COMPLETE CONTROLLABILITY OF SEMILINEAR STOCHASTIC SYSTEMS WITH DELAY IN BOTH STATE AND CONTROL

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This paper deals with the complete controllability of semilinear stochastic systems with delay in both state and control under the assumption that the corresponding linear system is completely controllable. The control function for this system is suitably constructed by using the controllability operator. With this control function, the sufficient conditions for the complete controllability of the proposed problem in finite dimensional are established. The results are obtained by using Banach fixed point theorem. Finally, one example is provided to illustrate the application of the obtained results.

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Key words: complete controllability, semilinear systems, stochastic control system, reachable set, state and control delay.

1. INTRODUCTION

Controllability concepts play a vital role in deterministic control theory. It is well known that controllability of deterministic equation is widely used in many fields of science and technology. But in many practical problems such as fluctuating stock prices or physical system subject to thermal fluctuations, Population dynamics etc, some randomness appear, so the system should be medelled in a stochastic form.

In setting of deterministic systems: Kalman [22] introduced the concept of controllability for finite dimensional deterministic linear control systems. The basic concepts of control theory in finite dimensional spaces has been introduced in [23]. In [10] Naito established sufficient conditions for approximate controllability of deterministic semilinear control system dominated by the linear part using Schuder's fixed point theorem. Balachandran and J.P. Dauer [9] obtained results for controllability of nonlinear systems in Banach spaces. In [12, 13] L. Wang extended the results of [9] and established sufficient conditions for delayed deterministic semilinear systems using Schauder's fixed point theorem and concept of fundamental solution. In [18, 19] Sukavanam *et. al* obtained the results for approximate controllability of a delayed semilinear control system with growing nonlinear term using Schauder's fixed point theorem.

In setting of stochastic systems: A.E. Bashirov and K.R. Kerimov [1] introduced controllability concepts for stochastic systems. In [14–17] N.I. Mahmudov, S. Zorlu and N. Semi established sufficient conditions for controllability of linear and nonlinear stochastic systems using fixed point theorems. J. Klamka in [7, 8] obtained some results for controllablity of linear systems with delay in control as well as delay in state in finite dimensional using Rank theorem. L. Shen *et. al* [11] extended the results of [8] in infinite dimensional using technique of [14] and obtained sufficient conditions for Relative controllability of stochastic nonlinear systems with delay in control. A. Shukla *et. al* in [4] extended the results of [11] and obtained complete controllability of semilinear stochastic systems with multiple delays in control using Banach fixed point theorem. P. Muthukumar and P. Balasubramaniam [20] obtained the results for approximate controllability of mixed stochastic Volterra-Fredholm type integrodifferential systems in Hilbert space using Banach fixed point theorem. Recently A. Shukla, Urvashi Arora and N. Sukavanam established some sufficient conditions for Approximate controllability of retarded semilinear stochastic system with non local conditions in infinite dimensional space using Banach fixed point theorem. However in best of our knowledge, there is no result on simultaneously delays in both state and control terms for deterministic or stochastic system. So it is interesting to see for which control the system will be completely controllable and for what conditions fixed point theorem will work. The present paper is devoted to study of complete controllability of semilinear stochastic systems with delay in both state and control terms.

In this paper, we adopt the following notations:

(i) $(\Omega, \mathcal{F}, P)$: The triple $(\Omega, \mathcal{F}, P)$ is probability space of the n-dimensional Wiener process ω.

(ii) $\{\mathcal{F}_t | t \in [0, T]\}$: The filtration generated by $\{\omega(s) : 0 \le s \le t\}$, here ω is Wiener Process.

(iii) $L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n)$: The Hilbert space of all $\mathcal{F}_T$-measurable square integrable random variables with values in $\mathbb{R}^n$.

(iv) $L_2^{\mathcal{F}}([0, T], \mathbb{R}^n)$: The Hilbert space of all square-integrable and $\mathcal{F}_t$-measurable processes with values in $\mathbb{R}^n$

(v) H_2: The Banach space of all square integrable and $\mathcal{F}_t$-adapted processes $\varphi(t)$ with norm

$$||\varphi||^2 = \sup_{t \in [0,T]} E||\varphi(t)||^2, \quad \textit{where} \quad E \quad \textit{is} \quad \textit{the} \quad \textit{expected} \quad \textit{value}.$$

(vi) $\mathbb{L}(X, Y)$: The space of all linear bounded operators from a Banach space X into a Banach space Y.

(vii) $U_{ad} = L_2^F([0,T], \mathbb{R}^m)$: The set of admissible controls.

The problem of controllability of linear stochastic system with state delay.

$$\mathrm{d}x(t) = [A_0 x(t) + A_1 x(t-h) + B_0 u(t)]\mathrm{d}t + \sigma \mathrm{d}\omega(t) \tag{1.1}$$

given the initial condition as a random function

$$x_0 \in L_2^F([-h, 0], L_2(\Omega, F_T, \mathbb{R}^n))$$

has been studied by many authors (see J. Klamka [7] and the references therein).

The problem of controllability of linear stochastic system with control delay

$$\mathrm{d}x(t) = [A_0 x(t) + B_0 u(t) + B_1 u(t-h)]\mathrm{d}t + \sigma \mathrm{d}\omega(t) \tag{1.2}$$

given the initial condition as a random function

$$x(0) = x_0 \in L_2(\Omega, F_T, \mathbb{R}^n) \quad and \quad u(t) = 0 \quad for \quad t \in [-h, 0]$$

has been studied by various authors (see J.Klamka [8] and the references therein).

In this paper we examine the controllability of the following semi-linear stochastic system with delay in both state and control term:

$$\begin{aligned}\mathrm{d}x(t) = [A_0 x(t) + A_1 x(t-h) + B_0 u(t) + B_1 u(t-h) + f(t, x(t))]\mathrm{d}t \\ + \sigma(t, x(t))\mathrm{d}\omega(t)\end{aligned} \tag{1.3}$$

with initial conditions

$$x(t) = \psi(t),\ x(0) = \psi(0) = x_0\ (say) \quad and \quad u(t) = 0 \quad for \quad t \in [-h, 0] \tag{1.4}$$

where the state $x(t) \in L_2(\Omega, F_t, \mathbb{R}^n) = X$ and the control $u(t) \in \mathbb{R}^m = U$, A_0 and A_1 are an $n \times n$ constant matrices, B_0, B_1 are an $n \times m$ constant matrices. $\sigma : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n \times n}$, $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$, ω is a n-dimensional Wiener process and $h > 0$ is a constant point delay.

2. PRELIMINARIES

It is well known [14, 15] that for a given initial condition (1.4) and any admissible control $u \in U_{ad}$ and suitable nonlinear functions $f(t, x(t))$ and $\sigma(t, x(t))$ for $t \in [0, T]$ (satisfies Lipschitz continuity condition) there exists a unique solution $x(t;x_0,u)\in L_2(\Omega,\digamma_T,\mathbb{R}^n)$ of the semi-linear stochastic differential state equation (1.3) which can be represented in every time interval $t\in[kh,(k+1)h), k=0,1,2,...$ by the following integral equation:

$$\begin{aligned}(2.1)\quad x(t;x_0,u)&=x(kh;x_0,u)+\int_{kh}^{t}(A_0x(s;x_0,u)+A_1x(s-h;x_0,u))\mathrm{d}s\\&+\int_{kh}^{t}(B_0u(s)+B_1u(s-h)+f(s,x(s)))\mathrm{d}s+\int_{kh}^{t}\sigma(s,x(s))\mathrm{d}\omega(s)\end{aligned}$$

taking into account the above integral formula and using the well-known method of steps $x(t;x_0,0)$ is given as for $t\in[0,T]$:

$$x(t;x_0,0)=\exp(A_0t)x_0+\int_{-h}^{0}F(t-s-h)A_1x_0(s)\mathrm{d}s$$

or, equivalently

$$(2.2)\qquad x(t;x_0,0)=\exp(A_0t)x_0+\int_{0}^{h}F(t-s)A_1x_0(s-h)\mathrm{d}s$$

where $F(t)$ is the $n\times n$ dimensional matrix for the delayed state equation (1.3), which satisfies the matrix integral equation.

$$(2.3)\qquad F(t)=I+\int_0^t F(s)A_0\mathrm{d}s+\int_0^{t-h}F(s)A_1\mathrm{d}s$$

for $t>0$, with the initial conditions

$$F(0)=I,\quad F(t)=\exp(A_0t)\quad for\quad t\in[0,h),\qquad F(t)=0\quad for\quad t<0$$

Using above concepts, we obtain the implicit solution of the delayed system (1.3) as

(2.4)

$$x(t;x_0,u)=\begin{cases}x(t;x_0,0)+\int_0^t F(t-s)(B_0u(s)+B_1u(s-h)+f(s,x(s)))\mathrm{d}s\\ \qquad+\int_0^t F(t-s)\sigma(s,x(s))\mathrm{d}\omega(s)\quad for\quad t>0\\ \psi(t)\quad for\quad t\in[-h,0]\end{cases}$$

Now we recall some definitions, lemmas and evaluate some results which will used in further sections:

Lemma 2.1. *Gronwall's inequality: Let $a\in L_1[t_0,\tau]$, $a(t)\geq 0$ and b be an absolutely continuous function on $[t_0,\tau]$. If $x\in L_\infty[t_0,\tau]$ satisfies*

$$x(t)\leq b(t)+\int_{t_0}^{t}a(s)x(s)\mathrm{d}s$$

then

$$x(t) \leq b(t_0)\exp\bigg(\int_{t_0}^{t} a(s)\mathrm{d}s\bigg) + \int_{t_0}^{t} b'(s)\exp\bigg(\int_{s}^{t} a(\eta)\mathrm{d}\eta\bigg)\mathrm{d}s$$

LEMMA 2.2. *Let $G : [0,T]\times R^n \to R^{n\times n}$ be a strongly measurable mapping such that $\int_0^T E||G(t)||^p\mathrm{d}t < \infty$. Then*

$$E\bigg|\bigg|\int_0^t G(s)\mathrm{d}\omega(s)\bigg|\bigg|^p \leq L_G\int_0^t E||G(s)||^p\mathrm{d}s, \tag{2.5}$$

for all $t \in [0,T]$ and $p \geq 2$, where L_G is the constant involving p and T.

Definition. A control system is said to be completely controllable in the interval $I = [0,T]$ if for every initial state x_0 and desired final state x_1, there exists a control $u(t)$ such that the solution $x(t)$ of the system corresponding to this control u satisfies $x(T) = x_1$.

From equation (2.3) we have

$$\begin{aligned} ||F(t)|| &= ||I + \int_0^t F(s)A_0\mathrm{d}s + \int_0^{t-h} F(s)A_1\mathrm{d}s|| \\ &\leq 1 + \int_0^t (||A_0|| + ||A_1||)||F(s)||\mathrm{d}s \\ &\leq \exp(t(||A_0|| + ||A_1||)) \quad \text{(using Gronwall's inequlity)} \end{aligned} \tag{2.6}$$

let $l_1 = \max(||F(t)||^2) \quad in \quad t \in [0,T]$
From equation (2.2) we have

$$\begin{aligned} E||x(t;x_0,0)||^2 &= E||\exp(A_0 t)x_0 + \int_0^h F(t-s)A_1 x_0(s-h)\mathrm{d}s||^2 \\ &\leq 2\bigg(E||\exp(A_0 t)x_0||^2 + E||\int_0^h F(t-s)A_1 x_0(s-h)\mathrm{d}s||^2\bigg) \\ &\leq 2\bigg(l_1||x_0||^2 + E\bigg(\int_0^h ||F(t-s)||^2||A_1||^2||x_0(s-h)||^2\mathrm{d}s\bigg)\bigg) \\ &\leq 2(l_1||x_0||^2 + ||A_1||^2||\psi(t)||^2 l_1). \end{aligned} \tag{2.7}$$

3. MAIN RESULTS

Now, for a given final time $T > h$, taking into account the form of the integral solution $x(t;x_0,u)$, let us introduce the following operators and sets.

Define the bounded linear operator $L_T : L_2([0,T],\mathbb{R}^m) \to L_2(\Omega,\mathcal{F}_T,\mathbb{R}^n)$ by

$$L_T u = \int_0^h \exp(A_0(T-s))B_0 u(s)\mathrm{d}s + \int_h^T F(T-s)B_0 u(s)\mathrm{d}s + \int_0^{T-h} F(T-s-h)B_1 u(s)\mathrm{d}s$$

Its adjoint bounded linear operator $L_T^* : L_2(\Omega, \digamma_T, \mathbb{R}^n) \to L_2([0,T], \mathbb{R}^m)$ has the following form:

$$L_T^* z(t) = \begin{cases} (B_0^* \exp(A_0^*(T-t)) + B_0^* F^*(T-t) + B_1^* F^*(T-t-h))E\{z|\digamma_t\} \\ \qquad \text{for} \quad t \in [h, T] \\ B_0^* \exp(A_0^*(T-t))E\{z|\digamma_T\} \quad \text{for} \quad t \in [0, h) \end{cases}$$

Define the set of all the states reachable in the final time T from a given initial state $x_0 \in L_2([-h, 0], \mathbb{R}^n)$, using a set of admissible controls, as follows

$$R_T(U_{ad}) \quad = \quad \{x(T; x_0, u) \in L_2(\Omega, \digamma_T, \mathbb{R}^n) : u \in U_{ad}\}$$

Now, we introduce the linear controllability operator

$$\Pi_0^T \in \mathbb{L}(L_2(\Omega, \digamma_T, \mathbb{R}^n), L_2(\Omega, \digamma_T, \mathbb{R}^n)),$$

which is strongly associated with the control operator L_T and is given the following equality:

$$\Pi_0^T\{.\} = L_T L_T^*\{.\}$$

$$= \begin{cases} \displaystyle\int_0^T \exp(A_0(T-t))B_0 B_0^* \exp(A_0^*(T-t))E\{.|\digamma_t\}\mathrm{d}t \quad for \quad T \leq h \\ \displaystyle\int_0^h \exp(A_0(T-t))B_0 B_0^* \exp(A_0^*(T-t))E\{.|\digamma_t\}\mathrm{d}t \\ \qquad \displaystyle + \int_h^T (F(T-t)B_0 B_0^* F^*(T-t))E\{.|\digamma_t\}\mathrm{d}t \\ \qquad \displaystyle + \int_0^{T-h} (F(T-t-h)B_1 B_1^* F^*(T-t-h))E\{.|\digamma_t\}\mathrm{d}t \quad for \quad T > h \end{cases}$$

Let us recall that the $n \times n$ deterministic controllability matrix is given by

$$\Gamma_s^T \quad = \quad L_T(s)L_T^*(s)$$

$$= \begin{cases} \displaystyle\int_s^T \exp(A_0(T-t))B_0 B_0^* \exp(A_0^*(T-t))\mathrm{d}t \quad for \quad T \leq h \\ \displaystyle\int_s^h \exp(A_0(T-t))B_0 B_0^* \exp(A_0^*(T-t))\mathrm{d}t \\ \qquad \displaystyle + \int_h^T F(T-t)B_0 B_0^* F^*(T-t)\mathrm{d}t \\ \qquad \displaystyle + \int_s^{T-h} F(T-t-h)B_1 B_1^* F^*(T-t-h)\mathrm{d}t \quad for \quad T > h \end{cases}$$

LEMMA 3.1. *Assume that the operator* (Π_0^T) *is invertible. Then for arbitrary* $x_T \in L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n)$, $f(\cdot) \in L_2([0,T], \mathbb{R}^n)$,$\sigma(\cdot) \in L_2([0,T], \mathbb{R}^{n\times n}$, *the control defined as:*

$$(3.1)\quad u(t) =$$

$$\begin{cases} (B_0^* F^*(T-t) + B_1^* F^*(T-t-h)) \times E\Big\{ (\Pi_0^T)^{-1}\Big(x_T - x(T;x_0,0) \\ -\displaystyle\int_h^T F(T-s)(f(s,x(s))\mathrm{d}s + \sigma(s,x(s))\mathrm{d}\omega(s)\Big) |\mathcal{F}_t\Big\} \quad \text{for} \quad t\in[h,T] \\ B_0^* \exp(A_0^*(T-t)) \times E\Big\{ (\Pi_0^T)^{-1}\Big(x_T - x(T;x_0,0) \\ -\displaystyle\int_0^h \exp(A(T-s))(f(s,x(s))\mathrm{d}s + \sigma(s,x(s))\mathrm{d}\omega(s))\Big) |\mathcal{F}_t\Big\} \quad \text{for} \quad t\in[0,h) \end{cases}$$

transfers the system (2.4) *from* $x_0 \in \mathbb{R}^n$ *to* x_T *at time* T *and*

$$(3.2)\quad x(t;x_0,u) = x(t;x_0,0) + \Pi_0^t \Big[F^*(T-t)(\Pi_0^T)^{-1}\Big(x_T - x(T;x_0,0) - \int_0^T F(T-s)f(s,x(s))\mathrm{d}s - \int_0^T F(T-s)\sigma(s,x(s))\mathrm{d}\omega(s)\Big)\Big] + \int_0^t F(t-s)f(s,x(s))\mathrm{d}s + \int_0^t F(t-s)\sigma(s,x(s))\mathrm{d}\omega(s)$$

provided the solution of (3.2) *exists.*

Proof. By substituting (3.1) in (2.4),we can easily obtain the following (see [7, 14])

For $T < h$

$$x(t;x_0,u) = x(t;x_0,0) + \int_0^t \exp(A_0(t-s))B_0 B_0^* \exp(A_0^*(t-s)) \times E\Big\{ (\Pi_0^T)^{-1}\Big(x_T - x(T;x_0,0) - \int_0^{T-h} F(T-s)(f(s,x(s))\mathrm{d}s + \sigma(s,x(s))\mathrm{d}\omega(s))\Big) |\mathcal{F}_s\Big\}\mathrm{d}s + \int_0^t F(t-s)f(s,x(s))\mathrm{d}s + \int_0^t F(t-s)\sigma(s,x(s))\mathrm{d}\omega(s).$$

In the same manner for $T \geq h$

$$x(t;x_0,u) = x(t;x_0,0) + \int_0^h (\exp(A_0(t-s))B_0 B_0^* \exp(A_0^*(T-s))$$

$$\times E\Big\{(\Pi_0^T)^{-1}\Big(x_T - x(T;x_0,0) - \int_0^h F(T-s)(f(s,x(s))\mathrm{d}s$$
$$+ \sigma(s)\mathrm{d}\omega(s,x(s)))\Big)|\digamma_s\Big\}\mathrm{d}s + \Big(\int_h^t F(t-s)B_0B_0^*F^*(T-s)$$
$$+ \int_0^{t-h} F(t-s-h)B_1B_1^*F(T-s-h)\Big) \times E\Big\{\Big((\Pi_0^T)^{-1}(x_T - x(T;x_0,0)$$
$$- \int_h^T (F(T-s)(f(s,x(s))\mathrm{d}s + \sigma(s,x(s))\mathrm{d}\omega(s))\Big)|\digamma_s\Big\}\mathrm{d}s$$
$$+ \int_0^t F(t-s)f(s,x(s))\mathrm{d}s + \int_0^t F(t-s)\sigma(s,x(s))\mathrm{d}\omega(s)$$

Thus,taking into account of the form of the operator Π_0^T we have

$$x(t;x_0,u) = x(t;x_0,0) + \Pi_0^t\Big[F^*(T-t)(\Pi_0^T)^{-1}\Big(x_T - x(T;x_0,0)$$
$$- \int_0^T F(T-s)(f(s,x(s))\mathrm{d}s + \sigma(s,x(s)))\mathrm{d}\omega(s)\Big)\Big]$$
$$+ \int_0^t F(t-s)f(s,x(s))\mathrm{d}s + \int_0^t F(t-s)\sigma(s,x(s))\mathrm{d}\omega(s)$$

Put $t = T$ in above equation we get

$$\begin{aligned} x(T;x_0,u) &= x(T;x_0,0) + \Pi_0^T\Big[F^*(T-T)(\Pi_0^T)^{-1}\Big(x_T - x(T;x_0,0) \\ &\quad - \int_0^T F(T-s)(f(s,x(s))\mathrm{d}s + \sigma(s,x(s)))\mathrm{d}\omega(s)\Big)\Big] \\ &\quad + \int_0^T F(T-s)f(s,x(s))\mathrm{d}s + \int_0^T F(T-s)\sigma(s,x(s))\mathrm{d}\omega(s) \\ x(T;x_0,u) &= x_T \quad \Box \end{aligned}$$

Remark. In THEOREM 3.3 sufficient condition are given for the existence and uniqueness of solution of (3.2).

LEMMA 3.2 (see [21]). *For every $z \in L_2(\Omega, \digamma_T, R^n)$, there exists a process $\varphi(.) \in L_2([0,T], R^{n\times n})$ such that*

$$z = \boldsymbol{E}z + \int_0^T \varphi(s)\mathrm{d}\omega(s)$$

$$\Pi_0^T z = \Gamma_0^T \boldsymbol{E}z + \int_0^T \Gamma_s^T \varphi(s)\mathrm{d}\omega(s)$$

Moreover

$$\begin{aligned} E\|\Pi_0^T z\|^2 &\leq ME\|E\{z|\digamma_T\}\|^2 \\ &\leq ME\|z\|^2, \quad z \in L_2(\Omega, \digamma_T, R^n) \end{aligned}$$

Note that if the assumption ($A3$) *holds, then for some* $\gamma > 0$

$$E\langle \Pi_0^T z, z\rangle \geq \gamma E\|z\|^2, \quad for \quad all \quad z \in L_2(\Omega, \digamma_T, R^n)$$

consequently

$$E\|(\Pi_0^T)^{-1}\|^2 \leq \frac{1}{\gamma} = l_4.$$

Now let us assume the following conditions

($A1$) (f, σ) satisfies the Lipschitz condition with respect to x *i.e.*,

$$\|f(t, x_1) - f(t, x_2)\|^2 \leq L_1\|x_1 - x_2\|^2, \quad \|\sigma(t, x_1) - \sigma(t, x_2)\|^2 \leq L_2\|x_1 - x_2\|^2$$

($A2$) (f, σ) is continuous on $[0, T] \times R^n$ and satisfies

$$\|f(t, x)\|^2 \leq L_3(\|x\|^2 + 1), \|\sigma(t, x)\|^2 \leq L_4(\|x\|^2 + 1)$$

($A3$) The linear systems (1.1) and (1.2) are completely controllable.

To apply the Banach fixed point theorem, define the operator $\mathbf{S}$ for (2.4) for $t \in [-h, T]$ as follows

$$\mathbf{S}(x)(t) = \begin{cases} \psi(t) \quad for \quad t \in [-h, 0] \\ x(t; x_0, 0) + \Pi_0^t[F^*(T-t)((\Pi_0^T)^{-1}(x_T - x(T; x_0, 0) \\ -\displaystyle\int_0^T F(T-r)f(r, x(r))\mathrm{d}r - \int_0^T F(T-r)\sigma(r, x(r))\mathrm{d}\omega(r))] \\ +\displaystyle\int_0^t F(t-s)f(s, x(s))\mathrm{d}s + \int_0^t F(t-s)\sigma(s, x(s))\mathrm{d}\omega(s) \quad \text{for} \quad t \in (0, T] \end{cases}$$

From LEMMA 3.1, the control $u(t)$ transfer the system (2.4) from the initial state x_0 to the final state x_T provided that the operator $\mathbf{S}$ has a fixed point. So, if the operator $\mathbf{S}$ has a fixed point then the system (1.3) is completely controllable.

Now for convenience, let us introduce the notation

$$l_1 = \max\|F(t)\|^2 : t \in [0, T], \qquad l_2 = \max(\|B_0\|^2, \|B_1\|^2)$$
$$l_3 = E\|x_T\|^2, \qquad M = \max\|\Gamma_s^T\|^2 : s \in [0, T]$$

THEOREM 3.3. *Assume that the conditions* ($A1$),($A2$) *and* ($A3$) *hold.In addition if the inequality*

$$4l_1(Ml_1l_4 + 1)(L_1T + L_2L_\sigma)T < 1 \tag{3.3}$$

holds, then the system (1.3) *is completely controllable.*

Proof. As mentioned above, to prove the complete controllability it is enough to show that $\mathbf{S}$ has a fixed point in H_2. To do this, we use the contraction mapping principle. To apply the contraction mapping principle, first we show that $\mathbf{S}$ maps H_2 into itself. Now by LEMMA 3.1 and equations (2.6) and (2.7) we have

$$
\begin{aligned}
E||(\mathbf{S}x)(t)||^2 = E\Bigg|\Bigg|\psi(t) + x(t;x_0,0) + \Pi_0^t[F^*(T-t)\times(\Pi_0^T)^{-1}(x_T - x(T;x_0,0)\\
- \int_0^T F(T-r)f(r,x(r))\mathrm{d}r - \int_0^T F(T-r)\sigma(r,x(r))\mathrm{d}\omega(r))]\\
+ \int_0^t F(t-s)f(s,x(s))\mathrm{d}s + \int_0^t F(t-s)\sigma(s,x(s))\mathrm{d}\omega(s)\Bigg|\Bigg|^2\\
\leq 5||\psi||^2 + 5(2(l_1||x_0||^2 + ||A_1||^2||\psi(t)||^2 l_1)\\
+ 5E\Bigg|\Bigg|\Pi_0^t[F^*(T-t)\times(\Pi_0^T)^{-1}(x_T - x(T;x_0,0)\\
- \int_0^T F(T-r)f(r,x(r))\mathrm{d}r - \int_0^T F(T-r)\sigma(r,x(r))\mathrm{d}\omega(r))]\Bigg|\Bigg|^2\\
+ 5t\int_0^t ||F(t-r)||^2 E||f(r,x(r)||^2\mathrm{d}r + 5\int_0^t ||F(t-r)||^2 E||\sigma(r,x(r))||^2\mathrm{d}r\\
\leq 5||\psi||^2 + 10l_1||x_0||^2 + 5(||A_1||^2||\phi(t)||^2 l_1)\\
+ 20Ml_1l_4(l_3 + 2(l_1||x_0||^2 + ||A_1||^2||\psi(t)||^2 l_1)\\
+ Tl_1\int_0^T E||f(r,x(r))||^2\mathrm{d}r + l_1L_\sigma\int_0^T E||\sigma(r,x(r))||^2\mathrm{d}r)\\
+ 5l_1\int_0^t (TE||f(r,x(r))||^2 + L_\sigma E||\sigma(r,x(r))||^2)\mathrm{d}r\\
\leq B_1 + B_2(\int_0^t (TE||f(r,x(r))||^2 + L_\sigma E||\sigma(r,x(r))||^2)\mathrm{d}r)
\end{aligned}
$$

where $B_1 > 0$ and $B_2 > 0$ are suitable constants. It follows from the above and the condition $(A2)$ that there exists $C_1 > 0$ such that

$$
\begin{aligned}
E||(Sx)(t)||^2 &\leq C_1(1 + \int_0^T E||x(r)||^2\mathrm{d}r)\\
\sup_{t\in[0,T]} E||(Sx)(t)||^2 &\leq C_1(1 + T\sup_{0\leq r\leq T} E||x(r)||^2)
\end{aligned}
$$

for all $t \in [-h,T]$. As $\sup\limits_{t\in[-h,T]} E||(Sx)(t)||^2 < \infty$, therefore $\mathbf{S}$ maps H_2 into itself.

Secondly, we show that $\mathbf{S}$ is a contraction mapping. Indeed

$$
\begin{aligned}
&E\Big|\Big|(\mathbf{S}x_1)(t)-(\mathbf{S}x_2)(t)\Big|\Big|^2\\
&\quad= E\Big|\Big|\Pi_0^t[F^*(T-t)(\Pi_0^T)^{-1}(\int_0^T F(T-s)(f(s,x_2(s))-f(s,x_1(s)))\mathrm{d}s\\
&\qquad\quad+\int_0^T F(T-s)(\sigma(s,x_2(s))-\sigma(s,x_1(s)))\mathrm{d}\omega(s))]\\
&\qquad\qquad+\int_0^t F(t-s)(f(s,x_1(s))-f(s,x_2(s)))\mathrm{d}s\\
&\qquad\quad+\int_0^t F(t-s)(\sigma(s,x_2(s))-\sigma(s,x_1(s)))\mathrm{d}\omega(s)\Big|\Big|^2\\
&\qquad\le 4Ml_1^2l_4\Big(T\int_0^T E\Big|\Big|f(s,x_1(s))-f(s,x_2(s))\Big|\Big|^2\mathrm{d}s\\
&\qquad\quad+L_\sigma\int_0^T E\Big|\Big|\sigma(s,x_1(s))-\sigma(s,x_2(s))\Big|\Big|^2\mathrm{d}s\Big)\\
&\qquad\quad+4l_1(T\int_0^t E\Big|\Big|f(s,x_1(s))-f(s,x_2(s))\Big|\Big|^2\mathrm{d}s\\
&\qquad\quad+L_\sigma\int_0^t E\Big|\Big|\sigma(s,x_1(s))-\sigma(s,x_2(s))\Big|\Big|^2\mathrm{d}s)\\
&\qquad= 4Ml_1^2l_4(L_1T+L_2L_\sigma)\int_0^T E\Big|\Big|x_1(s)-x_2(s)\Big|\Big|^2\mathrm{d}s\\
&\qquad\quad+4l_1(L_1T+L_2L_\sigma)\int_0^t E\Big|\Big|x_1(s)-x_2(s)\Big|\Big|^2\mathrm{d}s\\
&\qquad\qquad\le 4l_1(Ml_1l_4+1)(L_1T+L_2L_\sigma)\int_0^T E\Big|\Big|x_1(s)-x_2(s)\Big|\Big|^2\mathrm{d}s
\end{aligned}
$$

It results that

$$
\begin{aligned}
&\sup_{t\in[-h,T]} E\Big|\Big|(\mathbf{S}x_1)(t)-(\mathbf{S}x_2)(t)\Big|\Big|^2\\
&\qquad\le 4l_1(Ml_1l_4+1)(L_1T+L_2L_\sigma)T\sup_{t\in[-h,T]} E||x_1(t)-x_2(t)||^2
\end{aligned}
$$

Therefore $\mathbf{S}$ is a contraction mapping if the inequality (3.3) holds. Then the mapping $\mathbf{S}$ has a unique fixed point $x(\cdot)$ in H_2 which is the solution of the system (1.3). Thus, the system (1.3) is completely controllable. The theorem is proved. $\square$

4. EXAMPLE

Consider a two-dimensional semi-linear stochastic system with delay in state and control terms

$$\text{(4.1)}\quad \mathrm{d}x(t) = [A_0x(t) + A_1x(t-h) + B_0u(t) + B_1u(t-h) + f(t, x(t))]\mathrm{d}t \\ + \sigma(t, x(t))\mathrm{d}\omega(t); \quad t \in [0, T]$$

with initial condition (1.4)
where $\omega(t)$ is a one dimensional Wiener process and

$$A_0 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}, \; A_1 = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}, \; B_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$f(t, x(t)) = \frac{1}{a}\begin{bmatrix} \sin x(t) \\ x(t) \end{bmatrix}, \; \sigma(t, x(t)) = \frac{1}{b}\begin{bmatrix} x(t) & 0 \\ 0 & \cos x(t) \end{bmatrix}$$

If we take Euclidean norm then

$$\left|\left|f(t, x_1(t) - f(t, x_2(t))\right|\right|^2 \le \frac{2}{a^2}||x_1(t) - x_2(t)||^2 \quad \text{and}$$

$$\left|\left|\sigma(t, x_1(t) - \sigma(t, x_2(t))\right|\right|^2 \le \frac{2}{b^2}||x_1(t) - x_2(t)||^2 \quad \text{so,}$$

$$\text{(4.2)}\qquad \text{where} \quad L_1 = \frac{2}{a^2}, \quad L_2 = \frac{2}{b^2}$$

$$||A_0|| = 2, ||A_1|| = \sqrt{3}, ||B_0|| = \sqrt{2}, ||B_1|| = \sqrt{2}$$

We can see that conditions of THEOREM 3.3 with the help of equation (3.1), definition of M and LEMMA 2.2 for sufficiently small L_1, L_2, L_σ (*using equation* (4.2)) for any time T are satisfied. So system (4.1) is completely controllable.

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