AN ALGORITHM TO CONSTRUCT THE BASIC ALGEBRA OF A SKEW GROUP ALGEBRA

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We give an algorithm for the computation of the basic algebra Morita equivalent to a skew group algebra of a path algebra by obtaining formulas for the number of vertices and arrows of the new quiver Q_b. We apply this algorithm to compute the basic algebra corresponding to all simple quaternion actions.

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1. INTRODUCTION

Let $Q = (Q_0, Q_1)$ be a finite and connected quiver, with Q_0 denoting the set of vertices and Q_1 the set of arrows between them. Fix an algebraically closed field k. Let kQ be the path algebra of the quiver Q and let G be a finite group whose order is invertible in k.

Assume that the group G acts on kQ by permuting the set of primitive idempotents $\{e \in Q_0\}$ and permuting the set of arrows Q_1.

Consider the skew group algebra $(kQ)G$, that is an associative k-algebra whose underlying k-vector space is spanned by elements of the form αg, with $\alpha \in kQ$ and $g \in G$. The multiplication being defined by

$$\alpha g \cdot \beta h = \alpha g(\beta) gh,$$

for all $\alpha, \beta \in kQ$ and $g, h \in G$. There is a significant amount of literature on the study of skew group algebras, on the relationship between $(kQ)G$ and kQ, on structure and representation types of skew group algebras (see Funes [3]) and on which properties of kQ are inherited by $(kQ)G$ (see for instance [1, 2] and [4]).

In this article, we aim to describe the quiver $Q_b = ((Q_b)_0, (Q_b)_1)$ of the basic algebra Morita equivalent to $(kQ)G$, under some assumptions on the quiver Q.

A description of the basic algebra of a skew group algebra was first given by Reiten and Riedtmann in [4] for the case of cyclic groups. The existence of an appropriate quiver Q_b and a construction of it, in the general case, was given by Demonet in 2010 (see [2]). Although, in principle, the question is solved in [2], the article does not include a wider class of examples. The result presented in this paper is algorithmic, hence gives a construction which is easier to compute. We will apply this algorithm to compute the basic algebra corresponding to all simple quaternion actions.

The drawback of our method is that it works under the following assumption on the quiver Q:

Assume that one can choose a set of representatives of vertex orbits, denoted by O, such that

$$Q = \bigcup_{e,f\in O} \bigcup_{\alpha:e\to f} \mathrm{Orb}(\, e \xrightarrow{\alpha} f\,). \tag{1}$$

From this point on, we assume that every quiver Q satisfies this condition.

The input of the algorithm we present is the skew group algebra $(kQ)G$ and the output is the quiver $Q_b = ((Q_b)_0, (Q_b)_1)$, where the basic algebra of $(kQ)G$ is the path algebra of the quiver Q_b. Suppose Q satisfies condition 1, then the algorithm is as follows.

ALGORITHM 1(Construction of the quiver Q_b)

`Step 1`: According to condition 1, decompose

$$Q = \bigcup_{e,f\in O} \bigcup_{\alpha:e\to f} \mathrm{Orb}(\, e \xrightarrow{\alpha} f\,)$$

`Step 2`: Set $G_e = \mathrm{Stab}_G(e)$, to be the stabilizer subgroup of $e \in O$.

`Step 3`: Calculate $\mathrm{Irr}(G_e)$ a set of primitive orthogonal idempotents of the group algebra kG_e, for each $e \in O$.

`Step 4`:

For: each $e \in O$ and for each $\rho \in Irr(G_e)$
Do: add a vertex to $(Q_b)_0$ labeled by e_ρ.

`Step 5`:

For: each pair (not necessarily distinct) $e, f \in O$
Do:
 For: each orbit of arrows (in Q) $\mathrm{Orb}(\, e \xrightarrow{\alpha} f\,)$
 Do: add as many arrows (in Q_b) from e_ρ to f_σ as the dimension of
 $\langle e_\rho \alpha' f_\sigma \mid \alpha' \in \mathrm{Orb}_{\mathrm{G_e\cap G_f}}(\, e \xrightarrow{\alpha} f\,)\rangle_k$.

If in addition to condition 1 we assume that stabilizing the source and target of an arrow, also stabilizes the arrow, or in other words, one can choose a set of representatives of vertex orbits, denoted by O, such that

$$Q = \bigcup_{e,f\in O} \mathrm{Orb}(\ e \xrightarrow{\alpha} f\ ), \tag{2}$$

the above algorithm can be distilled to the following theorem.

THEOREM 1.1. *Suppose Q satisfies condition 2, then the basic algebra of $(kQ)G$ is the path algebra of the quiver Q_b, where the set of vertices $(Q_b)_0$ can be labeled by the elements of*

$$\bigcup_{e\in O} \bigcup_{\rho\in \mathrm{Irr}(G_e)} e_\rho.$$

Moreover for a pair $e_\rho \in \mathrm{Irr}(G_e)$ and $f_\sigma \in \mathrm{Irr}(G_f)$ and for every orbit of arrows $\mathrm{Orb}(\ e \xrightarrow{\alpha} f\ )$, *there is an arrow in Q_b between the vertices labeled by e_ρ and f_σ if and only if*

$$i_G(\mathrm{Ind}_{G_e}^G(e_\rho(kG_e)), \mathrm{Ind}_{G_f}^G(f_\sigma(kG_f))) \neq 0.$$

Here $\mathrm{Ind}_{G_e}^G$ stands for the induction functor from modules over kG_e to modules over kG and i_G stands for the intertwining number of the two kG-modules. We recall that if V and W are two modules of the group algebra kG the intertwining number of them (considered as kG-modules) is defined by

$$i_G(V, W) = \dim_k \mathrm{Hom}_{kG}(V, W).$$

Remark 1.2. We would like to mention, that the formula defining the number of arrows in the above theorem although it seems complicated at the first glance, it is easy to calculate with, since we have that (see the proof of Theorem 1.1)

$$i_G(\mathrm{Ind}_{G_e}^G(e_\rho(kG_e)), \mathrm{Ind}_{G_f}^G(f_\sigma(kG_f))) \neq 0,$$

if and only if $e_\rho \cdot f_\sigma$ is non-zero in the group algebra kG.

This paper is organized as follows. In Section 2, we give the proofs of Algorithm 1 and Theorem 1.1. In Section 3, we deal with some examples that exist in the literature (see [4]), and we will see that conditions 1 and 2 are not so restrictive as they seem, but rather allow a wide range of useful examples. Finally, in Section 4, we apply our method to compute the basic algebra corresponding to all simple quaternion actions.

Our notation is standard, and for general notions and results we refer the reader to [1].

2. PROOF OF ALGORITHM 1 AND THEOREM 1.1

Let kQ_0 be the subalgebra of kQ generated by the primitive idempotents, and let $kQ_1 \subset kQ$ be the linear subspace spanned by the arrows, regarded as a kQ_0-bimodule. Consider the tensor algebra

$$T(kQ_0, kQ_1) = \bigoplus_{i \geq 0} T_i,$$

endowed with the canonical product, where $T_0 = kQ_0$, and for every $n \in \mathbb{N}$, $T_n = T_{n-1} \otimes_{kQ_0} kQ_1$. It is well known that kQ is canonically isomorphic to $T(kQ_0, kQ_1)$, on which the action of G is graded. By [2] we have that

$$(kQ)G \cong T(kQ_0, kQ_1)G \cong T((kQ_0)G, (kQ_1)G).$$

This isomorphism allows us to compute the basic algebra of $(kQ_0)G$ and $(kQ_1)G$ separately.

We will start by determining the basic algebra of $(kQ_0)G$. Take two representatives $e, f \in O$ of vertex orbits and an orbit of arrows $\mathrm{Orb}(\ e \xrightarrow{\alpha} f\ )$ between them. The following construction must be done for each pair (not necessarily distinct) of vertex orbits ($e, f \in O$) and each orbit of arrows between them.

By [4, Prop. 1.6] we have that $(k\mathrm{Orb}(e))G$ is Morita equivalent to kG_e. It follows that the quiver corresponding to the basic algebra of $(k\mathrm{Orb}(e))G$ has as many vertices as the quiver of the basic algebra of the group algebra kG_e, which we know in general that is equal to the number of conjugacy classes of G_e, or in other words the number of irreducible representations of G_e, namely $|\mathrm{Irr}(G_e)|$. Therefore, the vertices of Q_b can be labeled by

$$(Q_b)_0 = \bigcup_{e \in O} \bigcup_{\rho \in \mathrm{Irr}(G_e)} e_\rho.$$

We continue by determining the basic algebra of $(kQ_1)G$. For this, we have to calculate the number of arrows from the vertices $\{e_\rho,\ \rho \in \mathrm{Irr}(G_e)\}$ to $\{f_\sigma,\ \sigma \in \mathrm{Irr}(G_f)\}$.

Let

$$\{g_{e,\rho} \in kG_e, \quad \rho \in \mathrm{Irr}(G_e)\}$$

and

$$\{g_{f,\sigma} \in kG_f, \quad \sigma \in \mathrm{Irr}(G_f)\}$$

be complete sets of primitive orthogonal idempotents of the group algebras kG_e and kG_f. Then we have that

$$\{eg_{e,\rho} \in (k\mathrm{Orb}(e))G, \quad \rho \in \mathrm{Irr}(G_e)\}$$

and

$$\{fg_{f,\sigma} \in (k\mathrm{Orb}(f))G, \quad \sigma \in \mathrm{Irr}(G_f)\}$$

will be the corresponding sets of primitive idempotents of $(k\mathrm{Orb}(e))G$ and $(k\mathrm{Orb}(f))G$.

In a similar fashion as in [2, 3.1] or [4, Prop 1.6], in order to determine the basic algebra corresponding to $(k\mathrm{Orb}(\ e \xrightarrow{\alpha} f\ ))G$, we have to determine $i(k\mathrm{Orb}(\alpha))i$, where

$$i = \sum_{\rho\in\mathrm{Irr}(G_e)} eg_{e,\rho} + \sum_{\sigma\in\mathrm{Irr}(G_f)} fg_{f,\sigma}.$$

Expanding the above expression we get

$$i(k\mathrm{Orb}(\alpha))i = \langle eg_{e,\rho}g(\alpha)fg_{f,\sigma},\ g \in G, \rho \in \mathrm{Irr}(G_e),\ \sigma \in \mathrm{Irr}(G_f)\rangle_k.$$

Suppose that a generator $eg_{e,\rho}g(\alpha)fg_{f,\sigma}$, of the space above, is non zero, then we must have that

$$g(\alpha) : g(e) \to f,$$

so $g(f) = f$, which means that $g \in G_f$ and moreover, we get that $(g_{e,\rho}g)(e) = e$. One should remember that $g_{e,\rho} \in kG_e$, so it follows that $g(e) = e$. We can conclude that

$$g \in G_e \cap G_f.$$

Putting all these together, we get that

$$i(k\mathrm{Orb}(\alpha))i = \langle g_{e,\rho}\alpha' g_{f,\sigma},\ \alpha' \in \mathrm{Orb}_{\mathrm{G_e\cap G_f}}(\alpha),\ \rho \in \mathrm{Irr}(G_e),\ \sigma \in \mathrm{Irr}(G_f)\rangle_k.$$

Thus for every orbit $\mathrm{Orb}(\ e \xrightarrow{\alpha} f\ )$, the number of arrows in Q_b from the vertex labeled by e_ρ to the vertex labeled by f_σ is equal to the dimension of the space

$$\langle g_{e,\rho}\alpha' g_{f,\sigma} \neq 0,\ \alpha' \in \mathrm{Orb}_{\mathrm{G_e\cap G_f}}(\alpha)\rangle_k.$$

This completes the proof of Algorithm 1.

We now assume that Q satisfies condition 2 and we proceed with the proof of Theorem 1.1.

We step back to the conclusion that $g \in G_e \cap G_f$. Then by condition 2, we have that $g(\alpha)$ must be equal to α and that $g_{e,\rho}(\alpha) \neq 0$ for any $\rho \in \mathrm{Irr}(G_e)$. So we get that

$$i(k\mathrm{Orb}(\alpha))i = \langle g_{e,\rho}(\alpha) \cdot (g_{e,\rho}g_{f,\sigma}),\ \rho \in \mathrm{Irr}(G_e),\ \sigma \in \mathrm{Irr}(G_f)\rangle_k.$$

This means that there is exactly one arrow between the vertex labeled by e_ρ to the vertex labeled by f_σ if and only if

$$g_{e,\rho}g_{f,\sigma} \neq 0.$$

So our problem reduces to determining whether the product of two idempotents is zero or not. This is equivalent to the question whether the corresponding induced kG-modules $\mathrm{Ind}_{G_e}^{G}(g_{e,\rho}kG_e)$ and $\mathrm{Ind}_{G_f}^{G}(g_{f,\sigma}kG_f)$ have or do not have common direct summands, since if they have a common direct summand (as kG-modules), then there exists a nonzero element

$$w \in \mathrm{Ind}_{G_e}^{G}(g_{e,\rho}kG_e) \cap \mathrm{Ind}_{G_f}^{G}(g_{f,\sigma}kG_f),$$

such that

$$g_{e,\rho} \cdot v_1 = w = g_{f,\sigma} \cdot v_2,$$

for some nonzero $v_1 \in \mathrm{Ind}_{G_e}^{G}(g_{e,\rho}kG_e)$ and $v_2 \in \mathrm{Ind}_{G_f}^{G}(g_{f,\sigma}kG_f)$. Then by multiplying from the left by $g_{e,\rho}$, we get

$$0 \neq v_1 = g_{e,\rho} \cdot w = (g_{e,\rho}g_{f,\sigma})v_2,$$

so the product is nonzero.

To test whether two kG-modules share direct summands, one should consider the intertwining number of them i_G, that is the k-dimension of the space of kG-homomorphisms between them. Now if we assume that $g_{e,\rho} \cdot g_{f,\rho} = 0$, then the two induced modules share no common summands, so we have that

$$i_G\left(\mathrm{Ind}_{G_e}^{G} g_{e,\rho}kG_e, \mathrm{Ind}_{G_f}^{G} g_{f,\sigma}kG_f\right) = 0.$$

Otherwise, if their product is nonzero, then by the Frobenius reciprocity, we get that

$$i_G\left(\mathrm{Ind}_{G_e}^{G} g_{e,\rho}kG_e, \mathrm{Ind}_{G_f}^{G} g_{f,\sigma}kG_f\right) = i_G\left(g_{e,\rho}kG_e, \mathrm{Res}_{G_e}^{G}\mathrm{Ind}_{G_f}^{G} g_{f,\sigma}kG_f\right) = 1,$$

since each term is considered as kG_e-modules and $g_{e,\rho}kG_e$ is an irreducible kG_e-module. Putting all together we have the following formula:

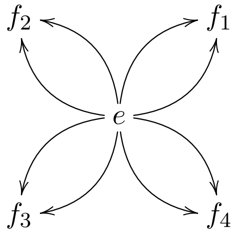
$$i_G\left(\mathrm{Ind}_{G_e}^{G} g_{e,\rho}kG_e, \mathrm{Ind}_{G_f}^{G} g_{f,\sigma}kG_f\right) = \begin{cases} 0, \text{if } g_{e,\rho} \cdot g_{f,\sigma} = 0; \\ 1, \text{if } g_{e,\rho} \cdot g_{f,\sigma} \neq 0 \end{cases}$$

And this completes the proof of Theorem 1.1.

3. EXAMPLES OF THE BASIC ALGEBRA CONSTRUCTION

In this section, we will present examples of the basic algebra construction given by Algorithm 1 and Theorem 1.1. First we reconsider an existing example of determining the basic algebra of a path algebra, to compare the classical results with the ones presented in this paper. This example corresponds to the one treated by Reiten and Riedtmann in [4, 2.6]. One can observe that the two results agree.

Example 3.1. Suppose k is an algebraically closed field and let Q be the following quiver



and let $D_8 = \langle r, t \,|\, r^4 = t^2 = 1, rtr = t \rangle$ be the dihedral group of order 8 acting on the quiver in a natural way, *i.e.* r being the anticlockwise rotation by $\pi/2$ and t the reflection fixing f_1. One can easily check that Q satisfies condition 2.

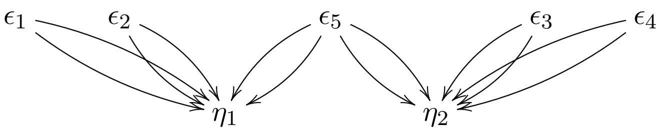
We have two orbits of vertices $\mathrm{Orb}(e) = \{e\}$ and $\mathrm{Orb}(f_1) = \{f_1, f_2, f_3, f_4\}$ and two orbits of arrows. For $\mathrm{Orb}(e)$ we have that the stabilizing subgroup is the hole group $G_e = D_8$. Consider the following set of five primitive orthogonal idempotents of the group algebra kD_8:

$$\begin{cases} \epsilon_1 = (1 + r + r^2 + r^3)(1 - t)/8, \\ \epsilon_2 = (1 + r + r^2 + r^3)(1 + t)/8, \\ \epsilon_3 = (1 - r + r^2 - r^3)(1 + t)/8, \\ \epsilon_4 = (1 - r + r^2 - r^3 - t + rt - r^2t + r^3t)/8, \\ \epsilon_5 = (1 - r^2)/8. \end{cases}$$

For $\mathrm{Orb}(f_1)$ the stabilizer is $G_{f_1} = \{1, t\}$. The corresponding group algebra has the following set of two primitive orthogonal idempotents:

$$\begin{cases} \eta_1 = (1 + t)/2, \\ \eta_2 = (1 - t)/2. \end{cases}$$

Now counting all the nonzero products of the form $\epsilon_i \cdot \eta_j$, we get the arrows from the ϵ_i-s to the η_j-s. A total of 6 arrows for each orbit of edges, that is a total of 12 arrows in Q_b. Putting all together we get that the quiver Q_b of the basic algebra Morita equivalent to $(kQ)D_8$ is the following



We want to remark here that the above construction easily generalizes to any dihedral group D_n and corresponding quiver Q and we get the same result as in [4, 2.6] for the general case.

The next example shows how our construction works in the case of a trivial action.

Remark 3.2. If G acts trivially, then any quiver Q satisfies condition 2.

Example 3.3. Suppose k is an algebraically closed field and let Q be the following quiver

$$\begin{tikzcd} e \arrow[loop left, "\alpha"] \arrow[loop right, "\beta"] \end{tikzcd}$$

Let G be a finite group acting on Q trivially, whose order is invertible in k. Then we have one orbit of vertices $Orb(e) = \{e\}$ and two orbits of arrows $Orb(\alpha) = \{\alpha\}$ and $Orb(\beta) = \{\beta\}$. The hole group G stabilizes e. Suppose that

$$\mathrm{Irr}(G) = \{\epsilon_1, ..., \epsilon_n\},$$

where n is the number of conjugacy classes of G. We also have that

$$i_G(e\epsilon_i(kG), e\epsilon_j(kG)) \neq 0$$

if and only if $i = j$. So in Q_b we get one arrow from ϵ_i to ϵ_i for each orbit of edges, that is two self-pointing arrows for each ϵ_i.
Putting all together, we get that Q_b has the following form.

$$\begin{tikzcd} \epsilon_1 \arrow[loop left] \arrow[loop right] & \epsilon_2 \arrow[loop left] \arrow[loop right] & \cdots & \epsilon_n \arrow[loop left] \arrow[loop right] \end{tikzcd}$$

Our next example shows how our construction distinguish between different orbits of arrows.

Example 3.4. Consider the quivers Q :

$$\begin{tikzcd} e_1 \arrow[r] & f_1 \\ e_2 \arrow[r] & f_2 \\ e_3 \arrow[r] & f_3 \end{tikzcd}$$

and Q' :

$$\begin{tikzcd} e_1 \arrow[dr] \arrow[ddr] & f_1 \\ e_2 \arrow[ur] \arrow[dr] & f_2 \\ e_3 \arrow[uur] \arrow[ur] & f_3 \end{tikzcd}$$

with the permutation group on three letters S_3 acting by permuting vertices e_1, e_2 and e_3 on the one hand, and f_1, f_2 and f_3 on the other hand. Both quivers satisfy condition 2 and they have the following orbit decompositions

$Q = \mathrm{Orb}(\ e_1 \longrightarrow f_1\ )$ and $Q' = \mathrm{Orb}(\ e_1 \longrightarrow f_2\ )$, corresponding to the requirements of condition 2. For Q we have that

$$\mathrm{Stab}_{S_3}(e_1) = \mathrm{Stab}_{S_3}(f_1) = \{(1), (2,3)\} \cong \mathbb{Z}_2.$$

We denote by $\{\epsilon_1, \epsilon_2\}$ the set of orthogonal idempotents of the group algebra of $\mathrm{Stab}_{S_3}(e_1)$ and by $\{\eta_1, \eta_2\}$ the set of orthogonal idempotents of the group algebra of $\mathrm{Stab}_{S_3}(f_1)$. More precisely we have

$$\begin{cases} \epsilon_1 = \eta_1 = ((1) + (2,3))/2; \\ \epsilon_2 = \eta_2 = ((1) - (2,3))/2. \end{cases}$$

There is one orbit of arrows from $\{e_1, e_2, e_3\}$ to $\{f_1, f_2, f_3\}$. Now counting all the nonzero products of the form $\epsilon_i \cdot \eta_j$, we get the arrows from the ϵ_i-s to the η_j-s. Hence Q_b has the following form.

$$\begin{tikzcd} \epsilon_1 \arrow[r] & \eta_1 \\ \epsilon_2 \arrow[r] & \eta_2 \end{tikzcd}$$

For Q' we have that

$$\mathrm{Stab}_{S_3}(e_1) = \mathrm{Stab}_{S_3}(f_1) = \{(1), (1,3)\} \cong \mathbb{Z}_2.$$

We denote by $\{\epsilon_1, \epsilon_2\}$ the set of orthogonal idempotents of the group algebra of $\mathrm{Stab}_{S_3}(e_1)$ and by $\{\eta_1, \eta_2\}$ the set of orthogonal idempotents of the group algebra of $\mathrm{Stab}_{S_3}(f_1)$. More precisely, we have

$$\begin{cases} \epsilon_1 = ((1) + (2,3))/2; \\ \epsilon_2 = ((1) - (2,3))/2. \end{cases}$$

and

$$\begin{cases} \eta_1 = ((1) + (1,3))/2; \\ \eta_2 = ((1) - (1,3))/2. \end{cases}$$

There is one orbit of arrows from $\{e_1, e_2, e_3\}$ to $\{f_1, f_2, f_3\}$. Now counting all the nonzero products of the form $\epsilon_i \cdot \eta_j$, we get the arrows from the ϵ_i-s to the η_j-s. Hence Q'_b has the following form.

$$\begin{tikzcd} \epsilon_1 \arrow[r] \arrow[dr] & \eta_1 \\ \epsilon_2 \arrow[r] \arrow[ur] & \eta_2 \end{tikzcd}$$

4. THE QUATERNION ACTION

In the following section, we deal with all possible simple settings for the quiver Q under condition 2 in the case when the action of the group is given by the quaternion group. We define the quaternion group to be

$$Q_8 = \{1,\ \overline{1},\ I,\ \overline{I},\ J,\ \overline{J},\ K,\ \overline{K}\},$$

satisfying the relations $\overline{1}^2 = 1$ and $I^2 = j^2 = K^2 = IJK = \overline{1}$. In the following sequence of examples each quiver will have two orbits of vertices and one orbit of arrows between these two. By the Orbit-Stabilizer theorem the possible cardinalities for the vertex orbits are $1, 2, 4$ and 8. We will deal with these possible combinations.

Example 4.1. Suppose k is an algebraically closed field with characteristic not equal to two and let Q be the following quiver

$$\begin{tikzcd}
 & f_1 \\
e_1 \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & \\
 & f_2
\end{tikzcd}$$

Let Q_8 be the quaternion group, as defined above, acting on the quiver Q by stabilizing e_1 and elements J and K permuting the vertices f_1 and f_2. We have two orbits of vertices $\mathrm{Orb}(e_1)$ and $\mathrm{Orb}(f_1)$ and one orbit of arrows $\mathrm{Orb}(\alpha)$ We follow the steps of Algorithm 1. For $\mathrm{Orb}(e_1)$ we have that the stabilizing subgroup is the hole group Q_8. Let us consider the following set of five primitive orthogonal idempotents of the group algebra kQ_8:

$$\begin{cases} \tilde{e_1} = (1 + \overline{1} + I + \overline{I} + J + \overline{J} + K + \overline{K})/8, \\ \tilde{e_2} = (1 + \overline{1} + I + \overline{I} - J - \overline{J} - K - \overline{K})/8, \\ \tilde{e_3} = (1 + \overline{1} - I - \overline{I} + J + \overline{J} - K - \overline{K})/8, \\ \tilde{e_4} = (1 + \overline{1} - I - \overline{I} - J - \overline{J} + K + \overline{K})/8, \\ \tilde{e_5} = (1 - \overline{1})/2. \end{cases}$$

For $\mathrm{Orb}(f_1)$ the stabilizer is $\{1, \overline{1}, I, \overline{I}\}$, isomorphic to $\mathbb{Z}_4$. The corresponding group algebra has the following set of two primitive orthogonal idempotents:

$$\begin{cases} \tilde{f_1} = (1 + I + \overline{1} + \overline{I})/4, \\ \tilde{f_2} = (1 + i \cdot I - \overline{1} - i \cdot \overline{I})/4, \\ \tilde{f_3} = (1 - I + \overline{1} - \overline{I})/4, \\ \tilde{f_4} = (1 - i \cdot I - \overline{1} + i \cdot \overline{I})/4. \end{cases}$$

Here i is a 4th root of unity. So we get that the quiver of the basic algebra Morita equivalent to the skew group algebra $(kQ)Q_8$ has as vertices $\tilde{e_1}, ..., \tilde{e_5}$ and $\tilde{f_1}, ..., \tilde{f_4}$ and there is an arrow from $\tilde{e_i}$ to $\tilde{f_j}$ if and only if $\tilde{e_i}\tilde{f_j} \neq 0$ in the group algebra kQ_8. Now counting these nonzero products of idempotents we get the following quiver $(Q)_b$.

$$\begin{tikzcd}
\tilde{e_1} \arrow[r] & \tilde{f_1} \\
\tilde{e_2} \arrow[ur] & \tilde{f_2} \\
\tilde{e_5} \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & \\
\tilde{e_3} \arrow[dr] & \tilde{f_4} \\
\tilde{e_4} \arrow[r] & \tilde{f_3}
\end{tikzcd}$$

Example 4.2. Suppose k is an algebraically closed field with characteristic not equal to two and let Q be the quiver

$$\begin{tikzcd}
 & g_1\ . \\
 & g_2 \\
e_1 \arrow[uur, "\alpha"] \arrow[ur, "\beta"'] \arrow[dr, "\gamma"] \arrow[ddr, "\delta"'] & \\
 & g_3 \\
 & g_4
\end{tikzcd}$$

Now Q_8 is acting on the quiver Q by stabilizing e_1 and permuting the g_i-s in the following way: $I = (g_1, g_2)(g_3, g_4)$, $J = (g_1, g_3)(g_2, g_4)$ and $K = (g_1, g_4)(g_2, g_3)$. Then again we have two orbits of vertices and one orbit of arrows. The stabilizer $\mathrm{Stab}(e_1)$ is again the hole group and we take the same set of primitive orthogonal idempotents as in Example 4.1. The stabilizer $\mathrm{Stab}(g_1)$ is $\{1, \overline{1}\}$, isomorphic to $\mathbb{Z}_2$, hence a suitable set of idempotents for its group algebra is

$$\begin{cases} \tilde{g_1} = (1+\overline{1})/2, \\ \tilde{g_2} = (1-\overline{1})/2. \end{cases}$$

The vertices of the new quiver Q_b will be labeled by $\tilde{e_1}, ..., \tilde{e_5}$ and by $\tilde{g_1},\ \tilde{g_2}$. We draw an arrow from $\tilde{e_i}$ to $\tilde{g_j}$ if and only if $\tilde{e_i} \cdot \tilde{g_j} \neq 0$ in the group algebra kQ_8. Counting the corresponding nonzero products we get that $(kQ)Q_8$ is Morita equivalent to the path algebra of the following quiver.

$$\begin{tikzcd}
 & \tilde{e_1} \arrow[ddl] & \\
 & \tilde{e_2} \arrow[dl] & \\
\tilde{g_1} & \tilde{e_5} \arrow[l] \arrow[r] & \tilde{g_2} \\
 & \tilde{e_3} \arrow[ul] & \\
 & \tilde{e_4} \arrow[uul] &
\end{tikzcd}$$

The next example covers the cases when the size of one of the orbits is exactly 8. In this case the stabilizer is only the identity, resulting only the trivial idempotent, hence none of the expected arrows will become zero in the arrow calculation process. We will deal with the case when the second orbit consists of one element, but the same calculations can be carried out when the size of the second orbit is 2, 4 or 8.

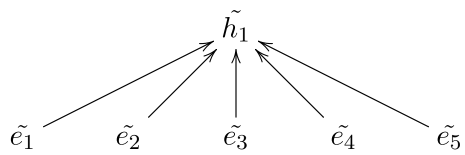
Example 4.3. Suppose k is an algebraically closed field with characteristic not equal to two and let Q be the following quiver

$$\begin{tikzcd}
h_1 & h_2 & h_3 & h_4 & & h_5 & h_6 & h_7 & h_8 \\
 & & & & e_1 \arrow[ullll] \arrow[ulll] \arrow[ull] \arrow[ul] \arrow[ur] \arrow[urr] \arrow[urrr] \arrow[urrrr] & & & &
\end{tikzcd}$$

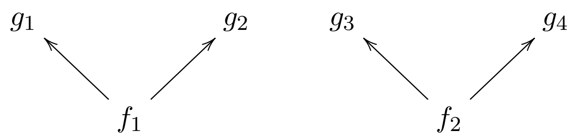
By Cayley's theorem Q_8 can be embedded to the group of permutations on 8 letters $h_1, ..., h_8$, then the group is generated by

$$\begin{cases} I = (h_1, h_3, h_2, h_4)(h_5, h_7, h_6, h_8), \\ J = (h_1, h_5, h_2, h_6)(h_3, h_8, h_4, h_7), \\ K = (h_1, h_7, h_2, h_8)(h_3, h_5, h_4, h_6). \end{cases}$$

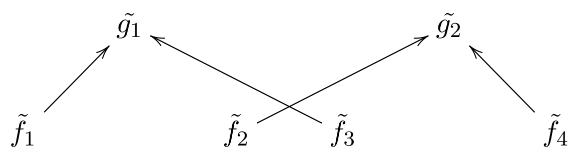
Now we take the action to be the natural one with respect to the above mentioned embedding, namely Q_8 stabilizes e_1 and permutes $h_1, ..., h_8$ in the above way. Now we proceed by the steps of Theorem 1.1. The stabilizer $\mathrm{Stab}(e_1)$ is again the whole group and we take the same set of primitive orthogonal idempotents as in Example 4.1. The stabilizer $\mathrm{Stab}(h_1)$ is just the identity. The vertices of the new quiver Q_b will be labeled by $\tilde{e_1}, ..., \tilde{e_5}$ and by $\tilde{h_1}$. Now each product of idempotents $\tilde{e_i} \cdot \tilde{h_1}$ is non-zero, so we draw one arrow from each $\tilde{e_i}$ to $\tilde{h_1}$. As expected Q_b is the following quiver.



Example 4.4. Suppose k is an algebraically closed field with characteristic not equal to two and consider the following quiver Q.



The action of Q_8 on the $g_1, ..., g_4$ is the same as in Example 4.2 and the action on f_1, f_2 is the same as in Example 4.1. The stabilizer $\mathrm{Stab}(g_1)$ is $\{1, \overline{1}\}$, isomorphic to $\mathbb{Z}_2$, and we take the same set of idempotents $\tilde{g_1}$, $\tilde{g_2}$ as in Example 4.2. The stabilizer $\mathrm{Stab}(f_1)$ is $\{1, \overline{1}, I, \overline{I}\}$, isomorphic to $\mathbb{Z}_4$. For this we take the same set of idempotents $\tilde{f_1}, ..., \tilde{f_4}$ as in Example 4.1. The vertices of the new quiver Q_8 are labeled by $\tilde{f_1}, ..., \tilde{f_4}$ and by $\tilde{g_1}$, $\tilde{g_2}$. We draw an arrow from $\tilde{f_i}$ to $\tilde{g_j}$ if and only if $f_i \cdot g_j \neq 0$ in kQ_8. We get that Q_b is the following quiver.



As seen in the above example, if all the stabilizer subgroups are cyclic, then we get that the number of vertices in Q_b, corresponding to one orbit of vertices, is exactly the cardinality of that stabilizer, since $|\mathrm{Irr}(\mathbb{Z}_n)| = n$. We also get that if one of the stabilizers is isomorphic to $\mathbb{Z}_n$ and the other one is isomorphic to $\mathbb{Z}_m$, then (for each arrow orbit) the total number of arrows in the new quiver Q_b between the two corresponding orbits is exactly the least common multiple of n and m. This phenomenon was already shows by Reiten and Riedtmann in [4, 2.3]. So our result is indeed a direct generalization of the aforementioned one.

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