be a nominal compact element in $(\mathcal{L}(G)_{nom}, \subseteq, \star)$. Then $H \ll_{nom} H$. The set A_H defined as in the proof of Proposition 4.13 is finitely supported and directed, and $H = \underset{K \in A_H}{\cup} K$. Since $H \ll_{nom} H$, there exists $H' \in A_H$ such that $H \subseteq H'$. However, because $H' \in A_H$, there exists a finite set $F \subseteq H$ such that $H' = [F]$. Since $F \subseteq H$, we have $[F] \subseteq H$. Therefore, $H' \subseteq H$, and so $H' = H$. We obtain that $H \in F(\mathcal{L}(G))$.

Conversely, by Proposition 4.10, any finitely generated subgroup of G is nominal compact. Thus, a finitely supported subgroup of G is nominal compact if and only if it is finitely generated.

Finally, according to Theorem 4.14, we get that $F(\mathcal{L}(G))$ is a nominal basis in $(\mathcal{L}(G)_{nom}, \subseteq, \star)$. Thus. $(\mathcal{L}(G)_{nom}, \subseteq, \star)$ is a nominal algebraic domain. □

5. CONCLUSION

The theory of nominal sets represents the mathematical framework for modelling renaming binding or fresh names. Atoms have the same properties as variables and names. The precise nature of names is unimportant because we focus only on their ability to identify and on their distinctness. The finite support requirement is motivated by the fact that syntax can only ever involve finitely many names. The applications of this theory have roots in various areas of computer science as semantics [4], database theory [9], programming [13, 11, 22], proof theory [24], game theory [1], algebra [6, 18], logic [19], topology [18], and automata theory [12].

Nominal algebraic structures (which are algebraic structures defined in the framework of nominal sets) have already been considered and studied in computer science. In [9], the monoids defined in the category of nominal sets (also called nominal monoids) are used in the study of languages over infinite alphabets. The theory of syntactic monoids for languages of data words represents the same theory as the theory of finite monoids in the category of nominal sets, and under certain conditions, a language of data words is definable in first-order logic if and only if its syntactic monoid is aperiodic [10]. A nominal theory for partially ordered sets and domains was first developed in [22] in order to describe a denotational semantics for a functional programming language incorporating facilities for manipulating syntax involving names and binding operations. Nominal partially ordered sets have also been used in [23] in order to develop the original path-based domain theory for concurrency within nominal set theory, in [18] in order to present a duality theory in the nominal settings, and in [3] in order to study the event structures in the framework of nominal sets.

Nominal groups were defined in [6], and used in [5] to define and study generalized multisets over possible infinite alphabets. However, in [6] we studied only the equivariant (empty supported) subgroups of a nominal group. In this paper we consider the finitely supported subgroups of a nominal group which have a possibly non-empty support. We present some algebraic properties of such subgroups, and prove some results stating that the family of all finitely supported subgroups of a nominal group forms a nominal complete lattice and nominal algebraic domain.

The subgroups lattice of a group has been studied in the ZF framework in [2], and in an alternative set theory with atoms named Extended Fraenkel-Mostowski (EFM) set theory in [7]. However, we cannot conclude that any property from the ZF framework or the EFM framework can be directly reformulated in terms of nominal sets. This is because we cannot prove an result in the theory of nominal sets only involving a ZF (or an EFM) result without an additional proof made in terms of finitely supported objects. If we work in the nominal settings, then all of the proofs have to be rephrased in order to be consistent with the finite support requirement; all of our proofs are presented only by using finitely supported objects in order to be sure that we remain in the framework of nominal sets. Translating a classical algebraic structure into the framework of nominal sets is not trivial, because we cannot always obtain a nominal result corresponding to a ZF result only by replacing 'structure' with 'finitely supported structure'. This is because, given a nominal set X, there could exist some subsets of X (and also some relations or functions involving subsets of X) which fail to be finitely supported (see Example 3.5(4)). Some related examples are the nominal embedding theorems for groups presented in [6] and proved only for a particular class of nominal groups, as well as the Tarski-like theorem for nominal complete lattices [3] which is valid only for a particular class of finitely supported monotone functions over a nominal complete lattice. Another example of a mathematical result which fails in the framework of nominal sets is the Stone representation theorem for Boolean lattices (claiming that every Boolean lattice is isomorphic to the dual algebra of its associated Stone space). According [18], Stone duality fails in the framework of nominal sets because its proof would require a choice principle, namely the ultrafilter theorem, and this theorem is false in the framework of nominal sets (Proposition 5.2.2 from [18]) even it is a valid result in some models of ZF set theory without choice like Howard-Rubin's first model (N38 in [16]) or Cohen's first model (M1 in [16]). Moreover, we conjecture that all the ZF choice principles (weaker forms of the axiom of choice) fail in the framework of nominal sets; this will be the topic of a future work. Other results which fail in the nominal settings, such as determinization of finite automata and equivalence of two-way and one-way finite automata, are presented in [12]. For a complete list of such examples we recommend [8].

The results in this paper generalize the classical results in the ZF framework. Indeed, since every ZF-set can be represented as a nominal set with the discrete group action described in Example 3.5(2), the results of this paper can be particularized in order to obtain the classical properties of the subgroups lattice of a group presented in [2].

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