UNIQUE INTEGRAL REPRESENTATION FOR THE CLASS OF BALANCED SEPARATELY SUPERHARMONIC FUNCTIONS IN A PRODUCT NETWORK

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In an infinite network X, the extremal elements for a base of the cone of positive superharmonic functions in X are determined and an integral representation for this cone of functions is given by using the Choquet integral representation theorem. Later, a similar representation is given to the class of non-negative separately superharmonic functions in a product network $X \times Y$, but without proving the uniqueness of the representing measures. However if we restrict to a subclass of non-negative separately superharmonic functions, called here the balanced functions, then the representing measure is unique.

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1. INTRODUCTION

In an infinite tree T, defining the Martin boundary, Cartier [8] obtains an integral representation for positive harmonic functions in T. In this note, we consider first the integral representation for positive superharmonic functions in a single infinite network X by using the Choquet theorem on integral representations. Then we consider the integral representation of positive separately superharmonic functions in a product $X \times Y$ of two infinite networks X and Y. In this general case, each non-negative separately superharmonic function can be expressed as an integral with respect to a Radon measure supported by extremal elements. However, the uniqueness of this measure cannot be guaranteed. We introduce a subclass $\mathfrak{B}$ of separately superharmonic functions in $X \times Y$ called balanced functions which have a certain mean-value property. This is analogous to the case of functions in a network X with a mean-value property which we call harmonic functions in X. We prove that every element $u(x, y)$ in $\mathfrak{B}^+$, consisting of non-negative elements of $\mathfrak{B}$, is the unique sum of three functions, one separately harmonic in $X \times Y$, one harmonic in the first variable and a potential in the second variable and one potential in the first variable and harmonic in the second variable. Finally, we prove if we restrict to this subclass $\mathfrak{B}^+$ then the representing measure is unique.

2. PRELIMINARIES

X is an infinite graph with countably infinite vertices and countably infinite edges. We say that x and y are neighbours, and write $x \sim y$, if and only if there is an edge joining x and y. Assume X is locally finite (that is, every vertex in X has only a finite number of neighbours), connected (that is, any two vertices x and y can be joined by a path $\{x = x_0, x_1, \dots, x_n = y\}$) and there are no self loops (that is, there is no edge from one vertex to itself). We refer to such a graph X as an infinite network if there is a *transition index* $t(x, y)$ associated with each pair of vertices x and y in X satisfying the conditions: $t(x, y)$ is a non-negative real number, $t(x, y) > 0$ if and only if $x \sim y$, $t(x, y)$ and $t(y, x)$ need not be the same. Then for any vertex x in X, $t(x) = \sum\limits_{x \sim y} t(x, y)$ is always a positive real number.

Let u be a real-valued function on X. Then the Laplacian of u is defined by

$$\Delta u(x) = \sum_{x \sim y} t(x, y)[u(y) - u(x)].$$

The function u is said to be *superharmonic* on X if $t(x)u(x) \geq \sum\limits_{x \sim y} t(x, y)u(y))$ for every $x \in X$; *harmonic* and *subharmonic* functions are defined accordingly. A superharmonic function $p \geq 0$ on X is said to be a *potential* if u subharmonic on X and $u \leq p$ imply $u \leq 0$. If there exists a positive potential on X, then X is called a hyperbolic network; otherwise X is called a parabolic network.

Discretising the notion of doubly subharmonic functions considered by Avanissian [6] in the context of several complex variables, we consider now doubly superharmonic functions in product networks.

Let $\{X, t_1\}$ and $\{Y, t_2\}$ be two infinite networks with Laplacians Δ_1 and Δ_2, respectively. Define their product as $\{X \times Y, t\}$ such that the neighbours of (x, y) are (x_i, y) and (x, y_j) where $x \sim x_i$ in X and $y \sim y_j$ in Y. Take $t\{(x, y), (x_i, y)\} = t_1(x, x_i)$ and $t\{(x, y), (x, y_j)\} = t_2(y, y_j)$. Then $X \times Y$ with transition index t becomes an infinite network. If $f(x, y)$ is defined on $X \times Y$, then for $(x, y) \in X \times Y$ define

$$\Delta f(x, y) = \sum_{(a,b) \sim (x,y)} t\{(x, y), (a, b)\}[f(a, b) - f(x, y)].$$

Let us denote by $f_y(x)$ the function $f(x,y)$ when y is fixed and by $f^x(y)$ the function $f(x,y)$ when x is fixed. Then,

$$\begin{aligned}\Delta f(x,y) &= \sum_{(x_i,y)\sim(x,y)} t\{(x,y),(x_i,y)\}[f(x_i,y)-f(x,y)]+ \\ &\quad \sum_{(x,y_j)\sim(x,y)} t\{(x,y),(x,y_j)\}[f(x,y_j)-f(x,y)] \\ &= \sum_{x_i\sim x} t_1(x,x_i)[f_y(x_i)-f_y(x)]+\sum_{y_j\sim y} t_2(y,y_j)[f^x(y_j)-f^x(y)] \\ &= \Delta_1 f_y(x)+\Delta_2 f^x(y).\end{aligned}$$

We say that $f(x,y)$ is superharmonic (respectively harmonic) in $X\times Y$ if and only if $\Delta f(x,y)\le 0$ (respectively $\Delta f(x,y)=0$) for every $(x,y)\in X\times Y$. A function $f(x,y)$ is said to be separately superharmonic in $X\times Y$, if for any fixed y, $f_y(x)=f(x,y)$ is superharmonic in X and for any fixed x, $f^x(y)=f(x,y)$ is superharmonic in Y.

Properties of separately superharmonic functions:

(1) A separately superharmonic (respectively separately harmonic) function in $X\times Y$ is $\Delta-$superharmonic (respectively $\Delta-$harmonic) in $X\times Y$.

Proof. If $(x,y)\in X\times Y$, then by using the definition of separately superharmonic functions in $X\times Y$, we see that $\Delta f(x,y)=\Delta_1 f_y(x)+\Delta_2 f^x(y)\le 0$. Similarly for separately harmonic functions. □

The converse need not be valid: for example, let X and Y be trees without terminal vertices. Then we can choose [2, Theorem 5.1.4.] $u(x)$ in X and $v(y)$ in Y such that $\Delta_1 u(x)=1$ and $\Delta_2 v(y)=3$. Let $s(x,y)=u(x)-v(y)$ in $X\times Y$. Then $s(x,y)$ is $\Delta-$ superharmonic but not separately superharmonic. Similarly, a harmonic function need not be separately harmonic. However, if a harmonic function is separately superharmonic, then it is separately harmonic.

(2) If u,v are separately superharmonic in $X\times Y$, then for non-negative numbers α,β we see that $\alpha u+\beta v$ and $\inf(u,v)$ are separately superharmonic in $X\times Y$.

(3) If $\{u_n\}$ is a sequence of separately superharmonic functions in $X\times Y$ such that $u(x,y)=\lim_{n\to\infty} u_n(x,y)$ is finite for each (x,y) in $X\times Y$, then $u(x,y)$ is separately superharmonic in $X\times Y$.

Proof. For any fixed $x\in X$, $u_n^x(y)$ is superharmonic in Y. We know that the limit of superharmonic functions is superharmonic if the limit is finite [2, p.46]. Hence, the fact that $u_n^x(y)\to u^x(y)$ in Y, for every fixed $x \in X$, implies that $u^x(y)$ is superharmonic in Y. Similarly, $u_y(x)$ is superharmonic in X. Hence, $u(x,y)$ is separately superharmonic in $X \times Y$. $\square$

(4) If $u \geq 0$ is Δ_1-superharmonic in X, and $v \geq 0$ is Δ_2-superharmonic in Y, then $s(x,y) = u(x)\,v(y)$ is a separately superharmonic function in $X \times Y$.

Let E denote a finite set in X and $\mathring{E}$ the interior of E. Then, for each $\alpha_i \in \partial E = E \setminus \mathring{E}$, there exists a unique function (Poisson Kernel [1, Theorem 11]) $P_E(x,\alpha_i)$ defined on E such that $\Delta_1 P_E(x,\alpha_i) = 0$ for each $x \in \mathring{E}$ and $P_E(\alpha_k,\alpha_i) = \delta(\alpha_k,\alpha_i)$. Note that if $\phi(x)$ is a function defined on the set $\partial E = \{\alpha_i\}$, then $h(x) = \sum\limits_i \phi(\alpha_i)P_E(x,\alpha_i)$ is defined on E such that $h(\alpha_i) = \phi(\alpha_i)$ for each α_i and $\Delta_1 h(x) = 0$ for $x \in \mathring{E}$. Similarly define $P_F(y,\beta_j)$ on a finite subset F in Y for which $\partial F = \{\beta_j\}$. Note that if $h(x)$ is harmonic in E then $h(x) = \sum\limits_i h(\alpha_i)P_E(x,\alpha_i)$.

3. INTEGRAL REPRESENTATION OF A POSITIVE SUPERHARMONIC FUNCTION

Using the Choquet integral representation theorem [9] an integral representation theorem for positive harmonic functions in X is given in [2, Theorem 3.2.15]. In this section, we obtain an integral representation for any positive superharmonic function in X, by using the Choquet integral representation theorem.

Let S^+ be the set of non-negative superharmonic functions in X. S^+ is a convex cone. A function $s \in S^+$ is said to be extremal if $s = u + v$ with $u, v \in S^+$, implies the existence of some $\lambda, 0 \leq \lambda \leq 1$, such that $u = \lambda s$ and $v = (1-\lambda)s$. Suppose $s \in S^+$ is extremal. By Riesz representation theorem, $s = p + h$ where p is a potential and h is harmonic. This implies that s is a potential or a harmonic function. From [2, Theorem 3.3.1] p is a potential if and only if $p(x) = \sum\limits_{y \in X}(-\Delta)p(y)G_y(x)$ where $G_y(x)$ is the unique potential in X such that $\Delta G_y(x) = -\delta_y(x)$. This implies that the extremal potential $p(x)$ has to be proportional to some $G_y(x)$. On the other hand if the extremal function $s \in S^+$ is harmonic, then s has to be minimal. Thus, if we denote by Ξ the extremal elements $s \in S^+$ such that $s(x_0) = 1$, then $\Xi = \Lambda_1 \cup (\Xi \setminus \Lambda_1)$ where Λ_1 consists of all minimal harmonic functions u in X such that $u(x_0) = 1$ and $\Xi \setminus \Lambda_1 = \{p_y(x), y \in X, p_y(x) = \frac{G_y(x)}{G_y(x_0)}\}$.

Let $S = S^+ - S^+$. For each $x \in X$, define the semi-norm $\|.\|_x$ on S as follows: $\|s_1 - s_2\|_x = |s_1(x) - s_2(x)|$. Provide S with the topology defined by the semi-norms $\|.\|_x, x \in X$. Since X has a countable number of vertices, these countable semi-norms define on S a locally convex metrisable topology. For a fixed $x_0 \in X$, let $B = \{s \in S^+ : s(x_o) = 1\}$. Then B is a compact metrisable base for the convex cone S^+. To show that B is compact, take a sequence $\{s_n\} \in S^+, s_n(x_0) = 1$. Then for any $a \in X$, there exists a constant α such that $s_n(a) \leq \alpha s_n(x_0)$ for every n (Harnack property [2, Page 47]). Since $\{s_n(a)\}$ is bounded we can extract a subsequence $\{s'_n(x)\}$ from $\{s_n(x)\}$ such that $\{s'_n(a)\}$ is convergent. Let b be another vertex in X. Then as before, we can extract a subsequence $\{s''_n(x)\}$ from $\{s'_n(x)\}$ which is convergent at $x = b$. Since X has a countable number of vertices this process produces a subsequence $\{s^*_n(x)\}$ of $\{s_n(x)\}$ such that $\lim\limits_{n\to\infty} s^*_n(x) = s(x)$ exists and is finite for each $x \in X$. Since the limit of a sequence of superharmonic functions (if the limit is finite at every vertex $x \in X$) is superharmonic, we conclude that $s(x)$ is non-negative superharmonic in X such that $s(x_0) = 1$. Consequently, B is a compact set.

THEOREM 3.1. *Let $s \geq 0$ be a positive superharmonic function. Then there exists a unique measure $\nu \geq 0$ with support in Ξ such that $s(x) = \int\limits_{\Xi} u(x)\mathrm{d}\nu(u)$.*

Proof. By Riesz representation theorem, a positive superharmonic function s can be uniquely written as $s = p + h$ where p is a positive potential and h is a positive harmonic function. Since $p(x)$ is a potential and h is non-negative harmonic, we have $p(x) = \sum\limits_{y\in X} (-\Delta)p(y)G_y(x)$, and by [2, Corollary 3.2.16] there exists a unique measure $\mu \geq 0$ with support in Λ_1 such that $h(x) = \int\limits_{\Lambda_1} v(x)\mathrm{d}\mu(v)$. Now to each $G_y(x), y \in X$, which is an extremal element, corresponds a unique element $q \in \Xi \setminus \Lambda_1$. Hence, if a measure λ is defined on $\Xi \setminus \Lambda_1$ such that $\lambda(q) = (-\Delta)p(y)$ then $\lambda \geq 0$ is a uniquely determined measure on $\Xi \setminus \Lambda_1$ such that $\sum\limits_{y\in X} (-\Delta)p(y)G_y(x) = \int\limits_{\Xi\setminus\Lambda_1} q(x)\mathrm{d}\lambda(q)$. Consequently $s(x) = \int\limits_{\Xi} u(x)\mathrm{d}\nu(u)$, where $\nu = \mu + \lambda$.

To prove the uniqueness, let $s(x) = \int\limits_{\Xi} u(x)\mathrm{d}\nu_1(u)$ be another representation, where $u(x)$ is either a minimal harmonic function or $u(x) = G_z(x)$ for some $z \in X$. Write $\mu_1 = \nu_1$ restricted to Λ_1 and $\lambda_1 = \nu_1$ restricted to $\Xi \setminus \Lambda_1$. Then $s(x) = \int\limits_{\Xi} u(x)\mathrm{d}\nu_1(u) = \int\limits_{\Xi\setminus\Lambda_1} q(x)\mathrm{d}\lambda_1(q) + \int\limits_{\Lambda_1} u(x)\mathrm{d}\mu_1(u)$. Here $\int\limits_{\Lambda_1} u(x)\mathrm{d}\mu_1(u)$ is a harmonic function in X and $\int\limits_{\Xi\setminus\Lambda_1} q(x)\mathrm{d}\lambda_1(q)$ is a potential in

X. Then the superharmonic function s has two representations, hence by the uniqueness of the Riesz representation $\int_{\Xi\setminus\Lambda_1} q(x)\mathrm{d}\lambda(q) = \int_{\Xi\setminus\Lambda_1} q(x)\mathrm{d}\lambda_1(q)$ and $\int_{\Lambda_1} u(x)\mathrm{d}\mu(u) = \int_{\Lambda_1} u(x)\mathrm{d}\mu_1(u)$. Now, again by the uniqueness of the Choquet representing measures, for harmonic function $\mu = \mu_1$. If $q(x)$ corresponds to $G_y(x)$ then $\lambda(q)$ and $\lambda_1(q)$ are equal to the same value $(-\Delta)p(y)$ hence $\lambda = \lambda_1$ on $\Xi \setminus \Lambda_1$. Hence, $\nu = \nu_1$. □

4. INTEGRAL REPRESENTATION OF POSITIVE SEPARATELY HARMONIC AND POSITIVE SEPARATELY SUPERHARMONIC FUNCTIONS IN A PRODUCT NETWORK

A representation for separately harmonic functions in the context of the Brelot axiomatic potential theory has been given by Gowrisankaran [11]. Similarly in [3] Anandam shows that if $h(x, y)$ is a non-negative separately harmonic function in the product network $X \times Y$, then there exists a unique measure μ on $\Lambda_1 \times \Lambda_2$ such that $h(x, y) = \int_{\Lambda_1\times\Lambda_2} h_1(x)h_2(y)\mathrm{d}\mu(h_1, h_2)$. This result is proved by using the Choquet integral representation theorem for the cone of non-negative separately harmonic functions in $X \times Y$. By this method, we have to find expressions for minimal separately harmonic functions $u(x, y)$ in $X \times Y$. The effort needed to prove that each such $u(x, y)$ is of the form $h_1(x)h_2(y)$ where $h_1(x)$ is minimal harmonic in $H^+(X)$(the set of non-negative harmonic functions in X) and $h_2(y)$ is minimal harmonic in $H^+(Y)$(the set of non-negative harmonic functions in Y) is explained in [3]. Here in Theorem 4.1 we want to avoid this calculation by considering the cones of non-negative harmonic functions in X and Y successively.

THEOREM 4.1. *Let $h(x, y)$ be positive separately harmonic. Then there exists a unique measure μ with support in $\Lambda_1 \times \Lambda_2$ such that $h(x, y) = \int_{\Lambda_1\times\Lambda_2} h_1(x)h_2(y)\mathrm{d}\mu(h_1, h_2)$.*

Proof. For fixed $y, h_y(x)$ is positive harmonic in X. Hence, $h(x, y) = h_y(x) = \int_{\Lambda_1} h_1(x)\mathrm{d}\lambda_y(h_1)$, where the representing measure λ_y is uniquely fixed on Λ_1. Now, for fixed $x, h(x, y)$ is harmonic in Y and for any $y \in Y$, let B denote $V(y)$ which is the set consisting of y and all its neighbours. Then $h(x, y) = \sum_{\beta\in\partial B} h(x, \beta)P_B(y, \beta)$. Consequently $\int_{\Lambda_1} h_1(x)\mathrm{d}\lambda_y(h_1) = \sum_{\beta\in\partial B} [\int_{\Lambda_1} h_1(x)\mathrm{d}\lambda_\beta(h_1)]$ $P_B(y, \beta) = \int_{\Lambda_1} h_1(x)\mathrm{d}[\sum_{\beta\in\partial B} \lambda_\beta(h_1)P_B(y, \beta)]$. Since the representing measure is uniquely fixed, we have $\lambda_y(h_1) = \sum_{\beta\in\partial B} \lambda_\beta(h_1)P_B(y, \beta)$. That is, for fixed

$h_1, \lambda_y(h_1)$ is harmonic in $y \in Y$. Hence, by the uniqueness of representation of positive harmonic functions, $\lambda_y(h_1) = \int_{\Lambda_2} h_2(y) \mathrm{d}\mu_{h_1}(h_2)$ for a uniquely determined measure μ_{h_1} on Λ_2. Hence, $h(x,y) = \int_{\Lambda_1} h_1(x) \mathrm{d}\lambda_y(h_1) = \int_{\Lambda_1 \times \Lambda_2} h_1(x) h_2(y)$ $\mathrm{d}\mu(h_1,h_2)$ for any $(x,y) \in X \times Y$.

To prove the uniqueness of the representing measure μ, suppose that for another measure $\nu(h_1,h_2)$ we have $h(x,y) = \int_{\Lambda_1 \times \Lambda_2} h_1(x) h_2(y) \mathrm{d}\nu(h_1,h_2)$. Then $\int_{\Lambda_1 \times \Lambda_2} h_1(x) h_2(y) \mathrm{d}\nu(h_1,h_2) = h(x,y) = \int_{\Lambda_1} h_1(x) \mathrm{d}\lambda_y(h_1)$. Hence, by the uniqueness of the representing measures for non-negative harmonic functions on $X, \mathrm{d}\lambda_y(h_1) = h_2(y) \mathrm{d}\nu_{h_1}(h_2)$ and hence $\lambda_y(h_1) = \int_{\Lambda_2} h_2(y) \mathrm{d}\nu_{h_1}(h_2)$. But $\lambda_y(h_1) = \int_{\Lambda_2} h_2(y) \mathrm{d}\mu_{h_1}(h_2)$. Hence, $\nu = \mu$. □

A representation for non-negative separately superharmonic functions is also possible up to the integral representation, but the uniqueness of the representing measure seems doubtful. In [10] Drinkwater has given an integral representation for multiply superharmonic functions in the product of Brelot spaces. But she did not prove the uniqueness.

LEMMA 4.2. *(Harnack property for non-negative separately superharmonic functions) Let (a,b) and (c,d) be two vertices in $X \times Y$. Then there exist two constants $\alpha > 0$ and $\beta > 0$ such that for any non-negative separately superharmonic function s, $\alpha s(c,d) \leq s(a,b) \leq \beta s(c,d)$.*

Proof. Since $X \times Y$ is a connected infinite network, there exists a path connecting (a,b) and (c,d). Suppose the path is of the form $\{(a,b) = (a_0,b_0), (a_1,b_0), (a_1,b_1), (a_2,b_1), (a_2,b_2), \ldots, (a_n,b_{n-1}), (a_n,b_n) = (c,d)\}$ connecting (a,b) and (c,d). Take any non-negative separately superharmonic function s in $X \times Y$. Then by fixing the vertex $b \in Y$, s is superharmonic at $a \in X$ implies $t_1(a)s(a,b) \geq t_1(a,a_1)s(a_1,b)$. Now fix $a_1 \in X$; then s is superharmonic at $b \in Y$. Hence, $t_1(a)s(a,b) \geq t_1(a,a_1)s(a_1,b) \geq t_1(a,a_1)\frac{t_2(b,b_1)}{t_2(b)}s(a_1,b_1)$. Proceeding further we arrive at the inequality $s(a,b) \geq \frac{t_1(a,a_1)}{t_1(a)}s(a_1,b) \geq \frac{t_1(a,a_1)}{t_1(a)} \times \frac{t_2(b,b_1)}{t_2(b)} \times \ldots \times \frac{t_1(a_{n-1},an)}{t_1(a_{n-1})} \times \frac{t_2(b_{n-1,b_n})}{t_2(b_{n-1})}s(a_n,b_n)$, which is of the form $s(a,b) \geq \alpha s(c,d)$. The other inequality $s(a,b) \leq \beta s(c,d)$ is proved similarly. Note that α, β do not depend on the choice of the superharmonic function s. □

THEOREM 4.3. *Let $\mathfrak{F}^+$ be the cone of non-negative separately superharmonic functions in $X \times Y$. Then given any $u \in \mathfrak{F}^+$, there exists a measure μ with support in the extremal set Π of elements of a base on $\mathfrak{F}^+$ such that*

$u(x,y) = \int\limits_{\Pi} s(x,y)\mathrm{d}\mu(s)$ *for any* $(x,y) \in X \times Y$.

Proof. Let $\mathfrak{F} = \mathfrak{F}^+ - \mathfrak{F}^+$. For each $(x,y) \in X \times Y$, define the semi-norm $\|.\|_{(x,y)}$ on $\mathfrak{F}$ as follows: $\|s_1 - s_2\|_{(x,y)} = |s_1(x,y) - s_2(x,y)|$. Provide $\mathfrak{F}$ with the topology defined by the semi-norms $\|.\|_{(x,y)}, (x,y) \in X \times Y$. Since $X \times Y$ has a countable number of vertices, these countable semi-norms define on $\mathfrak{F}$ a locally convex metrisable topology. For a fixed $(x_0, y_0) \in X \times Y$, let $B = \{s \in \mathfrak{F}^+ : s(x_0, y_0) = 1\}$. Then B is a compact metrisable base for the convex cone $\mathfrak{F}^+$. To show that B is compact, take a sequence $\{s_n(x,y)\} \in \mathfrak{F}^+, s_n(x_0,y_0) = 1$. Then by Harnack property for separately superharmonic functions (Lemma 4.2), for any $(a,b) \in X \times Y$, there exists a constant α such that $s_n(a,b) \le \alpha s_n(x_0,y_0)$ for every n. Since $\{s_n(a,b)\}$ is bounded, we can extract a subsequence $\{s'_n(x,y)\}$ from $\{s_n(x,y)\}$ which is convergent at $(x,y) = (a,b)$. Let (c,d) be another vertex in $X \times Y$. Then from $\{s'_n(x,y)\}$ we can extract a subsequence $\{s''_n(x,y)\}$ which is convergent at $(x,y) = (c,d)$. Since $X \times Y$ has a countable number of vertices this process produces a subsequence $\{s^*_n(x,y)\}$ of $\{s_n(x,y)\}$ such that $\lim\limits_{n\to\infty} s^*_n(x,y) = s(x,y)$ exists and is finite for each $(x,y) \in X \times Y$. By property (3) of separately superharmonic functions, we conclude that $s(x,y)$ is non-negative separately superharmonic on $X \times Y$ such that $s(x_0,y_0) = 1$. Consequently, B is a compact set. Hence, by the Choquet integral representation theorem there exists a measure ν with support in the extremal set Π of elements of the base B such that $\frac{u(x,y)}{u(x_0,y_0)} = \int\limits_{\Pi} s(x,y)\mathrm{d}\nu(s)$ for any $(x,y) \in X \times Y$. Write $\mathrm{d}\mu(s) = u(x_0,y_0)\mathrm{d}\nu(s)$. Then $u(x,y) = \int\limits_{\Pi} s(x,y)\mathrm{d}\mu(s)$ for any $(x,y) \in X \times Y$. However, whether the cone $\mathfrak{F}^+$ is a lattice for its own order has not been proved, so that the uniqueness of the representing measure μ is not asserted in the statement of the theorem. □

To obtain the uniqueness of the representing measure, Cairoli [7] considered representations for a subclass of non-negative separately superharmonic functions in the context of two standard processes in probability theory, and Gowrisankaran [12] in the context of the product of two Brelot harmonic spaces. Here the uniqueness of the representing measure μ can be established for a subclass of functions $\mathfrak{B}^+$ which consists of non-negative separately superharmonic functions having certain mean-value property. This is proved in Section 6.

5. BALANCED FUNCTIONS

For any x in X, let A denote $V(x)$ which is the set consisting of x and all its neighbours in X. Similarly for $y \in Y$, B is the set $V(y)$ in Y.

The simplest form of a separately superharmonic function in $X \times Y$ is $f(x)g(y)$ where $f(x)$ is non-negative superharmonic in X and $g(y)$ is non-negative superharmonic in Y. For such a separately superharmonic function $u(x,y) = f(x)g(y)$ we have

$$[f(x) - \sum_{\alpha\in\partial A} f(\alpha)P_A(x,\alpha)][g(y) - \sum_{\beta\in\partial B} g(\beta)P_B(y,\beta)] \geq 0$$

so that

$$u(x,y) + \sum_{\alpha\in\partial A, \beta\in\partial B} u(\alpha,\beta)P_A(x,\alpha)P_B(y,\beta) \geq \sum_{\alpha\in\partial A} u(\alpha,y)P_A(x,\alpha) + \sum_{\beta\in\partial B} u(x,\beta)P_B(y,\beta).$$

In this note, we are interested in the class of functions in $X \times Y$ for which the above inequality can be replaced by equality.

Definition 5.1. A real valued function $f(x,y)$ on $X \times Y$ is said to be balanced if and only if for any (x,y) in $X \times Y$,

$$f(x,y) + \sum_{\alpha\in\partial A, \beta\in\partial B} f(\alpha,\beta)P_A(x,\alpha)P_B(y,\beta) = \sum_{\alpha\in\partial A} f(\alpha,y)P_A(x,\alpha) + \sum_{\beta\in\partial B} f(x,\beta)P_B(y,\beta)$$

Example. If $f(x,y)$ is a real valued function that is harmonic in one variable (say x) when the other is fixed, then $f(x,y)$ is balanced.

Proof. If $f(x,y)$ is harmonic in X for fixed y, then

$$f(x,y) = \sum_{\alpha\in\partial A} f(\alpha,y)P_A(x,\alpha) \qquad \text{for any } x.$$

$$\sum_{\alpha\in\partial A, \beta\in\partial B} f(\alpha,\beta)P_A(x,\alpha)P_B(y,\beta) = \sum_{\beta\in\partial B} [\sum_{\alpha\in\partial A} f(\alpha,\beta)P_A(x,\alpha)]P_B(y,\beta)$$

$$= \sum_{\beta\in\partial B} f(x,\beta)P_B(y,\beta). \quad \text{This implies}$$

$$f(x,y) + \sum_{\alpha\in\partial A, \beta\in\partial B} f(\alpha,\beta)P_A(x,\alpha)P_B(y,\beta) = \sum_{\alpha\in\partial A} f(\alpha,y)P_A(x,\alpha) + \sum_{\beta\in\partial B} f(x,\beta)P_B(y,\beta).$$

Hence, $f(x,y)$ is balanced. □

Properties of balanced functions:

(1) If f, g are balanced on $X \times Y$, then for non-negative numbers a, b, $af + bg$ is balanced.

(2) If f_n is a sequence of balanced functions and if $f(x,y) = \lim_{n\to\infty} f_n(x,y)$ exists and is finite for every (x,y) in $X \times Y$, then f is balanced.

Proof. Since each f_n is balanced we have

$$f_n(x,y) + \sum_{\alpha\in\partial A,\beta\in\partial B} f_n(\alpha,\beta)P_A(x,\alpha)P_B(y,\beta) = \sum_{\alpha\in\partial A} f_n(\alpha,y)P_A(x,\alpha) + \sum_{\beta\in\partial B} f_n(x,\beta)P_B(y,\beta).$$

Taking limits on both sides

$$\lim_{n\to\infty} f_n(x,y) + \lim_{n\to\infty} \sum_{\alpha\in\partial A,\beta\in\partial B} f_n(\alpha,\beta)P_A(x,\alpha)P_B(y,\beta) = \lim_{n\to\infty} \sum_{\alpha\in\partial A} f_n(\alpha,y)P_A(x,\alpha) + \lim_{n\to\infty} \sum_{\beta\in\partial B} f_n(x,\beta)P_B(y,\beta).$$

Since the sums are finite we can take the limits inside the sums

$$f(x,y) + \sum_{\alpha\in\partial A,\beta\in\partial B} f(\alpha,\beta)P_A(x,\alpha)P_B(y,\beta) = \sum_{\alpha\in\partial A} f(\alpha,y)P_A(x,\alpha) + \sum_{\beta\in\partial B} f(x,\beta)P_B(y,\beta).$$

Hence, f is balanced. □

PROPOSITION 5.2. *For a real valued function $f(x,y)$ the following are equivalent:*

(1) $f(x,y)$ is balanced.

(2) $f(x,y) - \sum_{\alpha\in\partial A} f(\alpha,y)P_A(x,\alpha) = \varphi(y)$ is a harmonic function in Y for fixed x.

(3) $f(x,y) - \sum_{\beta\in\partial B} f(x,\beta)P_B(y,\beta) = \psi(x)$ is a harmonic function in X for fixed y.

Proof. Let $f(x,y)$ be a balanced function in $X \times Y$. Let us show that $\varphi(y)$ is a harmonic function in Y for fixed x.

$$\begin{aligned}\sum_{\beta\in\partial B} \varphi(\beta)P_B(y,\beta) &= \sum_{\beta\in\partial B} [f(x,\beta) - \sum_{\alpha\in\partial A} f(\alpha,\beta)P_A(x,\alpha)]P_B(y,\beta) \\ &= \sum_{\beta\in\partial B} f(x,\beta)P_B(y,\beta) - \sum_{\alpha\in\partial A,\beta\in\partial B} f(\alpha,\beta)P_A(x,\alpha)P_B(y,\beta) \\ &= f(x,y) - \sum_{\alpha\in\partial A} f(\alpha,y)P_A(x,\alpha) \quad \text{since } f(x,y) \text{ is balanced} \\ &= \varphi(y).\end{aligned}$$

Hence, $\varphi(y)$ is harmonic in Y for fixed x. On the other hand, let $\varphi(y)$ be harmonic in Y for fixed x. Then

$$\begin{aligned} f(x,y) - \sum_{\alpha\in\partial A} f(\alpha,y)P_A(x,\alpha) = \varphi(y) &= \sum_{\beta\in\partial B} \varphi(\beta)P_B(y,\beta) \\ &= \sum_{\beta\in\partial B} [f(x,\beta) - \sum_{\alpha\in\partial A} f(\alpha,\beta)P_A(x,\alpha)]P_B(y,\beta) \\ &= \sum_{\beta\in\partial B} f(x,\beta)P_B(y,\beta) - \sum_{\alpha\in\partial A,\beta\in\partial B} f(\alpha,\beta)P_A(x,\alpha)P_B(y,\beta) \end{aligned}$$

Hence, $f(x,y)$ is balanced. Similarly we can prove the other equivalent condition. □

6. BALANCED SEPARATELY SUPERHARMONIC FUNCTIONS

Definition 6.1. A real valued function $u(x,y)$ on $X \times Y$ is said to be balanced separately superharmonic if and only if for any (x,y) in $X \times Y$, $u(x,y)$ is separately superharmonic and

$$\begin{aligned} u(x,y) + \sum_{\alpha\in\partial A,\beta\in\partial B} u(\alpha,\beta)P_A(x,\alpha)P_B(y,\beta) &= \sum_{\alpha\in\partial A} u(\alpha,y)P_A(x,\alpha) \\ &\quad + \sum_{\beta\in\partial B} u(x,\beta)P_B(y,\beta) \end{aligned}$$

Let $\mathfrak{B}$ be the class of balanced separately superharmonic functions in $X \times Y$ and let $\mathfrak{B}^+$ denote the class of non-negative balanced separately superharmonic functions in $X \times Y$. Then

(1) If $f, g \in \mathfrak{B}$, then for non-negative numbers a, b the function $af + bg \in \mathfrak{B}$.
(2) If $v_n \in \mathfrak{B}$ and if $v(x,y) = \lim_{n\to\infty} v_n(x,y)$ exists and is finite for every (x,y) in $X \times Y$, then $v \in \mathfrak{B}$.
(3) If $u(x,y) \in \mathfrak{B}^+$ and if $u(x_0,y_0) = 0$ for some (x_0,y_0) in $X \times Y$, then $u = 0$.
(4) There are some non-negative separately superharmonic functions in $X \times Y$ that are not balanced.
Example. Let ξ_n be the set of vertices in a hyperbolic network. Let $G_{\xi_n}(x)$ be the Green potential in X with harmonic singularity at ξ_n([1, Theorem 9]). Since $G_{\xi_n}(x) \le G_{\xi_n}(\xi_n)$, the function $p(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{G_{\xi_n}(x)}{G_{\xi_n}(\xi_n)}$ as a convergent sum of potentials is a potential in X that is not harmonic at any vertex in X. Hence, $[p(x) - \sum_{\alpha\in\partial A} p(\alpha)P_A(x,\alpha)] > 0$ for any $x \in X$.
Similarly, let $q(y)$ be a potential in Y which is not harmonic at any

vertex in Y. Let $u(x,y) = p(x)q(y)$ which is a separately superharmonic function in $X \times Y$.
Now,

$$[p(x) - \sum_{\alpha \in \partial A} p(\alpha)P_A(x,\alpha)][q(y) - \sum_{\beta \in \partial B} q(\beta)P_B(y,\beta)] > 0$$

so that

$$u(x,y) + \sum_{\alpha \in \partial A, \beta \in \partial B} u(\alpha,\beta)P_A(x,\alpha)P_B(y,\beta) > \sum_{\alpha \in \partial A} u(\alpha,y)P_A(x,\alpha) + \sum_{\beta \in \partial B} u(x,\beta)P_B(y,\beta).$$

Hence, $u(x,y)$ is not a balanced function, that is $u \notin \mathfrak{B}^+$.

LEMMA 6.2. *Let $u(x,y)$ be a non-negative balanced separately superharmonic function in $X \times Y$. Then the non-negative function $\psi(x,y) = \sum_{\alpha \in \partial A} u(\alpha,y) P_A(x,\alpha)$ is superharmonic in Y for fixed x and harmonic in X for fixed y.*

Proof. When y is fixed, $u_y(x)$ is superharmonic in X and $\sum_{\alpha \in \partial A} u(\alpha,y) P_A(x,\alpha)$ is the Poisson modification of $u_y(x)$ at the vertex x so that $\sum_{\alpha \in \partial A} u(\alpha,y) P_A(x,\alpha)$ is harmonic at x. On the other hand, for fixed x, and $\alpha \sim x$, $u^\alpha(y) = u(\alpha,y)$ is superharmonic in Y and $P_A(x,\alpha)$ is a positive number. Hence, $u(\alpha,y)P_A(x,\alpha)$ is superharmonic in Y for fixed x and so is $\sum_{\alpha \in \partial A} u(\alpha,y)P_A(x,\alpha)$. The lemma is proved. $\square$

LEMMA 6.3. *Let $u(x,y)$ be a non-negative balanced separately superharmonic function in $X \times Y$. Let $\varphi(x,y) = u(x,y) - \sum_{\alpha \in \partial A} u(\alpha,y)P_A(x,\alpha)$. Then $\varphi(x,y)$ is non-negative superharmonic in X for fixed y and harmonic in Y for fixed x.*

Proof. By Proposition 5.2 $\varphi(x,y)$ is harmonic in Y for fixed x. When y is fixed $u_y(x)$ is superharmonic in X and by Lemma 6.2 $\sum_{\alpha \in \partial A} u(\alpha,y)P_A(x,\alpha)$ is harmonic at x. This implies $\varphi(x,y)$ is superharmonic at x for fixed y. Since $u(x,y)$ is superharmonic at x for fixed y, $\sum_{\alpha \in \partial A} u(\alpha,y)P_A(x,\alpha) \leq u(x,y)$ so that $\varphi(x,y) \geq 0$. $\square$

LEMMA 6.4. *Let $u(x,y)$ be a non-negative balanced separately superharmonic function in $X \times Y$ and let $\varphi(x,y) = u(x,y) - \sum_{\alpha \in \partial A} u(\alpha,y)P_A(x,\alpha)$. Then $\varphi(x,y) = h(x,y) + p(x,y)$ where $h(x,y)$ is separately harmonic in $X \times Y$; and $p(x,y)$ is harmonic in Y for fixed x and a potential in X for fixed y.*

Proof. Since $\varphi(x,y)$ is a non-negative separately superharmonic function in $X \times Y$, by ([3, Theorem 3.3]) there exists a unique non-negative separately harmonic function $h(x,y)$ in $X \times Y$ such that $h(x,y) \le \varphi(x,y)$; moreover, $h(x,y)$ is the greatest harmonic minorant of $\varphi(x,y)$ for fixed y. Thus, if we write $\varphi(x,y) = h(x,y) + p(x,y)$, then $p(x,y) \ge 0$. Since for fixed y, $h(x,y)$ is the greatest harmonic minorant of $\varphi(x,y)$, $p(x,y)$ is a potential in X for fixed y; further for fixed x, $\varphi(x,y)$ and $h(x,y)$ are harmonic in Y, so that $p(x,y)$ is harmonic in Y for fixed x. □

Let $\mathfrak{F}_1$ be the family of non-negative separately harmonic functions in $X \times Y$, $\mathfrak{F}_2$ be the family of non-negative separately superharmonic functions that are harmonic in X for fixed y and potentials in Y for fixed x. Similarly let $\mathfrak{F}_3$ be the family of non-negative separately superharmonic functions that are potentials in X for fixed y and harmonic in Y for fixed x.

THEOREM 6.5. $\mathfrak{B}^+ = \mathfrak{F}_1 \oplus \mathfrak{F}_2 \oplus \mathfrak{F}_3$.

Proof. Let $u \in \mathfrak{B}^+$. Then by Lemma 6.3 $\varphi(x,y) = u(x,y) - \sum_{\alpha \in \partial A} u(\alpha,y)$ $P_A(x,\alpha)$ is harmonic in Y for fixed x. Now by Lemma 6.4 $u(x,y) = h(x,y) + p(x,y) + \sum_{\alpha \in \partial A} u(\alpha,y) P_A(x,\alpha)$ where $h(x,y)$ is separately harmonic in $X \times Y$; and $p(x,y)$ is harmonic in Y for fixed x and a potential in X for fixed y. Then by Lemma 6.2 the non-negative function $\sum_{\alpha \in \partial A} u(\alpha,y) P_A(x,\alpha)$ is superharmonic in Y for fixed x and harmonic in X for fixed y. As in Lemma 6.4 we have $\sum_{\alpha \in \partial A} u(\alpha,y) P_A(x,\alpha) = h_1(x,y) + q(x,y)$ where $h_1(x,y)$ is separately harmonic in $X \times Y$; and $q(x,y)$ is harmonic in X for fixed y and a potential in Y for fixed x. Hence, $u(x,y) = H(x,y) + q(x,y) + p(x,y)$ where $H(x,y) = h(x,y) + h_1(x,y)$. The families $\mathfrak{F}_1$, $\mathfrak{F}_2$, $\mathfrak{F}_3$ are mutually exclusive. For, suppose $v(x,y) \in \mathfrak{F}_1 \cup \mathfrak{F}_2$ then $v(x,y)$ is non-negative separately harmonic in $X \times Y$ and potential in Y for fixed x. This cannot happen which implies $\mathfrak{F}_1 \cap \mathfrak{F}_2 = \phi$. Similarly we can prove $\mathfrak{F}_2 \cap \mathfrak{F}_3 = \phi$ and $\mathfrak{F}_1 \cap \mathfrak{F}_3 = \phi$. Hence, the uniqueness. □

Let Λ_1 and Λ_2 be the minimal boundaries of X and Y respectively.

LEMMA 6.6. *For any $u \in \mathfrak{F}_2$ there exists a unique measure μ_η on Λ_1 for each $\eta \in Y$ such that*

$$u(x,y) = \sum_{\eta \in Y} [\int_{\Lambda_1} h(x) \mathrm{d}\mu_\eta(h)] G'_\eta(y)$$

for any $(x,y) \in X \times Y$.

Proof. Let $u \in \mathfrak{F}_2$. Then for fixed x, $u(x,y)$ is a positive potential in Y.

Hence, by ([2, Theorem 3.3.1]) $u(x,y) = \sum_{\eta\in Y} \lambda^x(\eta) G'_\eta(y)$, where $\lambda^x(\eta) \geq 0$ is a constant for each η. Write

$$u(x,y) = \sum_{\eta\in Y} \lambda(x,\eta) G'_\eta(y). \tag{1}$$

For any fixed vertex in Y, $u(x,y)$ is harmonic in X; hence

$$\begin{aligned} u(x,y) &= \sum_{\alpha\in\partial A} u(\alpha,y) P_A(x,\alpha) \\ &= \sum_{\alpha\in\partial A} [\sum_{\eta\in Y} \lambda(\alpha,\eta) G'_\eta(y)] P_A(x,\alpha) \\ &= \sum_{\eta\in Y} [\sum_{\alpha\in\partial A} \lambda(\alpha,\eta) P_A(x,\alpha)] G'_\eta(y) \qquad (2) \end{aligned}$$

Now for any $\eta \in Y$, $\lambda(\alpha,\eta) = \lambda^\alpha(\eta)$ is a non-negative constant. Hence, $\sum_{\alpha\in\partial A} \lambda(\alpha,\eta) P_A(x,\alpha)$ is a non-negative constant for fixed x. Thus, for fixed x, the potential $u^x(y)$ has two series expansions ((1) and (2)). But the expansion for a potential to be unique implies $\lambda(x,\eta) = \sum_{\alpha\in\partial A} \lambda(\alpha,\eta) P_A(x,\alpha)$. That is, for any $\eta \in Y$, $\lambda(x,\eta)$ is harmonic in X and $\lambda(x,\eta) \geq 0$. By ([2, Corollary 3.2.16]) there exists a unique measure μ_η on Λ_1 for each $\eta \in Y$ such that

$$u(x,y) = \sum_{\eta\in Y} [\int_{\Lambda_1} h(x) \mathrm{d}\mu_\eta(h)] G'_\eta(y). \quad \square$$

Remark. For any $v \in \mathfrak{F}_3$ there exists a unique Radon measure ν_ξ on Λ_2 for each $\xi \in X$ such that

$$v(x,y) = \sum_{\xi\in X} [\int_{\Lambda_2} h'(y) \mathrm{d}\nu_\xi(h')] G_\xi(x)$$

for any $(x,y) \in X \times Y$.

THEOREM 6.7. *For every function in $\mathfrak{B}^+$, there exist a unique measure λ on $\Lambda_1 \times \Lambda_2$ and two families of uniquely determined associated measures: $\{\mu_\eta\}$ on Λ_1 for each $\eta \in Y$ and $\{\nu_\xi\}$ on Λ_2 for each $\xi \in X$ such that*

$$u(x,y) = \int_{\Lambda_1\times\Lambda_2} h(x) h'(y) \mathrm{d}\lambda(h,h') + \sum_{\eta\in Y} [\int_{\Lambda_1} h(x) \mathrm{d}\mu_\eta(h)] G'_\eta(y) + \sum_{\xi\in X} [\int_{\Lambda_2} h'(y) \mathrm{d}\nu_\xi(h')] G_\xi(x).$$

Proof. If $u \in \mathfrak{B}^+$, then by Theorem 6.5 u can be uniquely written as $u(x,y) = h(x,y) + p(x,y) + q(x,y)$ where h, p and q belong to $\mathfrak{F}_1$, $\mathfrak{F}_2$, and $\mathfrak{F}_3$, respectively. Then by ([3, Theorem 5.6]), Lemma 6.6 and the above Remark there exist a unique measure λ on $\Lambda_1 \times \Lambda_2$ and two families of associated measures uniquely determined: $\{\mu_\eta\}$ on Λ_1 for each $\eta \in Y$ and $\{\nu_\xi\}$ on Λ_2 for each $\xi \in X$ such that

$$u(x,y) = \int\limits_{\Lambda_1 \times \Lambda_2} h(x)h'(y)\mathrm{d}\lambda(h,h') + \sum_{\eta \in Y}[\int\limits_{\Lambda_1} h(x)\mathrm{d}\mu_\eta(h)]G'_\eta(y) + \sum_{\xi \in X}[\int\limits_{\Lambda_2} h'(y)\mathrm{d}\nu_\xi(h')]G_\xi(x). \quad \square$$

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