UNIQUE INTEGRAL REPRESENTATION FOR THE CLASS OF BALANCED SEPARATELY SUPERHARMONIC FUNCTIONS IN A PRODUCT NETWORK

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Communicated by Lucian Beznea

In an infinite network $X$, the extremal elements for a base of the cone of positive superharmonic functions in $X$ are determined and an integral representation for this cone of functions is given by using the Choquet integral representation theorem. Later, a similar representation is given to the class of non-negative separately superharmonic functions in a product network $X \times Y$, but without proving the uniqueness of the representing measures. However if we restrict to a subclass of non-negative separately superharmonic functions, called here the balanced functions, then the representing measure is unique.

AMS 2010 Subject Classification: 31C20, 31C10, 32U05.

Key words: product networks, separately superharmonic functions, mean-value property, balanced separately superharmonic functions, integral representations.

1. INTRODUCTION

In an infinite tree $T$, defining the Martin boundary, Cartier [8] obtains an integral representation for positive harmonic functions in $T$. In this note, we consider first the integral representation for positive superharmonic functions in a single infinite network $X$ by using the Choquet theorem on integral representations. Then we consider the integral representation of positive separately superharmonic functions in a product $X \times Y$ of two infinite networks $X$ and $Y$. In this general case, each non-negative separately superharmonic function can be expressed as an integral with respect to a Radon measure supported by extremal elements. However, the uniqueness of this measure cannot be guaranteed. We introduce a subclass $\mathcal{B}$ of separately superharmonic functions in $X \times Y$ called balanced functions which have a certain mean-value property. This is analogous to the case of functions in a network $X$ with a mean-value property which we call harmonic functions in $X$. We prove that every element $u(x, y)$ in $\mathcal{B}^+$, consisting of non-negative elements of $\mathcal{B}$, is the unique sum of three functions, one separately harmonic in $X \times Y$, MATH. REPORTS 18(68), 3 (2016), 299–313
one harmonic in the first variable and a potential in the second variable and one potential in the first variable and harmonic in the second variable. Finally, we prove if we restrict to this subclass $\mathcal{B}^+$ then the representing measure is unique.

2. PRELIMINARIES

$X$ is an infinite graph with countably infinite vertices and countably infinite edges. We say that $x$ and $y$ are neighbours, and write $x \sim y$, if and only if there is an edge joining $x$ and $y$. Assume $X$ is locally finite (that is, every vertex in $X$ has only a finite number of neighbours), connected (that is, any two vertices $x$ and $y$ can be joined by a path $\{x = x_0, x_1, \ldots, x_n = y\}$ and there are no self loops (that is, there is no edge from one vertex to itself). We refer to such a graph $X$ as an infinite network if there is a transition index $t(x,y)$ associated with each pair of vertices $x$ and $y$ in $X$ satisfying the conditions:

- $t(x,y)$ is a non-negative real number,
- $t(x,y) > 0$ if and only if $x \sim y$, and $t(x,y)$ and $t(y,x)$ need not be the same. Then for any vertex $x$ in $X$, $t(x) = \sum_{x \sim y} t(x,y)$ is always a positive real number.

Let $u$ be a real-valued function on $X$. Then the Laplacian of $u$ is defined by

$$\Delta u(x) = \sum_{x \sim y} t(x,y)[u(y) - u(x)].$$

The function $u$ is said to be superharmonic on $X$ if $t(x)u(x) \geq \sum_{x \sim y} t(x,y)u(y))$ for every $x \in X$; harmonic and subharmonic functions are defined accordingly. A superharmonic function $p \geq 0$ on $X$ is said to be a potential if $u$ subharmonic on $X$ and $u \leq p$ imply $u \leq 0$. If there exists a positive potential on $X$, then $X$ is called a hyperbolic network; otherwise $X$ is called a parabolic network.

Discretising the notion of doubly subharmonic functions considered by Avanissian [6] in the context of several complex variables, we consider now doubly superharmonic functions in product networks.

Let $\{X, t_1\}$ and $\{Y, t_2\}$ be two infinite networks with Laplacians $\Delta_1$ and $\Delta_2$, respectively. Define their product as $\{X \times Y, t\}$ such that the neighbours of $(x,y)$ are $(x_i, y)$ and $(x,y_j)$ where $x \sim x_i$ in $X$ and $y \sim y_j$ in $Y$. Take $t\{(x,y), (x_i, y)\} = t_1(x, x_i)$ and $t\{(x,y), (x, y_j)\} = t_2(y, y_j)$. Then $X \times Y$ with transition index $t$ becomes an infinite network. If $f(x,y)$ is defined on $X \times Y$, then for $(x,y) \in X \times Y$ define

$$\Delta f(x,y) = \sum_{(a,b) \sim (x,y)} t\{(x,y), (a,b)\}[f(a, b) - f(x,y)].$$
Let us denote by $f_y(x)$ the function $f(x, y)$ when $y$ is fixed and by $f^x(y)$ the function $f(x, y)$ when $x$ is fixed. Then,

$$\Delta f(x, y) = \sum_{(x_i, y) \sim (x, y)} \left[ f(x_i, y) - f(x, y) \right] + \sum_{(x, y_j) \sim (x, y)} \left[ f(x, y_j) - f(x, y) \right]$$

$$= \sum_{x_i \sim x} t_1(x_i, x) - f(x, y) + \sum_{y_j \sim y} t_2(y, y_j) - f^x(y)$$

$$= \Delta_1 f_y(x) + \Delta_2 f^x(y).$$

We say that $f(x, y)$ is superharmonic (respectively harmonic) in $X \times Y$ if and only if $\Delta f(x, y) \leq 0$ (respectively $\Delta f(x, y) = 0$) for every $(x, y) \in X \times Y$. A function $f(x, y)$ is said to be separately superharmonic in $X \times Y$, if for any fixed $y$, $f_y(x) = f(x, y)$ is superharmonic in $X$ and for any fixed $x$, $f^x(y) = f(x, y)$ is superharmonic in $Y$.

**Properties of separately superharmonic functions:**

1. A separately superharmonic (respectively separately harmonic) function in $X \times Y$ is $\Delta-$superharmonic (respectively $\Delta-$harmonic) in $X \times Y$.

   **Proof.** If $(x, y) \in X \times Y$, then by using the definition of separately superharmonic functions in $X \times Y$, we see that $\Delta f(x, y) = \Delta_1 f_y(x) + \Delta_2 f^x(y) \leq 0$. Similarly for separately harmonic functions. □

The converse need not be valid: for example, let $X$ and $Y$ be trees without terminal vertices. Then we can choose [2, Theorem 5.1.4.] $u(x)$ in $X$ and $v(y)$ in $Y$ such that $\Delta_1 u(x) = 1$ and $\Delta_2 v(y) = 3$. Let $s(x, y) = u(x) - v(y)$ in $X \times Y$. Then $s(x, y)$ is $\Delta-$superharmonic but not separately superharmonic. Similarly, a harmonic function need not be separately harmonic. However, if a harmonic function is separately superharmonic, then it is separately harmonic.

2. If $u, v$ are separately superharmonic in $X \times Y$, then for non-negative numbers $\alpha, \beta$ we see that $\alpha u + \beta v$ and $\inf(u, v)$ are separately superharmonic in $X \times Y$.

3. If $\{u_n\}$ is a sequence of separately superharmonic functions in $X \times Y$ such that $u(x, y) = \lim_{n \to \infty} u_n(x, y)$ is finite for each $(x, y)$ in $X \times Y$, then $u(x, y)$ is separately superharmonic in $X \times Y$.

   **Proof.** For any fixed $x \in X$, $u^x_n(y)$ is superharmonic in $Y$. We know that the limit of superharmonic functions is superharmonic if the limit is finite [2, p.46]. Hence, the fact that $u^x_n(y) \to u^x(y)$ in $Y$, for every
fixed } x \in X, \text{ implies that } u^x(y) \text{ is superharmonic in } Y. \text{ Similarly, } u_y(x) \text{ is superharmonic in } X. \text{ Hence, } u(x, y) \text{ is separately superharmonic in } X \times Y. \quad \square

(4) If } u \geq 0 \text{ is } \Delta_1-\text{superharmonic in } X, \text{ and } v \geq 0 \text{ is } \Delta_2-\text{superharmonic in } Y, \text{ then } s(x, y) = u(x)v(y) \text{ is a separately superharmonic function in } X \times Y.

Let } E \text{ denote a finite set in } X \text{ and } \hat{E} \text{ the interior of } E. \text{ Then, for each } \alpha_i \in \partial E = E \setminus \hat{E}, \text{ there exists a unique function (Poisson Kernel [1, Theorem 11]) } P_E(x, \alpha_i) \text{ defined on } E \text{ such that } \Delta_1 P_E(x, \alpha_i) = 0 \text{ for each } x \in \hat{E} \text{ and } P_E(\alpha_k, \alpha_i) = \delta(\alpha_k, \alpha_i). \text{ Note that if } \phi(x) \text{ is a function defined on the set } \partial E = \{\alpha_i\}, \text{ then } h(x) = \sum_i \phi(\alpha_i) P_E(x, \alpha_i) \text{ is defined on } E \text{ such that } h(\alpha_i) = \phi(\alpha_i)

for each } \alpha_i \text{ and } \Delta_1 h(x) = 0 \text{ for } x \in \hat{E}. \text{ Similarly define } P_F(y, \beta_j) \text{ on a finite subset } F \text{ in } Y \text{ for which } \partial F = \{\beta_j\}. \text{ Note that if } h(x) \text{ is harmonic in } E \text{ then } h(x) = \sum_i h(\alpha_i) P_E(x, \alpha_i).

3. INTEGRAL REPRESENTATION OF A POSITIVE SUPERHARMONIC FUNCTION

Using the Choquet integral representation theorem [9] an integral representation theorem for positive harmonic functions in } X \text{ is given in [2, Theorem 3.2.15]. In this section, we obtain an integral representation for any positive superharmonic function in } X, \text{ by using the Choquet integral representation theorem.

Let } S^+ \text{ be the set of non-negative superharmonic functions in } X. \text{ } S^+ \text{ is a convex cone. A function } s \in S^+ \text{ is said to be extremal if } s = u + v \text{ with } u, v \in S^+, \text{ implies the existence of some } \lambda, 0 \leq \lambda \leq 1, \text{ such that } u = \lambda s \text{ and } v = (1 - \lambda)s. \text{ Suppose } s \in S^+ \text{ is extremal. By Riesz representation theorem, } s = p + h \text{ where } p \text{ is a potential and } h \text{ is harmonic. This implies that } s \text{ is a potential or a harmonic function. From [2, Theorem 3.3.1] } p \text{ is a potential if and only if } p(x) = \sum_{y \in X} (-\Delta)p(y)G_y(x) \text{ where } G_y(x) \text{ is the unique potential in } X \text{ such that } \Delta G_y(x) = -\delta_y(x). \text{ This implies that the extremal potential } p(x) \text{ has to be proportional to some } G_y(x). \text{ On the other hand if the extremal function } s \in S^+ \text{ is harmonic, then } s \text{ has to be minimal. Thus, if we denote by } \Xi \text{ the extremal elements } s \in S^+ \text{ such that } s(x_0) = 1, \text{ then } \Xi = \Lambda_{1} \cup (\Xi \setminus \Lambda_{1}) \text{ where } \Lambda_{1} \text{ consists of all minimal harmonic functions } u \text{ in } X \text{ such that } u(x_0) = 1 \text{ and } \Xi \setminus \Lambda_{1} = \{p_y(x), y \in X, p_y(x) = \frac{G_y(x)}{G_y(x_0)}\}.

Let } S = S^+ - S^+. \text{ For each } x \in X, \text{ define the semi-norm } \| . \|_x \text{ on } S \text{ as
follows: $\|s_1 - s_2\|_x = |s_1(x) - s_2(x)|$. Provide $S$ with the topology defined by the semi-norms $\|\cdot\|_x$, $x \in X$. Since $X$ has a countable number of vertices, these countable semi-norms define on $S$ a locally convex metrisable topology. For a fixed $x_0 \in X$, let $B = \{s \in S^+ : s(x_0) = 1\}$. Then $B$ is a compact metrisable base for the convex cone $S^+$. To show that $B$ is compact, take a sequence $\{s_n\} \subset S^+, s_n(x_0) = 1$. Then for any $a \in X$, there exists a constant $\alpha$ such that $s_n(a) \leq \alpha s_n(x_0)$ for every $n$ (Harnack property [2, Page 47]). Since $\{s_n(a)\}$ is bounded we can extract a subsequence $\{s'_n(x)\}$ from $\{s_n(x)\}$ such that $\{s'_n(a)\}$ is convergent. Let $b$ be another vertex in $X$. Then as before, we can extract a subsequence $\{s''_n(x)\}$ from $\{s'_n(x)\}$ which is convergent at $x = b$. Since $X$ has a countable number of vertices this process produces a subsequence $\{s^*_n(x)\}$ of $\{s_n(x)\}$ such that $\lim_{n \to \infty} s^*_n(x) = s(x)$ exists and is finite for each $x \in X$. Since the limit of a sequence of superharmonic functions (if the limit is finite at every vertex $x \in X$) is superharmonic, we conclude that $s(x)$ is non-negative superharmonic in $X$ such that $s(x_0) = 1$. Consequently, $B$ is a compact set.

**Theorem 3.1.** Let $s \geq 0$ be a positive superharmonic function. Then there exists a unique measure $\nu \geq 0$ with support in $\Xi$ such that $s(x) = \int u(x) d\nu(u)$.

**Proof.** By Riesz representation theorem, a positive superharmonic function $s$ can be uniquely written as $s = p + h$ where $p$ is a positive potential and $h$ is a positive harmonic function. Since $p(x)$ is a potential and $h$ is non-negative harmonic, we have $p(x) = \sum_{y \in X} (-\Delta)p(y)G_y(x)$, and by [2, Corollary 3.2.16] there exists a unique measure $\mu \geq 0$ with support in $\Lambda_1$ such that $h(x) = \int \nu(x) \, d\mu(v)$. Now to each $G_y(x), y \in X$, which is an extremal element, corresponds a unique element $q \in \Xi \setminus \Lambda_1$. Hence, if a measure $\lambda$ is defined on $\Xi \setminus \Lambda_1$ such that $\lambda(q) = (-\Delta)p(y)$ then $\lambda \geq 0$ is a uniquely determined measure on $\Xi \setminus \Lambda_1$ such that $\sum_{y \in X} (-\Delta)p(y)G_y(x) = \int q(x) \, d\lambda(q)$. Consequently $s(x) = \int u(x) \, d\nu(u)$, where $\nu = \mu + \lambda$.

To prove the uniqueness, let $s(x) = \int u(x) \, d\nu_1(u)$ be another representation, where $u(x)$ is either a minimal harmonic function or $u(x) = G_z(x)$ for some $z \in X$. Write $\mu_1 = \nu_1$ restricted to $\Lambda_1$ and $\lambda_1 = \nu_1$ restricted to $\Xi \setminus \Lambda_1$. Then $s(x) = \int u(x) \, d\nu_1(u) = \int q(x) \, d\lambda_1(q) + \int u(x) \, d\mu_1(u)$. Here $\int u(x) \, d\mu_1(u)$ is a harmonic function in $X$ and $\int q(x) \, d\lambda_1(q)$ is a potential in $\Lambda_1$. Therefore $\mu_1 = \nu_1$ and $\lambda_1 = \nu_1$. Since $\nu_1$ is a positive measure and $\Lambda_1$ is a compact set, the measure $\nu_1$ is a finite measure. Consequently, $s(x) = \int u(x) \, d\nu_1(u)$ is a potential in $\Xi \setminus \Lambda_1$.
X. Then the superharmonic function $s$ has two representations, hence by the uniqueness of the Riesz representation $\int q(x) d\lambda(q) = \int q(x) d\lambda_1(q)$ and $\int u(x) d\mu(u) = \int u(x) d\mu_1(u)$. Now, again by the uniqueness of the Choquet representing measures, for harmonic function $\mu = \mu_1$. If $q(x)$ corresponds to $G_y(x)$ then $\lambda(q)$ and $\lambda_1(q)$ are equal to the same value $(-\Delta)p(y)$ hence $\lambda = \lambda_1$ on $\Xi \setminus \Lambda_1$. Hence, $\nu = \nu_1$. □

4. INTEGRAL REPRESENTATION OF POSITIVE SEPARATELY HARMONIC AND POSITIVE SEPARATELY SUPERHARMONIC FUNCTIONS IN A PRODUCT NETWORK

A representation for separately harmonic functions in the context of the Brelot axiomatic potential theory has been given by Gowrisankaran [11]. Similarly in [3] Anandam shows that if $h(x, y)$ is a non-negative separately harmonic function in the product network $X \times Y$, then there exists a unique measure $\mu$ on $\Lambda_1 \times \Lambda_2$ such that $h(x, y) = \int_{\Lambda_1 \times \Lambda_2} h_1(x) h_2(y) d\mu(h_1, h_2)$. This result is proved by using the Choquet integral representation theorem for the cone of non-negative separately harmonic functions in $X \times Y$. By this method, we have to find expressions for minimal separately harmonic functions $u(x, y)$ in $X \times Y$. The effort needed to prove that each such $u(x, y)$ is of the form $h_1(x) h_2(y)$ where $h_1(x)$ is minimal harmonic in $H^+(X)$ (the set of non-negative harmonic functions in $X$) and $h_2(y)$ is minimal harmonic in $H^+(Y)$ (the set of non-negative harmonic functions in $Y$) is explained in [3]. Here in Theorem 4.1 we want to avoid this calculation by considering the cones of non-negative harmonic functions in $X$ and $Y$ successively.

**Theorem 4.1.** Let $h(x, y)$ be positive separately harmonic. Then there exists a unique measure $\mu$ with support in $\Lambda_1 \times \Lambda_2$ such that $h(x, y) = \int_{\Lambda_1 \times \Lambda_2} h_1(x) h_2(y) d\mu(h_1, h_2)$.

**Proof.** For fixed $y$, $h_y(x)$ is positive harmonic in $X$. Hence, $h(x, y) = h_y(x) = \int_{\Lambda_1} h_1(x) d\lambda_y(h_1)$, where the representing measure $\lambda_y$ is uniquely fixed on $\Lambda_1$. Now, for fixed $x$, $h(x, y)$ is harmonic in $Y$ and for any $y \in Y$, let $B$ denote $V(y)$ which is the set consisting of $y$ and all its neighbours. Then $h(x, y) = \sum_{\beta \in \partial B} h(x, \beta) P_B(y, \beta)$. Consequently $\int_{\Lambda_1} h_1(x) d\lambda_y(h_1) = \int_{\Lambda_1} [\sum_{\beta \in \partial B} \lambda_\beta(h_1) P_B(y, \beta)]$. Since the representing measure is uniquely fixed, we have $\lambda_y(h_1) = \sum_{\beta \in \partial B} \lambda_\beta(h_1) P_B(y, \beta)$. That is, for fixed...
h_1, \lambda_y(h_1) is harmonic in y \in Y. Hence, by the uniqueness of representation of positive harmonic functions, \lambda_y(h_1) = \int_{\Lambda_2} h_2(y)d\mu_{h_1}(h_2) for a uniquely determined measure \mu_{h_1} on \Lambda_2. Hence, h(x, y) = \int_{\Lambda_1} h_1(x)d\lambda_y(h_1) = \int_{\Lambda_1 \times \Lambda_2} h_1(x)h_2(y)d\mu(h_1, h_2) for any (x, y) \in X \times Y.

To prove the uniqueness of the representing measure \mu, suppose that for another measure \nu(h_1, h_2) we have h(x, y) = \int_{\Lambda_1 \times \Lambda_2} h_1(x)h_2(y)d\nu(h_1, h_2). Then \int_{\Lambda_1 \times \Lambda_2} h_1(x)h_2(y)d\nu(h_1, h_2) = h(x, y) = \int_{\Lambda_1} h_1(x)d\lambda_y(h_1). Hence, by the uniqueness of the representing measures for non-negative harmonic functions on X, d\lambda_y(h_1) = h_2(y)d\nu_{h_1}(h_2) and hence \lambda_y(h_1) = \int_{\Lambda_2} h_2(y)d\nu_{h_1}(h_2). But \lambda_y(h_1) = \int_{\Lambda_2} h_2(y)d\mu_{h_1}(h_2). Hence, \nu = \mu. \quad \square

A representation for non-negative separately superharmonic functions is also possible up to the integral representation, but the uniqueness of the representing measure seems doubtful. In [10] Drinkwater has given an integral representation for multiply superharmonic functions in the product of Brelot spaces. But she did not prove the uniqueness.

**Lemma 4.2.** (Harnack property for non-negative separately superharmonic functions) Let (a, b) and (c, d) be two vertices in X × Y. Then there exist two constants \alpha > 0 and \beta > 0 such that for any non-negative separately superharmonic function s, \alpha s(c, d) \leq s(a, b) \leq \beta s(c, d).

**Proof.** Since X × Y is a connected infinite network, there exists a path connecting (a, b) and (c, d). Suppose the path is of the form \{(a, b) = (a_0, b_0), (a_1, b_0), (a_1, b_1), (a_2, b_1), (a_2, b_2), ..., (a_n, b_n-1), (a_n, b_n) = (c, d)\} connecting (a, b) and (c, d). Take any non-negative separately superharmonic function s in X × Y. Then by fixing the vertex b \in Y, s is superharmonic at a \in X implies \ t_1(a)s(a, b) \geq t_1(a, a_1)s(a_1, b). Now fix a_1 \in X; then s is superharmonic at b \in Y. Hence, \ t_1(a)s(a, b) \geq t_1(a, a_1)s(a_1, b) \geq t_1(a, a_1)\frac{t_2(b, b_1)}{t_2(b)}s(a_1, b_1). Proceeding further we arrive at the inequality s(a, b) \geq \frac{t_1(a,a_1)}{t_1(a)}s(a_1, b) \geq \frac{t_1(a,a_1)}{t_1(a)}\times \frac{t_2(b,b_1)}{t_2(b)} \times ... \times \frac{t_1(a_{n-1},a_n)}{t_1(a_{n-1})} \times \frac{t_2(b_{n-1},b_n)}{t_2(b_{n-1})} s(a_n, b_n), which is of the form s(a, b) \geq \alpha s(c, d). The other inequality s(a, b) \leq \beta s(c, d) is proved similarly. Note that \alpha, \beta do not depend on the choice of the superharmonic function s. \quad \square

**Theorem 4.3.** Let \mathfrak{F}^+ be the cone of non-negative separately superharmonic functions in X × Y. Then given any u ∈ \mathfrak{F}^+, there exists a measure \mu with support in the extremal set \Pi of elements of a base on \mathfrak{F}^+ such that
\[ u(x, y) = \int s(x, y)d\mu(s) \text{ for any } (x, y) \in X \times Y. \]

**Proof.** Let \( \mathfrak{F} = \mathfrak{F}^+ - \mathfrak{F}^+ \). For each \((x, y) \in X \times Y\), define the semi-norm \( \|\cdot\|(x,y)\) on \( \mathfrak{F} \) as follows: \( \|s_1 - s_2\|(x,y) = |s_1(x,y) - s_2(x,y)| \). Provide \( \mathfrak{F} \) with the topology defined by the semi-norms \( \|\cdot\|(x,y)\), \((x, y) \in X \times Y\). Since \( X \times Y \) has a countable number of vertices, these countable semi-norms define on \( \mathfrak{F} \) a locally convex metrisable topology. For a fixed \((x_0, y_0) \in X \times Y\), let \( B = \{s \in \mathfrak{F}^+: s(x_0, y_0) = 1\} \). Then \( B \) is a compact metrisable base for the convex cone \( \mathfrak{F}^+ \). To show that \( B \) is compact, take a sequence \( \{s_n(x, y)\} \in \mathfrak{F}^+, s_n(x_0, y_0) = 1 \). Then by Harnack property for separately superharmonic functions (Lemma 4.2), for any \((a, b) \in X \times Y\), there exists a constant \( \alpha \) such that \( s_n(a, b) \leq \alpha s_n(x_0, y_0) \) for every \( n \). Since \( \{s_n(a, b)\} \) is bounded, we can extract a subsequence \( \{s'_n(x, y)\} \) from \( \{s_n(x, y)\} \) which is convergent at \((x, y) = (a, b)\). Let \((c, d)\) be another vertex in \( X \times Y \). Then from \( \{s'_n(x, y)\} \) we can extract a subsequence \( \{s''_n(x, y)\} \) which is convergent at \((x, y) = (c, d)\). Since \( X \times Y \) has a countable number of vertices this process produces a subsequence \( \{s^*_n(x, y)\} \) of \( \{s_n(x, y)\} \) such that \( \lim_{n \to \infty} s^*_n(x, y) = s(x, y) \) exists and is finite for each \((x, y) \in X \times Y\). By property (3) of separately superharmonic functions, we conclude that \( s(x, y) \) is non-negative separately superharmonic on \( X \times Y \) such that \( s(x_0, y_0) = 1 \). Consequently, \( B \) is a compact set. Hence, by the Choquet integral representation theorem there exists a measure \( \nu \) with support in the extremal set \( \Pi \) of elements of the base \( B \) such that \( \frac{u(x, y)}{u(x_0, y_0)} = \int s(x, y)d\nu(s) \) for any \((x, y) \in X \times Y\). Write \( d\mu(s) = u(x_0, y_0)d\nu(s) \). Then \( \int_{\Pi} u(x, y) = \int_{\Pi} s(x, y)d\mu(s) \) for any \((x, y) \in X \times Y\). However, whether the cone \( \mathfrak{F}^+ \) is a lattice for its own order has not been proved, so that the uniqueness of the representing measure \( \mu \) is not asserted in the statement of the theorem. \( \square \)

To obtain the uniqueness of the representing measure, Cairoli [7] considered representations for a subclass of non-negative separately superharmonic functions in the context of two standard processes in probability theory, and Gowrisankaran [12] in the context of the product of two Brelot harmonic spaces. Here the uniqueness of the representing measure \( \mu \) can be established for a subclass of functions \( \mathfrak{B}^+ \) which consists of non-negative separately superharmonic functions having certain mean-value property. This is proved in Section 6.

### 5. BALANCED FUNCTIONS

For any \( x \) in \( X \), let \( A \) denote \( V(x) \) which is the set consisting of \( x \) and all its neighbours in \( X \). Similarly for \( y \in Y \), \( B \) is the set \( V(y) \) in \( Y \).
The simplest form of a separately superharmonic function in \( X \times Y \) is \( f(x)g(y) \) where \( f(x) \) is non-negative superharmonic in \( X \) and \( g(y) \) is non-negative superharmonic in \( Y \). For such a separately superharmonic function \( u(x,y) = f(x)g(y) \) we have

\[
[f(x) - \sum_{\alpha \in \partial A} f(\alpha)P_A(x,\alpha)][g(y) - \sum_{\beta \in \partial B} g(\beta)P_B(y,\beta)] \geq 0
\]

so that

\[
u(x,y) + \sum_{\alpha \in \partial A, \beta \in \partial B} u(\alpha,\beta)P_A(x,\alpha)P_B(y,\beta) \geq \sum_{\alpha \in \partial A} u(\alpha,y)P_A(x,\alpha) + \sum_{\beta \in \partial B} u(x,\beta)P_B(y,\beta).
\]

In this note, we are interested in the class of functions in \( X \times Y \) for which the above inequality can be replaced by equality.

**Definition 5.1.** A real valued function \( f(x,y) \) on \( X \times Y \) is said to be balanced if and only if for any \((x,y)\) in \( X \times Y \),

\[
f(x,y) + \sum_{\alpha \in \partial A, \beta \in \partial B} f(\alpha,\beta)P_A(x,\alpha)P_B(y,\beta) = \sum_{\alpha \in \partial A} f(\alpha,y)P_A(x,\alpha) + \sum_{\beta \in \partial B} f(x,\beta)P_B(y,\beta)
\]

**Example.** If \( f(x,y) \) is a real valued function that is harmonic in one variable (say \( x \)) when the other is fixed, then \( f(x,y) \) is balanced.

**Proof.** If \( f(x,y) \) is harmonic in \( X \) for fixed \( y \), then

\[
f(x,y) = \sum_{\alpha \in \partial A} f(\alpha,y)P_A(x,\alpha) \quad \text{for any } x.
\]

\[
\sum_{\alpha \in \partial A, \beta \in \partial B} f(\alpha,\beta)P_A(x,\alpha)P_B(y,\beta) = \sum_{\beta \in \partial B} \left[ \sum_{\alpha \in \partial A} f(\alpha,\beta)P_A(x,\alpha) \right]P_B(y,\beta)
\]

\[= \sum_{\beta \in \partial B} f(x,\beta)P_B(y,\beta). \quad \text{This implies}
\]

\[
f(x,y) + \sum_{\alpha \in \partial A, \beta \in \partial B} f(\alpha,\beta)P_A(x,\alpha)P_B(y,\beta) = \sum_{\alpha \in \partial A} f(\alpha,y)P_A(x,\alpha) + \sum_{\beta \in \partial B} f(x,\beta)P_B(y,\beta).
\]

Hence, \( f(x,y) \) is balanced. \( \square \)

**Properties of balanced functions:**

1. If \( f, g \) are balanced on \( X \times Y \), then for non-negative numbers \( a, b \), \( af + bg \) is balanced.
(2) If \( f_n \) is a sequence of balanced functions and if \( f(x, y) = \lim_{n \to \infty} f_n(x, y) \) exists and is finite for every \((x, y)\) in \( X \times Y \), then \( f \) is balanced.

**Proof.** Since each \( f_n \) is balanced we have

\[
 f_n(x, y) + \sum_{\alpha \in \partial A, \beta \in \partial B} f_n(\alpha, \beta)P_A(x, \alpha)P_B(y, \beta) = \sum_{\alpha \in \partial A} f_n(\alpha, y)P_A(x, \alpha) + \sum_{\beta \in \partial B} f_n(x, \beta)P_B(y, \beta).
\]

Taking limits on both sides

\[
\lim_{n \to \infty} f_n(x, y) + \lim_{n \to \infty} \sum_{\alpha \in \partial A, \beta \in \partial B} f_n(\alpha, \beta)P_A(x, \alpha)P_B(y, \beta) = \sum_{\alpha \in \partial A} f(\alpha, y)P_A(x, \alpha) + \sum_{\beta \in \partial B} f(x, \beta)P_B(y, \beta).
\]

Since the sums are finite we can take the limits inside the sums

\[
f(x, y) + \sum_{\alpha \in \partial A, \beta \in \partial B} f(\alpha, \beta)P_A(x, \alpha)P_B(y, \beta) = \sum_{\alpha \in \partial A} f(\alpha, y)P_A(x, \alpha) + \sum_{\beta \in \partial B} f(x, \beta)P_B(y, \beta).
\]

Hence, \( f \) is balanced. \( \square \)

**Proposition 5.2.** For a real valued function \( f(x, y) \) the following are equivalent:

1. \( f(x, y) \) is balanced.
2. \( f(x, y) - \sum_{\alpha \in \partial A} f(\alpha, y)P_A(x, \alpha) = \varphi(y) \) is a harmonic function in \( Y \) for fixed \( x \).
3. \( f(x, y) - \sum_{\beta \in \partial B} f(x, \beta)P_B(y, \beta) = \psi(x) \) is a harmonic function in \( X \) for fixed \( y \).

**Proof.** Let \( f(x, y) \) be a balanced function in \( X \times Y \). Let us show that \( \varphi(y) \) is a harmonic function in \( Y \) for fixed \( x \).

\[
\sum_{\beta \in \partial B} \varphi(\beta)P_B(y, \beta) = \sum_{\beta \in \partial B} [f(x, \beta) - \sum_{\alpha \in \partial A} f(\alpha, \beta)P_A(x, \alpha)]P_B(y, \beta)
\]

\[
= \sum_{\beta \in \partial B} f(x, \beta)P_B(y, \beta) - \sum_{\alpha \in \partial A, \beta \in \partial B} f(\alpha, \beta)P_A(x, \alpha)P_B(y, \beta)
\]

\[
= f(x, y) - \sum_{\alpha \in \partial A} f(\alpha, y)P_A(x, \alpha) \quad \text{since } f(x, y) \text{ is balanced}
\]

\[
= \varphi(y).
\]
Hence, $\varphi(y)$ is harmonic in $Y$ for fixed $x$. On the other hand, let $\varphi(y)$ be harmonic in $Y$ for fixed $x$. Then

$$f(x, y) - \sum_{\alpha \in \partial A} f(\alpha, y) P_A(x, \alpha) = \varphi(y) = \sum_{\beta \in \partial B} \varphi(\beta) P_B(y, \beta)$$

$$= \sum_{\beta \in \partial B} [f(x, \beta) - \sum_{\alpha \in \partial A} f(\alpha, \beta) P_A(x, \alpha)] P_B(y, \beta)$$

$$= \sum_{\beta \in \partial B} f(x, \beta) P_B(y, \beta) - \sum_{\alpha \in \partial A, \beta \in \partial B} f(\alpha, \beta) P_A(x, \alpha) P_B(y, \beta)$$

Hence, $f(x, y)$ is balanced. Similarly we can prove the other equivalent condition. \(\square\)

6. BALANCED SEPARATELY SUPERHARMONIC FUNCTIONS

Definition 6.1. A real valued function $u(x, y)$ on $X \times Y$ is said to be balanced separately superharmonic if and only if for any $(x, y)$ in $X \times Y$, $u(x, y)$ is separately superharmonic and

$$u(x, y) + \sum_{\alpha \in \partial A, \beta \in \partial B} u(\alpha, \beta) P_A(x, \alpha) P_B(y, \beta) = \sum_{\alpha \in \partial A} u(\alpha, y) P_A(x, \alpha)$$

$$+ \sum_{\beta \in \partial B} u(x, \beta) P_B(y, \beta)$$

Let $\mathcal{B}$ be the class of balanced separately superharmonic functions in $X \times Y$ and let $\mathcal{B}^+$ denote the class of non-negative balanced separately superharmonic functions in $X \times Y$. Then

(1) If $f, g \in \mathcal{B}$, then for non-negative numbers $a, b$ the function $af + bg \in \mathcal{B}$.

(2) If $v_n \in \mathcal{B}$ and if $v(x, y) = \lim_{n \to \infty} v_n(x, y)$ exists and is finite for every $(x, y)$ in $X \times Y$, then $v \in \mathcal{B}$.

(3) If $u(x, y) \in \mathcal{B}^+$ and if $u(x_0, y_0) = 0$ for some $(x_0, y_0)$ in $X \times Y$, then $u = 0$.

(4) There are some non-negative separately superharmonic functions in $X \times Y$ that are not balanced.

Example. Let $\xi_n$ be the set of vertices in a hyperbolic network. Let $G_{\xi_n}(x)$ be the Green potential in $X$ with harmonic singularity at $\xi_n$ (\cite[Theorem 9]{1}). Since $G_{\xi_n}(x) \leq G_{\xi_n}(\xi_n)$, the function $p(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} G_{\xi_n}(x)$ as a convergent sum of potentials is a potential in $X$ that is not harmonic at any vertex in $X$. Hence, $[p(x) - \sum_{\alpha \in \partial A} p(\alpha) P_A(x, \alpha)] > 0$ for any $x \in X$.

Similarly, let $q(y)$ be a potential in $Y$ which is not harmonic at any
vertex in $Y$. Let $u(x, y) = p(x)q(y)$ which is a separately superharmonic function in $X \times Y$.

Now,

$$[p(x) - \sum_{\alpha \in \partial A} p(\alpha)P_A(x, \alpha)][q(y) - \sum_{\beta \in \partial B} q(\beta)P_B(y, \beta)] > 0$$

so that

$$u(x, y) + \sum_{\alpha \in \partial A, \beta \in \partial B} u(\alpha, \beta)P_A(x, \alpha)P_B(y, \beta) > \sum_{\alpha \in \partial A} u(\alpha, y)P_A(x, \alpha)$$

$$+ \sum_{\beta \in \partial B} u(x, \beta)P_B(y, \beta).$$

Hence, $u(x, y)$ is not a balanced function, that is $u \notin \mathcal{B}^+.$

**Lemma 6.2.** Let $u(x, y)$ be a non-negative balanced separately superharmonic function in $X \times Y$. Then the non-negative function $\psi(x, y) = \sum_{\alpha \in \partial A} u(\alpha, y)P_A(x, \alpha)$ is superharmonic in $Y$ for fixed $x$ and harmonic in $X$ for fixed $y$.

**Proof.** When $y$ is fixed, $u_y(x)$ is superharmonic in $X$ and $\sum_{\alpha \in \partial A} u(\alpha, y)P_A(x, \alpha)$ is the Poisson modification of $u_y(x)$ at the vertex $x$ so that $\sum_{\alpha \in \partial A} u(\alpha, y)P_A(x, \alpha)$ is harmonic at $x$. On the other hand, for fixed $x$, and $\alpha \sim x$, $u^\alpha(y) = u(\alpha, y)$ is superharmonic in $Y$ and $P_A(x, \alpha)$ is a positive number. Hence, $u(\alpha, y)P_A(x, \alpha)$ is superharmonic in $Y$ for fixed $x$ and so is $\sum_{\alpha \in \partial A} u(\alpha, y)P_A(x, \alpha)$. The lemma is proved. \(\square\)

**Lemma 6.3.** Let $u(x, y)$ be a non-negative balanced separately superharmonic function in $X \times Y$. Let $\varphi(x, y) = u(x, y) - \sum_{\alpha \in \partial A} u(\alpha, y)P_A(x, \alpha)$. Then $\varphi(x, y)$ is non-negative superharmonic in $X$ for fixed $y$ and harmonic in $Y$ for fixed $x$.

**Proof.** By Proposition 5.2 $\varphi(x, y)$ is harmonic in $Y$ for fixed $x$. When $y$ is fixed $u_y(x)$ is superharmonic in $X$ and by Lemma 6.2 $\sum_{\alpha \in \partial A} u(\alpha, y)P_A(x, \alpha)$ is harmonic at $x$. This implies $\varphi(x, y)$ is superharmonic at $x$ for fixed $y$. Since $u(x, y)$ is superharmonic at $x$ for fixed $y$, $\sum_{\alpha \in \partial A} u(\alpha, y)P_A(x, \alpha) \leq u(x, y)$ so that $\varphi(x, y) \geq 0$. \(\square\)

**Lemma 6.4.** Let $u(x, y)$ be a non-negative balanced separately superharmonic function in $X \times Y$ and let $\varphi(x, y) = u(x, y) - \sum_{\alpha \in \partial A} u(\alpha, y)P_A(x, \alpha)$. Then $\varphi(x, y) = h(x, y) + p(x, y)$ where $h(x, y)$ is separately harmonic in $X \times Y$; and $p(x, y)$ is harmonic in $Y$ for fixed $x$ and a potential in $X$ for fixed $y$. 

Proof. Since \( \varphi(x, y) \) is a non-negative separately superharmonic function in \( X \times Y \), by ([3, Theorem 3.3]) there exists a unique non-negative separately harmonic function \( h(x, y) \) in \( X \times Y \) such that \( h(x, y) \leq \varphi(x, y) \); moreover, \( h(x, y) \) is the greatest harmonic minorant of \( \varphi(x, y) \) for fixed \( y \). Thus, if we write \( \varphi(x, y) = h(x, y) + p(x, y) \), then \( p(x, y) \geq 0 \). Since for fixed \( y \), \( h(x, y) \) is the greatest harmonic minorant of \( \varphi(x, y) \), \( p(x, y) \) is a potential in \( X \) for fixed \( y \); further for fixed \( x \), \( \varphi(x, y) \) and \( h(x, y) \) are harmonic in \( Y \), so that \( p(x, y) \) is harmonic in \( Y \) for fixed \( x \). \( \Box \)

Let \( \mathcal{F}_1 \) be the family of non-negative separately harmonic functions in \( X \times Y \), \( \mathcal{F}_2 \) be the family of non-negative separately superharmonic functions that are harmonic in \( X \) for fixed \( y \) and potentials in \( Y \) for fixed \( x \). Similarly let \( \mathcal{F}_3 \) be the family of non-negative separately superharmonic functions that are potentials in \( X \) for fixed \( y \) and harmonic in \( Y \) for fixed \( x \).

**Theorem 6.5.** \( \mathcal{B}^+ = \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_3 \).

**Proof.** Let \( u \in \mathcal{B}^+ \). Then by Lemma 6.3 \( \varphi(x, y) = u(x, y) - \sum_{\alpha \in \partial A} u(\alpha, y) \) \( P_A(x, \alpha) \) is harmonic in \( Y \) for fixed \( x \). Now by Lemma 6.4 \( u(x, y) = h(x, y) + p(x, y) + \sum_{\alpha \in \partial A} u(\alpha, y)P_A(x, \alpha) \) where \( h(x, y) \) is separately harmonic in \( X \times Y \); and \( p(x, y) \) is harmonic in \( Y \) for fixed \( x \) and a potential in \( X \) for fixed \( y \). Then by Lemma 6.2 the non-negative function \( \sum_{\alpha \in \partial A} u(\alpha, y)P_A(x, \alpha) \) is superharmonic in \( Y \) for fixed \( x \) and harmonic in \( X \) for fixed \( y \). As in Lemma 6.4 we have \( \sum_{\alpha \in \partial A} u(\alpha, y)P_A(x, \alpha) = h_1(x, y) + q(x, y) \) where \( h_1(x, y) \) is separately harmonic in \( X \times Y \); and \( q(x, y) \) is harmonic in \( X \) for fixed \( y \) and a potential in \( Y \) for fixed \( x \). Hence, \( u(x, y) = H(x, y) + q(x, y) + p(x, y) \) where \( H(x, y) = h(x, y) + h_1(x, y) \). The families \( \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3 \) are mutually exclusive. For, suppose \( v(x, y) \in \mathcal{F}_1 \cup \mathcal{F}_2 \) then \( v(x, y) \) is non-negative separately harmonic in \( X \times Y \) and potential in \( Y \) for fixed \( x \). This cannot happen which implies \( \mathcal{F}_1 \cap \mathcal{F}_2 = \phi \). Similarly we can prove \( \mathcal{F}_2 \cap \mathcal{F}_3 = \phi \) and \( \mathcal{F}_1 \cap \mathcal{F}_3 = \phi \). Hence, the uniqueness. \( \Box \)

Let \( \Lambda_1 \) and \( \Lambda_2 \) be the minimal boundaries of \( X \) and \( Y \) respectively.

**Lemma 6.6.** For any \( u \in \mathcal{F}_2 \) there exists a unique measure \( \mu_\eta \) on \( \Lambda_1 \) for each \( \eta \in Y \) such that

\[
u(x, y) = \sum_{\eta \in Y} \int_{\Lambda_1} h(x) d\mu_\eta(h) G_\eta^y(y)
\]

for any \( (x, y) \in X \times Y \).

**Proof.** Let \( u \in \mathcal{F}_2 \). Then for fixed \( x \), \( u(x, y) \) is a positive potential in \( Y \).
Hence, by (2, Theorem 3.3.1) \( u(x, y) = \sum_{\eta \in Y} \lambda^x(\eta)G'_{\eta}(y) \), where \( \lambda^x(\eta) \geq 0 \) is a constant for each \( \eta \). Write

\[
(1) \quad u(x, y) = \sum_{\eta \in Y} \lambda(x, \eta)G'_{\eta}(y).
\]

For any fixed vertex in \( Y \), \( u(x, y) \) is harmonic in \( X \); hence

\[
(2) \quad u(x, y) = \sum_{\eta \in Y} \sum_{\alpha \in \partial A} \lambda(\alpha, \eta)P_A(x, \alpha)G_{\eta}(y).
\]

Now for any \( \eta \in Y \), \( \lambda(\alpha, \eta) = \lambda^\alpha(\eta) \) is a non-negative constant. Hence, \( \sum_{\alpha \in \partial A} \lambda(\alpha, \eta)P_A(x, \alpha) \) is a non-negative constant for fixed \( x \). Thus, for fixed \( x \), the potential \( u^x(y) \) has two series expansions ((1) and (2)). But the expansion for a potential to be unique implies \( \lambda(x, \eta) = \sum_{\alpha \in \partial A} \lambda(\alpha, \eta)P_A(x, \alpha) \). That is, for any \( \eta \in Y \), \( \lambda(x, \eta) \) is harmonic in \( X \) and \( \lambda(x, \eta) \geq 0 \). By (2, Corollary 3.2.16) there exists a unique measure \( \mu_{\eta} \) on \( \Lambda_1 \) for each \( \eta \in Y \) such that

\[
u(x, y) = \sum_{\eta \in Y} \int_{\Lambda_1} h(x)d\mu_{\eta}(h)]G'_{\eta}(y). \quad \Box
\]

**Remark.** For any \( v \in \mathcal{F}_3 \) there exists a unique Radon measure \( \nu_\xi \) on \( \Lambda_2 \) for each \( \xi \in X \) such that

\[
v(x, y) = \sum_{\xi \in X} \int_{\Lambda_2} h'(y)d\nu_\xi(h')]G_{\xi}(x)
\]

for any \((x, y) \in X \times Y\).

**Theorem 6.7.** For every function in \( \mathcal{B}^+ \), there exist a unique measure \( \lambda \) on \( \Lambda_1 \times \Lambda_2 \) and two families of uniquely determined associated measures: \( \{\mu_{\eta}\} \) on \( \Lambda_1 \) for each \( \eta \in Y \) and \( \{\nu_\xi\} \) on \( \Lambda_2 \) for each \( \xi \in X \) such that

\[
u(x, y) = \int_{\Lambda_1 \times \Lambda_2} h(x)h'(y)d\lambda(h, h') + \sum_{\eta \in Y} \int_{\Lambda_1} h(x)d\mu_{\eta}(h)]G'_{\eta}(y) + \sum_{\xi \in X} \int_{\Lambda_2} h'(y)d\nu_\xi(h')]G_{\xi}(x).
\]
Proof. If \( u \in \mathfrak{B}^+ \), then by Theorem 6.5 \( u \) can be uniquely written as \( u(x, y) = h(x, y) + p(x, y) + q(x, y) \) where \( h, p \) and \( q \) belong to \( \mathcal{F}_1, \mathcal{F}_2, \) and \( \mathcal{F}_3, \) respectively. Then by ([3, Theorem 5.6]), Lemma 6.6 and the above Remark there exist a unique measure \( \lambda \) on \( \Lambda_1 \times \Lambda_2 \) and two families of associated measures uniquely determined: \( \{\mu_\eta\} \) on \( \Lambda_1 \) for each \( \eta \in Y \) and \( \{\nu_\xi\} \) on \( \Lambda_2 \) for each \( \xi \in X \) such that

\[
\begin{align*}
  u(x, y) &= \int_{\Lambda_1 \times \Lambda_2} h(x)h'(y)d\lambda(h, h') + \sum_{\eta \in Y} \left[ \int_{\Lambda_1} h(x)d\mu_\eta(h) \right] G'_\eta(y) \\
  &\quad+ \sum_{\xi \in X} \left[ \int_{\Lambda_2} h'(y)d\nu_\xi(h') \right] G_\xi(x)
\end{align*}
\]

\( \square \)

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