A FAMILY OF INTEGER SOMOS SEQUENCES

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Somos sequences are sequences of rational numbers defined by a bilinear recurrence relation. Remarkably, although the recurrences describing the Somos sequences are rational, some Somos sequences turn out to have only integer terms. In this paper, a family of Somos 4 sequences is given and it is proved that all Somos 4 sequences associated to Tate normal forms with $h_{-1} = \pm 1$ consist entirely of integers for $n \geq 0$. It is also shown that there are infinitely many squares and infinitely many cubes in Somos 4 sequences associated to Tate normal forms.

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1. INTRODUCTION

For an integer $k \geq 4$, a *Somos k sequence* is a sequence (h_n) which satisfies the recurrence relation

$$h_n h_{n-k} = \sum_{i=1}^{\lfloor k/2 \rfloor} \tau_i h_{n-i} h_{n-k+i}, \qquad \text{for } n \in \mathbb{Z} \tag{1}$$

where the coefficients τ_i and the initial values $h_0, ..., h_{k-1}$ are given integers. Especially, a *Somos*(k) *sequence* is a sequence whose coefficients and initial values are all equal to 1. These sequences were named after Micheal Somos. He first introduced the sequence Somos(6) which begins

$$1, 1, 1, 1, 1, 1, 3, 5, 9, 23, 75, ...$$

and observed that the terms of the sequence consist entirely of integers even though the terms are obtained from a rational recursion, see [19] for more details.

Note that the recurrences describing the Somos sequences involve divisions by another term. It is clear that these sequences turn out to have rational terms. The surprising fact is that there are sequences which contain only integer terms. Indeed, the Somos(k) sequences have only integer terms for $k \leq 7$

but not for $k = 8$ [6, 26]. The question of when a Somos sequence has only integer terms has received much attention in the literature [5–7, 13, 16]. Some history about the integrality properties of these sequences can be found in [6,7]. Gale also mentioned that there are many families of Somos sequences that appear to have only integer terms.

In this work, we are interested in Somos 4 sequences which satisfy a recurrence relation of the form

$$h_{n+2}h_{n-2} = \tau_1 h_{n+1}h_{n-1} + \tau_2 h_n^2, \qquad \text{for } n \in \mathbb{Z} \tag{2}$$

where τ_1, τ_2 are given integers. These sequences are generalisations of elliptic divisibility sequences (EDSs) which were first introduced by M. Ward [30]. He also showed that the terms of EDSs consist entirely of integers if the sequence begins $1, h_2, h_3, h_2c, ...(h_2, h_3, c \in \mathbb{Z})$. For more details see [4, 30].

In [26], Robinson was interested in the properties of the Somos(4) sequence reduced modulo a prime power p^r where $r \geq 1$. Swart [29] extended his results to Somos 4 sequences and proved some of his conjectures. In particular, Somos 4 sequences are quite interesting because of the close relation with elliptic curves. In fact, Somos 4 sequences with the first coefficient square can be expressed in terms of the x-coordinates of the points $(x_n,\ y_n) = Q + nP$ where $P = (0, 0)$ and $Q = (x, y)$ is a suitable point on an elliptic curve. Somos 4 sequences are also closely related to cluster algebras [5], to integrable systems [12] and to continued fractions [20, 21]. Fomin and Zelevinsky [5] used the theory of cluster algebras to prove that all elements of the Somos 4 sequences are Laurent polynomials, *i.e.*,

$$h_n \in \mathbb{Z}[\tau_1, \tau_2, h_1^{\pm 1}, h_2^{\pm 1}, h_3^{\pm 1}, h_4^{\pm 1}]$$

for all $n \in \mathbb{Z}$. Hone and Swart [13] extended the known results on integrality of Somos 4 sequences. They used the relation between Somos 4 sequences and sequences of points on elliptic curves. This gave a stronger Laurent phenomenon for Somos 4 sequences. Hence they obtained integrality results for Somos 4 sequences.

In this paper, using the Tate normal form having one parameter $\alpha \in \mathbb{Z}$ of elliptic curves with torsion points, a family of Somos 4 sequences is given by means of Mazur's theorem. It is shown that all elements of the Somos 4 sequences associated to Tate normal forms with $h_{-1} = \pm 1$ are elements of the ring of polynomials in x, y, α and h_0 with integral coefficients, *i.e.*, $h_n \in \mathbb{Z}[x,\ y,\ \alpha, h_0]$ for all $n \geq 0$.

Our first main theorem shows that all Somos 4 sequences associated to Tate normal forms with $h_{-1} = \pm 1$ consist entirely of integers for $n \geq 0$.

Theorem 1. *Let E_N denote a Tate normal form of an elliptic curve having integral points $P = (0,0)$ and $Q = (x,\ y)$ with P torsion point of maximal order N and $Q + nP \neq O$ for all $n \in \mathbb{Z}$. Let (h_n) denote a Somos 4 sequence associated to a Tate normal form with $h_{-1} = \pm\,1$. Then the Somos 4 sequence (h_n) consists entirely of integers for $n \geq 0$.*

The question of when a term of a Lucas sequence can be square has generated interest in the literature [1, 2, 24, 25]. Similar results concerning cubes were also obtained for specific sequences such as Fibonacci, Lucas and Pell numbers [18, 23]. It has been proved that the only perfect powers in the Fibonnaci sequence are 1, 8 and 144 [3]. Authors wrote many papers when a term of an integer sequence generated by linear recurrence can be a perfect power. However, not so much is known about nonlinear recurrence sequences. In [9, 10], we describe when a term of an elliptic divisibility sequence can be a square or a cube, if one of the first six terms is zero. We [11] extended these results to elliptic divisibility sequences associated to Tate normal forms. Reynolds [22] showed that there are finitely many perfect powers in an elliptic divisibility sequence whose first term is divisible by 2 or 3. The following question arises: Are there finitely or infinitely many squares (perfect powers) in a Somos 4 sequence?

In the second main theorem we gave a partial answer to this question: There are infinitely many squares and infinitely many cubes in Somos 4 sequences associated to Tate normal forms.

Theorem 2. *Let E_N denote a Tate normal form of an elliptic curve and let P, Q be points as defined in Theorem 1.1. Let (h_n) denote a Somos 4 sequence associated to a Tate normal form with $h_{-1} = \pm\,1$. There are infinitely many squares and infinitely many cubes in (h_n).*

In Lemma 3.1, we give explicit formulas for the terms of h_n in terms of x, y, α, h_{-1}, h_0. Theorem 1.1 and Theorem 1.2 are proven by using these formulas.

2. SOME PRELIMINARIES

Let E denote an elliptic curve given by a Weierstrass equation

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \qquad (3)$$

with coefficients $a_1, ..., a_6$ in $\mathbb{Q}$. Let $E(\mathbb{Q})$ denote the set of rational points on E together with a point O, called the point at infinity. The set $E(\mathbb{Q})$ forms an abelian group, with the point at infinity as the identity. For more details on elliptic curves in general, see [27, 28]. One of the most important theorems in the theory of elliptic curves is the Mordell-Weil theorem, which implies that, if $\mathbb{K}$ is a number field containing $\mathbb{Q}$, then $E(\mathbb{K})$ is a finitely generated abelian group. Also, the Mordell-Weil theorem shows that $E_{tors}(\mathbb{K})$, the *torsion subgroup* of $E(\mathbb{K})$, is finitely generated and abelian, hence it is finite, since its generators are of finite order. It is always interesting to characterize the torsion subgroup of a given elliptic curve. The question of a uniform bound on $E_{tors}(\mathbb{Q})$ was studied from the point of view of modular curves by Shimura, Ogg, and others. In 1976, B. Mazur proved the following strongest result which had been conjectured by Ogg:

THEOREM 3 (Mazur [17]). *Let E be an elliptic curve defined over $\mathbb{Q}$. Then the torsion subgroup $E_{tors}(\mathbb{Q})$ is either isomorphic to $\mathbb{Z}/N\mathbb{Z}$ for $N = 1, 2, ..., 10, 12$ or to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2N\mathbb{Z}$ for $N = 1, 2, 3, 4$. Further, each of these groups does occur as an $E_{tors}(\mathbb{Q})$.*

It is well known that all elliptic curves with a torsion point of order N lie in one parameter family where $N \in \{4, ..., 10, 12\}$. The *Tate normal form* of an elliptic curve E with a torsion point $P = (0, 0)$ is defined by

$$E_N : y^2 + (1 - c)xy - by = x^3 - bx^2.$$

If an elliptic curve in Weierstrass form has a point of order $N > 3$, then there is an admissible change of variables that transforms the curve to the Tate normal form. In this case the point $P = (0, 0)$ is a torsion point of maximal order. Especially, if we want a classification with respect to the order of the torsion points, the use of Tate normal form of elliptic curves is unavoidable.

In [15], Kubert listed one parameter family of elliptic curves E defined over $\mathbb{Q}$ with a torsion point of order N where $N = 4, ..., 10, 12$. Most cases can be found in [14]. Also some algorithms are given by using the existence of such a family [8]. To decide when an elliptic curve defined over $\mathbb{Q}$ has a point of given order N, we need a result on parametrization of torsion structures:

THEOREM 4 ([8]). *Every elliptic curve with a point P of order $N = 4, ..., 10, 12$ can be written in the following Tate normal form*

$$E_N : y^2 + (1 - c)xy - by = x^3 - bx^2,$$

with the following relations:
1. *If* $N = 4,\ b = \alpha, c = 0$.
2. *If* $N = 5,\ b = \alpha, c = \alpha$.
3. *If* $N = 6,\ b = \alpha + \alpha^2,\ c = \alpha$.
4. *If* $N = 7,\ b = \alpha^3 - \alpha^2,\ c = \alpha^2 - \alpha$.
5. *If* $N = 8,\ b = (2\alpha - 1)(\alpha - 1),\ c = b/\alpha$.
6. *If* $N = 9,\ c = \alpha^2(\alpha - 1),\ b = c(\alpha(\alpha - 1) + 1)$.

7. If $N = 10$, $c = (2\alpha^3 - 3\alpha^2 + \alpha)/(\alpha - (\alpha - 1)^2)$, $b = c\alpha^2/(\alpha - (\alpha - 1)^2)$.
8. If $N = 12$, $c = (3\alpha^2 - 3\alpha + 1)(\alpha - 2\alpha^2)/(\alpha - 1)^3$, $b = c(-2\alpha^2 + 2\alpha - 1)/(\alpha - 1)$.

Theorem 2.2 states that, if any elliptic curve has a point of finite order then this curve is birationally equivalent to one of the Tate normal forms given in the theorem above. Therefore, in this work, we are only interested in the elliptic curves in Tate normal forms with one parameter $\alpha \in \mathbb{Z}$.

The relation between an elliptic curve and a Somos 4 sequence is established independently by N. Elkies (for more details see [19]) and N. Stephens. In [29], some unpublished works of N. Stephens are given. See also [12] for a different approach.

Let $P = (0, 0)$ and $Q = (x, y)$ be integral points on E as in Theorem 1.1. Then the terms of Somos 4 sequence (h_n) can be defined as follows: Let h_{-1} and h_0 arbitrary non-zero integers and

$$h_{n+1} = -\frac{x_n h_n^2}{h_{n-1}} \tag{4}$$

for all $n \geq 0$ (see [29] for more details). The following result due to Nelson Stephens.

THEOREM 5 ([29]). *Let E denote an elliptic curve given by a Weierstrass equation*

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x \tag{5}$$

with integral coefficients $a_1, ..., a_4$. Let $P = (0, 0)$ and $Q = (x, y)$ be integral points on E such that $Q + nP \neq O$ for all $n \in \mathbb{Z}$ and write $Q + nP = (x_n, y_n)$. Then the coefficients τ_1, τ_2 and the initial values of the Somos 4 sequence associated to the elliptic curve E given by

$$\tau_1 = a_3^2, \quad \tau_2 = a_4(a_4 + a_1a_3) - a_3^2a_2$$

and h_{-1} and h_0 arbitrary non-zero integers,

$$h_1 = -xh_0^2/h_{-1}, h_2 = -(a_4x - a_3y)h_0^3/h_{-1}^2.$$

Let E_N denote a Tate normal form of an elliptic curve E with a point P of order N. We can use E_N to give general terms of associated Somos 4 sequences. In Theorem 2.3 we assume that the coefficients of E_N are chosen to lie in $\mathbb{Z}$. However for $N = 8$, 10 or 12, E_N has rational coefficients. In these cases, we transform E_N into a birationally equivalent curve E'_N having an equation with integral coefficients. The equations of the birationally equivalent curves for $N = 8$, 10 or 12 are given as follows.

$$E'_8 \; : \; y^2 + (\alpha - \beta)xy - \alpha^3\beta y = x^3 - \alpha^2\beta x^2, \tag{6}$$

(7) E'_{10} : $y^2+(\zeta^2-\alpha\beta\zeta)xy-\alpha^3\beta\zeta^4y = x^3-\alpha^3\beta\zeta^2x^2,$
(8) E'_{12} : $y^2+(\alpha-1)((\alpha-1)^3-\lambda)xy-(\alpha-1)^8\lambda\theta y = x^3-(\alpha-1)^4\lambda\theta x^2,$

where

$$\begin{aligned} \beta &= (2\alpha-1)(\alpha-1), \\ \zeta &= \alpha-(\alpha-1)^2, \\ \lambda &= (3\alpha^2-3\alpha+1)(\alpha-2\alpha^2), \\ \theta &= 2\alpha-2\alpha^2-1. \end{aligned} \tag{9}$$

From now on, for simplicity of notation, we continue to write E_8, E_{10}, E_{12} for E'_8, E'_{10}, E'_{12}, respectively.

3. PROOF OF THEOREM 1.1

Let P and Q be points as in Theorem 1.1, and let h_{-1}, h_0 be arbitrary non-zero integers. Let $\beta, \zeta, \lambda, \theta$ be as in (2.7) and let

$$\gamma = \alpha^2(\alpha-1)(\alpha^2-\alpha+1).$$

We present below all the initial values h_1, h_2 and the coefficients τ_1, τ_2 of the Somos 4 sequences associated to E_N by using Theorem 2.3.

Table 1

The values τ_1, τ_2, h_1 and h_2

N	τ_1	τ_2	h_1	h_2
4	α^2	α^3	$-xh_{-1}^{-1}h_0^2$	$-\alpha yh_{-1}^{-2}h_0^3$
5	α^2	α^3	$-xh_{-1}^{-1}h_0^2$	$-\alpha yh_{-1}^{-2}h_0^3$
6	$\alpha^2(\alpha+1)^2$	$\alpha^3(\alpha+1)^3$	$-xh_{-1}^{-1}h_0^2$	$-\alpha(\alpha+1)yh_{-1}^{-2}h_0^3$
7	$\alpha^4(\alpha-1)^2$	$\alpha^6(\alpha-1)^3$	$-xh_{-1}^{-1}h_0^2$	$-\alpha^2(\alpha-1)yh_{-1}^{-2}h_0^3$
8	$\alpha^6\beta^2$	$\alpha^8\beta^3$	$-xh_{-1}^{-1}h_0^2$	$-\alpha^3\beta yh_{-1}^{-2}h_0^3$
9	γ^2	γ^3	$-xh_{-1}^{-1}h_0^2$	$-\gamma yh_{-1}^{-2}h_0^3$
10	$\alpha^6\beta^2\zeta^8$	$\alpha^9\beta^3\zeta^{10}$	$-xh_{-1}^{-1}h_0^2$	$-\alpha^3\beta\zeta^4 yh_{-1}^{-2}h_0^3$
12	$(\alpha-1)^{16}\lambda^2\theta^2$	$(\alpha-1)^{20}\lambda^3\theta^3$	$-xh_{-1}^{-1}h_0^2$	$-(\alpha-1)^8\lambda\theta yh_{-1}^{-2}h_0^3$

We consider only the case $N = 8$. The remaining cases can be dealt with similarly. If $N = 8$ then the coefficients τ_1, τ_2 and the initial values h_1, h_2 of the Somos 4 sequences associated to E_8 are

$$\tau_1 = \alpha^6\beta^2, \quad \tau_2 = \alpha^8\beta^3$$

and

$$h_1 = -xh_0^2/h_{-1}, \qquad h_2 = -\alpha^3\beta yh_0^3/h_{-1}^2$$

as shown in Table 1. Using the relation (2) we obtain

$$h_3 = \alpha^8\beta^3(x^2-\alpha y)h_0^4/h_{-1}^3,$$

$$h_4 = -\alpha^{14}\beta^5(x^3 - \alpha xy - y^2)h_0^5/h_{-1}^4,$$

and

$$(10) \quad h_5 = -\alpha^{23}\beta^8(\alpha\beta x^4 + x^3y - 2\alpha^2\beta x^2y - \alpha xy^2 + \alpha^3\beta y^2 - y^3)h_0^6/xh_{-1}^5.$$

Note that both h_3 and $h_4 \in \mathbb{Z}[\alpha, x, y, h_{-1}^{\pm 1}, h_0]$, but $h_5 \notin \mathbb{Z}[\alpha, x, y, h_{-1}^{\pm 1}, h_0]$ since x appears in the denominator of h_5. Now using the equation (6) we see that x divides the numerator of h_5. The equation (10) may therefore be rewritten as

$$h_5 = -\alpha^{23}\beta^9(\alpha x^3 - \alpha^2 xy - y^2)h_0^6/h_{-1}^5.$$

Similarly using equation (2) we find that

$$\begin{aligned}(11)\ h_6 \ &= \ \alpha^{33}\beta^{12}(-x^6 + \alpha^2\beta x^5 + 2\alpha x^4y + 2x^3y^2 - 2\alpha^3\beta x^3y - \alpha\beta x^2y^2 \\ &\quad -\alpha^2x^2y^2 + \alpha^4\beta xy^2 - 2\alpha xy^3 + \alpha^2\beta y^3 - y^4)h_0^7/yh_{-1}^6.\end{aligned}$$

In this case, the value y appears in the denominator of h_6. Now applying (6) to (11), we obtain

$$h_6 = \alpha^{33}\beta^{13}(\alpha^2\beta x^3 + (\alpha^2 - \alpha - \beta)x^2y - \alpha^3\beta xy - (\alpha^3 - \alpha^2)y^2)h_0^7/h_{-1}^6.$$

In the same manner we see that

$$\begin{aligned}h_7 \ &= \ -\alpha^{45}\beta^{18}((\alpha^4\beta - \alpha^5\beta)x^3 + (\alpha^4 - \alpha^3 + \alpha^3\beta - 2\alpha^2\beta)x^2y - (\alpha^3 - \alpha^2 \\ &\quad -\beta)xy^2 + (\alpha^6\beta - \alpha^5\beta)xy - (\alpha^5 - \alpha^4 - \alpha^3\beta)y^2)h_0^8/h_{-1}^7.\end{aligned}$$

Now since P is a point of order 8 every subsequent term of the sequence can be expressed as products of the previous 9 terms. Thus, the general term of the Somos 4 sequence associated to E_8 can be given as

$$\begin{aligned}(12) \quad h_n \ &= \ \varepsilon\alpha^{\{(15n^2-p)/16\}}(\alpha-1)^{\{(7n^2-6n-q)/16\}}(2\alpha-1)^{\{(3n^2-r)/8\}} \\ &\quad \times P_8(\alpha, x, y)[Q_8(\alpha, x, y)]^{\{(n-m)/8\}}h_{-1}^{-n}h_0^{n+1};\end{aligned}$$

where

$$\varepsilon = \begin{cases} +1 & \text{if } n \equiv 0, 3, 6, 8, 9, 13, 14, 15\ (16) \\ -1 & \text{if } n \equiv 1, 2, 4, 5, 7, 10, 11, 12\ (16), \end{cases}$$

$$p = \begin{cases} 0 & \text{if } n \equiv 0\ (8) \\ 7 & \text{if } n \equiv 3, 5\ (8) \\ 12 & \text{if } n \equiv 2, 6\ (8) \\ 15 & \text{if } n \equiv 1, 7\ (8) \\ 16 & \text{if } n \equiv 4\ (8), \end{cases} \quad q = \begin{cases} 0 & \text{if } n \equiv 0, 2\ (8) \\ -3 & \text{if } n \equiv 3\ (8) \\ 8 & \text{if } n \equiv 4, 6\ (8) \\ 1 & \text{if } n \equiv 1, 5\ (8) \\ 13 & \text{if } n \equiv 7\ (8), \end{cases} \quad r = \begin{cases} 0 & \text{if } n \equiv 0\ (8) \\ 3 & \text{if } n \equiv 1, 3, 5, 7\ (8) \\ 4 & \text{if } n \equiv 2, 6\ (8) \\ 8 & \text{if } n \equiv 4\ (8), \end{cases}$$

$$P_8(\alpha,x,y)=\begin{cases} 1 & \text{if } n\equiv 0,7\ (8)\\ x & \text{if } n\equiv 1\ (8)\\ y & \text{if } n\equiv 2\ (8)\\ x^2-\alpha y & \text{if } n\equiv 3\ (8)\\ x^3-\alpha xy-y^2 & \text{if } n\equiv 4\ (8)\\ \alpha x^3-\alpha^2xy-y^2 & \text{if } n\equiv 5\ (8)\\ \alpha^2\beta x^3-(\alpha-1)^2x^2y-\alpha^3\beta xy-\alpha^2(\alpha-1)y^2, & \text{if } n\equiv 6\ (8), \end{cases}$$

$$Q_8=-\alpha^4\beta(\alpha-1)x^3+2\alpha^2(\alpha-1)^3x^2y+\alpha^5\beta(\alpha-1)xy-(\alpha-1)^3xy^2 \\ +\alpha^3(\alpha-1)^2y^2,$$

and

$$m=\begin{cases} k & \text{if } n\equiv k\ (8) \text{ and } k\neq 7\\ -1 & \text{if } n\ \equiv k\ (8) \text{ and } k=7. \end{cases}$$

As an example, the 1222^{th}, ..., 1231^{th} terms of the Somos 4 sequences associated to E_8 can be obtained by using the general term formula (12). The values in (12) can be found in Table 2.

Table 2

The values in (12) for the 1222^{th}, ..., 1231^{th} terms

n	$\equiv (16)$	$\equiv (8)$	ε	p	q	r	$P_8(\alpha,x,y)$	m
1222	6	6	+1	12	8	4	$\alpha^2\beta x^3-(\alpha-1)^2x^2y$ $-\alpha^3\beta xy-\alpha^2(\alpha-1)y^2$	6
1223	7	7	−1	15	13	3	1	−1
1224	8	0	+1	0	0	0	1	0
1225	9	1	+1	15	1	3	x	1
1226	10	2	−1	12	0	4	y	2
1227	11	3	−1	7	−3	3	$x^2-\alpha y$	3
1228	12	4	−1	16	8	8	$x^3-\alpha xy-y^2$	4
1229	13	5	+1	7	1	3	$\alpha x^3-\alpha^2xy-y^2$	5
1230	14	6	+1	12	8	4	$\alpha^2\beta x^3-(\alpha-1)^2x^2y$ $-\alpha^3\beta xy-\alpha^2(\alpha-1)y^2$	6
1231	15	7	+1	15	13	3	1	−1

Thus by using the equation (12) we find the 1222th term in the sequence as

$$h_{1222} = \varepsilon\alpha^{\{(15\cdot 1222^2-p)/16\}}(\alpha-1)^{\{(7\cdot 1222^2-6\cdot 1222-q)/16\}}(2\alpha-1)^{\{(3\cdot 1222^2-r)/8\}} \\ \times P_8(\alpha,x,y)[Q_8(\alpha,x,y)]^{\{(1222-m)/8\}}h_{-1}^{-n}h_0^{n+1}.$$

Consider the elliptic curve

$$E_8: y^2-199xy+138510y=x^3-15390x^2,$$

for $\alpha = -9$ and the points $P = (0, 0)$, $Q = (-210, -3900)$ on this curve. Now, the 1222th term of the Somos 4 sequence associated to E_8 is given by

$$\begin{aligned} h_{1222} &= (+1)(-9)^{\{(15\cdot 1222^2-12)/16\}}(-10)^{\{(7\cdot 1222^2-6\cdot 1222-8)/16\}} \\ &\quad (-19)^{\{(3\cdot 1222^2-4)/8\}}\times P_8(-9,-210,-3900) \\ &\quad [Q_8(-9,-210,-3900)]^{\{(1222-6)/8\}} \\ &= -2^{65431}\ 3^{2800365}\ 5^{653923}\ 19^{559981} \end{aligned}$$

for $h_{-1} = h_0 = 1$. In the same manner, the other nine terms are obtained as follows:

$$\begin{aligned} h_{1223} &= -2^{655299}\ 3^{2804949}\ 5^{654993}\ 19^{560898}, \\ h_{1224} &= 2^{656370}\ 3^{2809539}\ 5^{656064}\ 19^{561816}, \\ h_{1225} &= -2^{657442}\ 3^{2814130}\ 5^{657136}\ 7\ 19^{562734}, \\ h_{1226} &= -2^{658515}\ 3^{2818726}\ 5^{658209}\ 13\ 19^{563653}, \\ h_{1227} &= -2^{659589}\ 3^{2823327}\ 5^{659283}\ 19^{564573}, \\ h_{1228} &= 2^{660664}\ 3^{2827929}\ 5^{660358}\ 19^{565494}, \\ h_{1229} &= -2^{661740}\ 3^{2832537}\ 5^{661433}\ 19^{566415}, \\ h_{1230} &= -2^{662819}\ 3^{2837148}\ 5^{662509}\ 19^{567337}, \\ h_{1231} &= 2^{663894}\ 3^{2841762}\ 5^{663586}\ 19^{568260}. \end{aligned}$$

The general terms of Somos 4 sequences associated to Tate normal forms are given in the following lemma. For the convenience of the reader, we have given the polynomials $P_N(\alpha, x, y)$, $Q_N(\alpha, x, y)$ and the values $\varepsilon, p, q, r, s, t, m$ in Appendix A.

Lemma 1. *Let E_N be a Tate normal form of an elliptic curve. Let P and Q be points on a Tate normal form as in Theorem 1.1 and let $\zeta, \lambda, \theta, \gamma$ be as defined above. Let (h_n) be a Somos 4 sequence associated to a Tate normal form with the initial values and the coefficients as above. Then the general terms of (h_n) are given by the following formulas:*

1. If $N = 4$,

$$h_n = \varepsilon\alpha^{\{(3n^2-p)/8\}}P_4(\alpha, x, y)[Q_4(\alpha, x, y)]^{\{(n-m)/4\}}h_{-1}^{-n}h_0^{n+1},$$

2. if $N = 5$,

$$h_n = \varepsilon\alpha^{\{(2n^2-p)/5\}}P_5(\alpha, x, y)[Q_5(\alpha, x, y)]^{\{(n-m)/5\}}h_{-1}^{-n}h_0^{n+1},$$

3. if $N = 6$,

$$h_n = \varepsilon\alpha^{\{(5n^2-2n-p)/12\}}(\alpha+1)^{\{(n^2-q)/3\}}P_6(\alpha, x, y)[Q_6(\alpha, x, y)]^{\{(n-m)/6\}}h_{-1}^{-n}h_0^{n+1},$$

4. if $N = 7$,

$$h_n = \varepsilon\alpha^{\{(5n^2-p)/7\}}(\alpha-1)^{\{(3n^2-n-q)/7\}}P_7(\alpha,x,y)[Q_7(\alpha,x,y)]^{\{(n-m)/7\}}h_{-1}^{-n}h_0^{n+1},$$

5. if $N = 8$,

$$\begin{aligned} h_n &= \varepsilon\alpha^{\{(15n^2-p)/16\}}(\alpha-1)^{\{(7n^2-6n-q)/16\}}(2\alpha-1)^{\{(3n^2-r)/8\}} \\ &\quad \times P_8(\alpha,x,y)[Q_8(\alpha,x,y)]^{\{(n-m)/8\}}h_{-1}^{-n}h_0^{n+1}, \end{aligned}$$

6. if $N = 9$,

$$\begin{aligned} h_n &= \varepsilon\alpha^{\{(7n^2-2n-p)/9\}}(\alpha-1)^{\{(4n^2-2n-q)/9\}}(\alpha^2-\alpha+1)^{\{(n^2-r)/3\}} \\ &\quad \times P_9(\alpha,x,y)[Q_9(\alpha,x,y)]^{\{(n-m)/9\}}h_{-1}^{-n}h_0^{n+1} \end{aligned}$$

7. if $N = 10$,

$$\begin{aligned} h_n &= \varepsilon\alpha^{\{(21n^2-p)/20\}}(\alpha-1)^{\{(9n^2-2n-q)/20\}}(2\alpha-1)^{\{(2n^2-r)/5\}}\zeta^{\{(5n^2-s)/4\}} \\ &\quad \times P_{10}(\alpha,x,y)[Q_{10}(\alpha,x,y)]^{\{(n-m)/10\}}h_{-1}^{-n}h_0^{n+1}, \end{aligned}$$

8. if $N = 12$,

$$\begin{aligned} h_n &= \varepsilon\alpha^{\{(n^2-2n-p)/12\}}(\alpha-1)^{\{(59n^2-q)/24\}}(2\alpha-1)^{\{(n^2-r)/24\}} \\ &\quad \lambda^{\{(3n^2-s)/8\}}\theta^{\{(n^2-t)/3\}}\times P_{12}(\alpha,x,y)[Q_{12}(\alpha,x,y)]^{\{(n-m)/12\}}h_{-1}^{-n}h_0^{n+1}, \end{aligned}$$

where $P_N(\alpha,x,y)$, $Q_N(\alpha,x,y)$ *are polynomials in* $\mathbb{Z}[\alpha,x,y]$.

Proof. We give the proof only for the case $N = 8$ by induction on n as follows. It is clear that the result is true for $n = 7$. Hence we assume that $n > 7$. First suppose that $n \equiv 0 \ (8)$, $n \not\equiv 0 \ (16)$ and (12) is true for $n+1$. Then we have

$$h_{n+2} = -\alpha^{60k^2+30k+3}(\alpha-1)^{28k^2+11k+1}(2\alpha-1)^{24k^2+12k+1}y\ Q_8^k\ h_{-1}^{-(8k+2)}h_0^{8k+3}$$

($k \in \mathbb{N}$) by (12). Indeed, we see that

$$\begin{aligned} h_{n-2} &= \alpha^{60k^2-30k+3}(\alpha-1)^{28k^2-17k+2}(2\alpha-1)^{24k^2-12k+1} \\ &\quad \times(\alpha^2\beta x^3-(\alpha-1)^2x^2y-\alpha^3\beta xy-\alpha^2(\alpha-1)y^2)\ Q_8^{k-1}h_{-1}^{-(8k-2)}h_0^{8k-1} \\ h_n &= \alpha^{60k^2}(\alpha-1)^{28k^2-3k}(2\alpha-1)^{24k^2}Q_8^k\ h_0^{8k+1}h_{-1}^{-8k} \\ h_{n-1} &= -\alpha^{60k^2-15k}(\alpha-1)^{28k^2-10k}(2\alpha-1)^{24k^2-6k}\ Q_8^k\ h_{-1}^{-(8k-1)}h_0^{8k} \\ h_{n+1} &= \alpha^{60k^2+15k}(\alpha-1)^{28k^2+4k}(2\alpha-1)^{24k^2+6k}\ x\ Q_8^k\ h_{-1}^{-(8k+1)}h_0^{8k+2} \end{aligned}$$

Substituting these expressions and $\tau_1 = \alpha^6\beta^2$, $\tau_2 = \alpha^8\beta^3$ into (2) and then using equation (6) we obtain

$$h_{n+2} = -\alpha^{60k^2+30k+3}(\alpha-1)^{28k^2+11k+1}(2\alpha-1)^{24k^2+12k+1}y\ Q_8^k\ h_{-1}^{-(8k+2)}h_0^{8k+3}.$$

A similar result can be obtained when $n \equiv 0$ (16). Thus we proved the conclusion (12) is true for $n+2$ which completes the proof for $n \equiv 0$ (8).

The remaining parts of the theorem can be proved in a similar manner. □

From Lemma 3.1 we deduce that any Somos 4 sequence associated to a Tate normal form is determined by the values $\alpha, x, y, h_{-1}, h_0$.

COROLLARY 1. *Let (h_n) be a Somos 4 sequence associated to a Tate normal form with the initial values and the coefficients as above. Then each h_n can be expressed as elements of the ring*

$$\mathcal{R} = \mathbb{Z}[\alpha, x, y, h_{-1}^{\pm 1}, h_0].$$

In particular if $h_{-1} = \pm 1$ the Somos 4 sequence consists entirely of integers for $n \geq 0$.

Remark 1. It is also possible to use the recursion (2) to extend the Somos 4 sequences to negative indices. However, for $n < 0$ the terms of the Somos 4 sequences associated to Tate normal forms may not be integral. For instance, consider the elliptic curve

$$E_{12} : y^2 + 586xy - 948480y = x^3 - 59280x^2,$$

and the points $P = (0, 0)$, $Q = (-21945, 9828225)$ on this curve. The Somos 4 sequence (h_n) associated to E_{12} is the sequence with coefficients

$$\tau_1 = 899614310400, \tau_2 = 53329136320512000$$

and the initial values

$$h_{-1} = h_0 = 1, h_1 = 21945, h_2 = -9321874848000$$

which begins

$$1, 1, 21945, -9321874848000, 172963147768509136896000000, ...$$

the sequence extends backward as

$$..., \frac{757510329397973625077760000000}{14641}, \frac{2806796648448000}{1331}, \frac{-948480}{121}, 1, 1, 21945, ...$$

4. CASES $N = 2, 3$

There is no Tate normal form of an elliptic curve with the torsion point of order two or three, but Kubert in [15] listed that the elliptic curves with torsion point of order two or three are

$$E_2 : y^2 = x^3 + a_2x^2 + a_4x \tag{13}$$

and

$$E_3 : y^2 + a_1xy + a_3y = x^3, \tag{14}$$

respectively. Let h_{-1}, h_0 arbitrary nonzero integers. The initial values h_1, h_2 and the coefficients τ_i of the Somos 4 sequences associated to the elliptic curves E_2 and E_3 are

$$\tau_1 = 0,\ \tau_2 = a_4^2 \text{ and } h_1 = -xh_{-1}^{-1}h_0^2,\ h_2 = -a_4xh_{-1}^{-2}h_0^3\ ,$$

and

$$\tau_1 = a_3^2,\ \tau_2 = 0 \text{ and } h_1 = -xh_{-1}^{-1}h_0^2,\ h_2 = a_3yh_{-1}^{-2}h_0^3,$$

respectively.

Under these considerations, an easy computation gives the general terms of these sequences.

THEOREM 6. *Let E_N be an elliptic curve as in (13) or (14) and let P, Q be points on E_N as in Theorem 1.1. Let (h_n) be a Somos 4 sequence associated to E_N with the initial values and the coefficients as above. Then the general terms of (h_n) are given by the following formulas:*
i. If $N = 2$,

$$h_n = \varepsilon a_4^{\{(n^2-p)/4\}} x^{\{(n-q)/2\}} h_{-1}^{-n} h_0^{n+1}$$

where

$$\varepsilon = \begin{cases} +1 & \text{if } n \equiv 0, 3\ (4) \\ -1 & \text{if } n \equiv 1, 2\ (4), \end{cases}$$

$$p = \begin{cases} 0 & \text{if } n \equiv 0\ (2) \\ 1 & \text{if } n \equiv 1\ (2), \end{cases} \qquad q = \begin{cases} 0 & \text{if } n \equiv 0\ (2) \\ -1 & \text{if } n \equiv 1\ (2). \end{cases}$$

ii. If $N = 3$,

$$h_n = \varepsilon a_3^{\{(n^2-p)/3\}} x^q y^{\{(n-r)/3\}} h_{-1}^{-n} h_0^{n+1}$$

where

$$\varepsilon = \begin{cases} +1 & \text{if } n \equiv 0, 2\ (3) \\ -1 & \text{if } n \equiv 1\ (3), \end{cases}$$

$$p = \begin{cases} 0 & \text{if } n \equiv 0\ (3) \\ 1 & \text{if } n \equiv 1, 2\ (3), \end{cases} \quad q = \begin{cases} 0 & \text{if } n \equiv 0, 2\ (3) \\ 1 & \text{if } n \equiv 1\ (3), \end{cases} \quad r = \begin{cases} 0 & \text{if } n \equiv 0\ (3) \\ 1 & \text{if } n \equiv 1\ (3) \\ -1 & \text{if } n \equiv 2\ (3). \end{cases}$$

5. PROOF OF THEOREM 1.2

Lemma 3.1 gives us the general terms of Somos 4 sequences associated to Tate normal forms. So, we can say there are infinitely many squares and infinitely many cubes in (h_n) by using these explicit formulas. The symbols $\square$ and C mean a square and a cube of a non-zero rational number.

Lemma 2. *Let (h_n) denote a Somos 4 sequence associated to a Tate normal form with $h_{-1} = \pm 1$.*
i. If $n \equiv -1\ (2N)$, then $h_n = \square$ for every non-zero α, x, y,
ii. if $n \equiv -1\ (3N)$, then $h_n = C$ for every non-zero α, x, y,
where $N \in \{4, ..., 10, 12\}$.

Proof. Let (h_n) be a Somos 4 sequence associated to a Tate normal form with $h_{-1} = \pm 1$. Lemma 3.1 shows that each term of (h_n) can be expressed as products of the polynomials $p(\alpha, x, y, h_0)$. It is easy to check that these polynomials are pairwise relatively prime. This implies that if each factor of the product is a square then h_n is a square. Clearly, the same is true for the cube case. We give the proof only for the case $N = 8$ based on this fact.
For (i), if $n \equiv -1\ (16)$ then $n = 16k - 1 (k \in \mathbb{N})$. Thus we have

$$h_n = \alpha^{30k(8k-1)} (\alpha - 1)^{4k(28k-5)} (2\alpha - 1)^{12k(8k-1)}\ Q_8^{2k}\ h_0^{16k},$$

by (12). Hence $h_n = \square$.
For(ii), if $n \equiv -1\ (24)$ then $n = 24k - 1 (k \in \mathbb{N})$. So we have

$$h_n = \varepsilon \alpha^{45k(12k-1)} (\alpha - 1)^{6k(42k-5)} (2\alpha - 1)^{18k(12k-1)}\ Q_8^{3k}\ h_0^{24k}.$$

Therefore $h_n = C$. The remaining cases can be proved in the same way as above. $\square$

Lemma 5.1 tells us that there are infinitely many squares and infinitely many cubes in (h_n) which proves the Theorem 1.2.

Remark 2. The same conclusion can be drawn for the cases $N = 2$ or 3, *i.e.*, let (h_n) denote a Somos 4 sequence associated to E_N with $h_{-1} = \pm 1$,
i. if $n \equiv -1\ (2N)$, then $h_n = \square$ for every non-zero α, x, y,
ii. if $n \equiv -1\ (3N)$, then $h_n = C$ for every non-zero α, x, y.

APPENDIX A

The polynomials $P_N(\alpha, x, y)$, $Q_N(\alpha, x, y)$ and the values $\varepsilon, p, q, r, s, t, m$ in Lemma 3.1 are given as follows. 1. If $N = 4$,

$$P_4 = \begin{cases} 1 & \text{if } n \equiv 0, 3\ (4) \\ x & \text{if } n \equiv 1\ (4) \\ y & \text{if } n \equiv 2\ (4), \end{cases} \qquad Q_4 = x^2 - y,$$

and

$$\varepsilon = \begin{cases} -1 & \text{if } n \equiv 1, 2, 4, 5\ (8) \\ +1 & \text{if } n \equiv 0, 3, 6, 7\ (8), \end{cases}$$

$$p = \begin{cases} 0 & \text{if } n \equiv 0 \ (4) \\ 3 & \text{if } n \equiv 1,3 \ (4) \\ 4 & \text{if } n \equiv 2 \ (4), \end{cases} \quad m = \begin{cases} k & \text{if } n \equiv k \ (4) \text{ and } k \neq 3 \\ -1 & \text{if } n \ \equiv k \ (4) \text{ and } k = 3. \end{cases}$$

2. If $N = 5$,

$$P_5 = \begin{cases} 1 & \text{if } n \equiv 0,4 \ (5) \\ x & \text{if } n \equiv 1 \ (5) \\ y & \text{if } n \equiv 2 \ (5) \\ x^2 - y & \text{if } n \equiv 3 \ (5), \end{cases} \quad Q_5 = x^2 - xy - y,$$

and

$$\varepsilon = \begin{cases} -1 & \text{if } n \equiv 1,2,4,5,8 \ (10) \\ +1 & \text{if } n \equiv 0,3,6,7,9 \ (10), \end{cases}$$

$$p = \begin{cases} 0 & \text{if } n \equiv 0 \ (5) \\ 2 & \text{if } n \equiv 1,4 \ (5) \\ 3 & \text{if } n \equiv 2,3 \ (5), \end{cases} \quad m = \begin{cases} k & \text{if } n \equiv k \ (5) \text{ and } k \neq 4 \\ -1 & \text{if } n \ \equiv k \ (5) \text{ and } k = 4. \end{cases}$$

3. If $N = 6$,

$$P_6 = \begin{cases} 1 & \text{if } n \equiv 0,5 \ (6) \\ x & \text{if } n \equiv 1 \ (6) \\ y & \text{if } n \equiv 2 \ (6) \\ x^2 - y & \text{if } n \equiv 3 \ (6) \\ x^3 - xy - y^2 & \text{if } n \equiv 4 \ (6), \end{cases} \quad Q_6 = (\alpha+1)x^3 - (\alpha+1)xy - y^2$$

and

$$\varepsilon = \begin{cases} +1 & \text{if } n \equiv 0,3,6,7,11 \ (12) \\ -1 & \text{if } n \equiv 1,2,4,5,8,9,10 \ (12), \end{cases}$$

$$p = \begin{cases} 0 & \text{if } n \equiv 0 \ (6) \\ 3 & \text{if } n \equiv 1,3 \ (6) \\ 4 & \text{if } n \equiv 2 \ (6) \\ 7 & \text{if } n \equiv 5 \ (6) \\ 12 & \text{if } n \equiv 4 \ (6), \end{cases} \quad q = \begin{cases} 0 & \text{if } n \equiv 0,3 \ (6) \\ 1 & \text{if } n \equiv 1,2,4,5 \ (6), \end{cases}$$

$$m = \begin{cases} k & \text{if } n \equiv k \ (6) \text{ and } k \neq 5 \\ -1 & \text{if } n \ \equiv k \ (6) \text{ and } k = 5. \end{cases}$$

4. If $N = 7$,

$$P_7 = \begin{cases} 1 & \text{if } n \equiv 0,6 \ (7) \\ x & \text{if } n \equiv 1 \ (7) \\ y & \text{if } n \equiv 2 \ (7) \\ x^2 - y & \text{if } n \equiv 3 \ (7) \\ x^3 - xy - y^2 & \text{if } n \equiv 4 \ (7) \\ \alpha x^3 - \alpha xy - y^2 & \text{if } n \equiv 5 \ (7), \end{cases} \quad Q_7 = \alpha^2 x^3 - (\alpha-1)x^2 y - \alpha^2 xy - y^2,$$

and

$$\varepsilon = \begin{cases} +1 & \text{if } n \equiv 0,3,6,7,10,13\ (14) \\ -1 & \text{if } n \equiv 1,2,4,5,8,9,11,12\ (14), \end{cases}$$

$$p = \begin{cases} 0 & \text{if } n \equiv 0\ (7) \\ 3 & \text{if } n \equiv 3\ (7) \\ 5 & \text{if } n \equiv 1,6\ (7) \\ 6 & \text{if } n \equiv 2,5\ (7) \\ 10 & \text{if } n \equiv 4\ (7), \end{cases} \qquad q = \begin{cases} 0 & \text{if } n \equiv 0\ (7) \\ 2 & \text{if } n \equiv 1\ (7) \\ 3 & \text{if } n \equiv 2,3\ (7) \\ 4 & \text{if } n \equiv 6\ (7) \\ 7 & \text{if } n \equiv 5\ (7) \\ 9 & \text{if } n \equiv 4\ (7), \end{cases}$$

$$m = \begin{cases} k & \text{if } n \equiv k\ (7) \text{ and } k \neq 6 \\ -1 & \text{if } n \equiv k\ (7) \text{ and } k = 6. \end{cases}$$

5. If $N = 9$,

$P_9 =$

1	if $n \equiv 0,8\ (9)$
x	if $n \equiv 1\ (9)$
y	if $n \equiv 2\ (9)$
$x^2 - y$	if $n \equiv 3\ (9)$
$x^3 - xy - y^2$	if $n \equiv 4\ (9)$
$(\alpha^2-\alpha+1)x^3-(\alpha^2-\alpha+1)xy-y^2$	if $n \equiv 5\ (9)$
$(\alpha^3-\alpha^2+\alpha)x^3-(\alpha-1)\,x^2y-(\alpha^3-\alpha^2+\alpha)xy-y^2$	if $n \equiv 6\ (9)$
$(\alpha^4-\alpha^3+\alpha^2)x^4+x^3y-(\alpha^4-\alpha^3+\alpha^2)x^2y$ $-(\alpha^2+1)xy^2-y^3$	if $n \equiv 7\ (9)$

$$Q_9 = \alpha^2(\alpha^2-\alpha+1)^2x^4 - (\alpha^4-2\alpha^3+\alpha^2-1)x^3y - \alpha^2(\alpha^2-\alpha+1)^2x^2y - (\alpha^3+1)xy^2 - y^3,$$

and

$$\varepsilon = \begin{cases} +1 & \text{if } n \equiv 0,3,6,7,10,11,13,14,17\ (18) \\ -1 & \text{if } n \equiv 1,2,4,5,8,9,12,15,16\ (18), \end{cases}$$

$$p = \begin{cases} 0 & \text{if } n \equiv 0\ (9) \\ 5 & \text{if } n \equiv 1\ (9) \\ 6 & \text{if } n \equiv 2\ (9) \\ 3 & \text{if } n \equiv 3,5\ (9) \\ 14 & \text{if } n \equiv 4,7\ (9) \\ -3 & \text{if } n \equiv 6\ (9) \\ 9 & \text{if } n \equiv 8\ (9), \end{cases} \qquad q = \begin{cases} 0 & \text{if } n \equiv 0\ (9) \\ 2 & \text{if } n \equiv 1\ (9) \\ 3 & \text{if } n \equiv 2,3\ (9) \\ 11 & \text{if } n \equiv 4,7\ (9) \\ 9 & \text{if } n \equiv 5\ (9) \\ 6 & \text{if } n \equiv 6,8\ (9), \end{cases}$$

$$r = \begin{cases} 1 & \text{if } n \equiv 1,2,4,5,7,8\ (9) \\ 0 & \text{if } n \equiv 0,3,6\ (9), \end{cases} \qquad m = \begin{cases} k & \text{if } n \equiv k\ (9) \text{ and } k \neq 8 \\ -1 & \text{if } n \equiv k\ (9) \text{ and } k = 8. \end{cases}$$

6. If $N = 10$,

$$P_{10} = \begin{cases} 1 & \text{if } n \equiv 0,9\ (10) \\ x & \text{if } n \equiv 1\ (10) \\ y & \text{if } n \equiv 2\ (10) \\ x^2 - \zeta^2 y & \text{if } n \equiv 3\ (10) \\ x^3 - \zeta^2 xy - y^2 & \text{if } n \equiv 4\ (10) \\ \alpha^2 x^3 - \alpha^2\zeta^2 xy - \zeta y^2 & \text{if } n \equiv 5\ (10) \\ \alpha^3\zeta x^3 - (\alpha-1)x^2 y - \alpha^3\zeta^3 xy - \zeta^2 y^2 & \text{if } n \equiv 6\ (10) \\ \alpha^4(2\alpha-1)\zeta^2 x^3 - \alpha(\alpha-1)(3\alpha-1)\zeta x^2 y \\ \quad +(\alpha-1)xy^2 - \alpha^4(2\alpha-1)\zeta^4 xy - \alpha\zeta^4 y^2 & \text{if } n \equiv 7\ (10) \\ \alpha^7(2\alpha-1)\zeta^3 x^3 - (\alpha-1)x^3 y \\ \quad -\alpha^3(\alpha-1)(2\alpha^2+2\alpha-1)\zeta^2 x^2 y \\ \quad -\alpha^7(2\alpha-1)\zeta^5 xy \\ \quad +(\alpha-1)(2\alpha^2+2\alpha-1)\zeta xy^2 - \alpha^3\zeta^5 y^2 & \text{if } n \equiv 8\ (10) \end{cases}$$

$$\begin{aligned} Q_{10} \;=\; & -\alpha^9(2\alpha-1)\zeta^4 x^3 + (\alpha-1)(2\alpha^2+2\alpha-1)\zeta x^3 y \\ & +\alpha^3(\alpha-1)(2\alpha^4+6\alpha^2-5\alpha+1)\zeta^3 x^2 y - (\alpha-1)x^2 y^2 \\ & +\alpha^9(2\alpha-1)\zeta^6 xy + (\alpha^4-10\alpha^3+4\alpha-1)(\alpha-1)\zeta^2 xy^2 + \alpha^3\zeta^7 y^2, \end{aligned}$$

and

$$\varepsilon = \begin{cases} +1 & \text{if } n \equiv 0,3,6,7,9,12,13,14,17,18,19\ (20) \\ -1 & \text{if } n \equiv 1,2,4,5,8,10,11,15,16\ (20), \end{cases}$$

$$p = \begin{cases} 0 & \text{if } n \equiv 0\ (10) \\ 21 & \text{if } n \equiv 1,9\ (10) \\ 24 & \text{if } n \equiv 2,8\ (10) \\ 9 & \text{if } n \equiv 3,7\ (10) \\ 36 & \text{if } n \equiv 4\ (10) \\ 25 & \text{if } n \equiv 5\ (10) \\ 16 & \text{if } n \equiv 6\ (10), \end{cases} \quad q = \begin{cases} 0 & \text{if } n \equiv 0\ (10) \\ 7 & \text{if } n \equiv 1\ (10) \\ 12 & \text{if } n \equiv 2\ (10) \\ 15 & \text{if } n \equiv 3\ (10) \\ 36 & \text{if } n \equiv 4\ (10) \\ 35 & \text{if } n \equiv 5\ (10) \\ 32 & \text{if } n \equiv 6\ (10) \\ 27 & \text{if } n \equiv 7\ (10) \\ 20 & \text{if } n \equiv 8\ (10) \\ 11 & \text{if } n \equiv 9\ (10) \end{cases} \quad r = \begin{cases} 0 & \text{if } n \equiv 0\ (10) \\ 2 & \text{if } n \equiv 1,6,9\ (10) \\ 3 & \text{if } n \equiv 2,3,7,8\ (10) \\ 7 & \text{if } n \equiv 4\ (10) \\ 5 & \text{if } n \equiv 5\ (10), \end{cases}$$

$$s = \begin{cases} 0 & \text{if } n \equiv 0\ (10) \\ 5 & \text{if } n \equiv 1,3,5,7,9\ (10) \\ 4 & \text{if } n \equiv 2,6,8\ (10) \\ 8 & \text{if } n \equiv 4\ (10), \end{cases} \quad m = \begin{cases} k & \text{if } n \equiv k\ (10) \text{ and } k \neq 9 \\ -1 & \text{if } n \equiv k\ (10) \text{ and } k = 9. \end{cases}$$

7. If $N = 12$,

$P_{12} =$

1	if $n \equiv 0, 11\ (12)$
x	if $n \equiv 1\ (12)$
y	if $n \equiv 2\ (12)$
$x^2-(\alpha-1)^4 y$	if $n \equiv 3\ (12)$
$x^3-(\alpha-1)^4 xy - y^2$	if $n \equiv 4\ (12)$
$\theta x^3 - \theta(\alpha-1)^4 xy - (\alpha-1)y^2$	if $n \equiv 5\ (12)$
$\theta(\alpha-1)^3(3\alpha^2-3\alpha+1)x^3 - (2\alpha^2-\alpha)x^2 y$ $-\theta(\alpha-1)^7(3\alpha^2-3\alpha+1)\, xy - (\alpha-1)^6 y^2$	if $n \equiv 6\ (12)$
$\theta(\alpha-1)x^4 + \theta(\alpha^2+\alpha-1)(\alpha-1)^3 x^2 y$ $-(3\alpha^2-3\alpha+1)xy^2 - \theta\alpha(2\alpha-1)(\alpha-1)^7 y^2$	if $n \equiv 7\ (12)$
$\theta^2(\alpha-1)^3 x^4 + \alpha^2 x^3 y + \theta(\alpha^2-\alpha+1)(\alpha-1)^7 x^2 y$ $+(2\alpha-1)(\alpha-1)^5 xy^2 + \theta\alpha(\alpha-1)^{12} y^2$	if $n \equiv 8\ (12)$
$\theta^2(3\alpha^2-3\alpha+1)(\alpha-1)^6 x^4$ $+ (6\alpha^2-7\alpha+3)\alpha^2(\alpha-1)^3 x^3 y + \alpha^2 x^2 y^2$ $+ \theta(2\alpha^2-\alpha+1)(3\alpha^2-3\alpha+1)(\alpha-1)^{10} x^2 y$ $-(\alpha-1)^7(8\alpha^3-11\alpha^2+6\alpha-1)xy^2$ $- \theta\alpha(3\alpha^2-3\alpha+1)(\alpha-1)^{14} y^2$	if $n \equiv 9\ (12)$
$2\alpha^2(3\alpha^2-4\alpha+2)(\alpha-1)^3 x^5 + \theta(3\alpha^2-3\alpha+1)$ $\times(14\alpha^5-36\alpha^4+40\alpha^3-23\alpha^2+7\alpha-1)(\alpha-1)^7 x^4$ $+ \alpha^2 x^4 y - \alpha^2(3\alpha^2-4\alpha+2)$ $\times(12\alpha^3-24\alpha^2+18\alpha-5)(\alpha-1)^4\, x^3 y$ $-\theta(3\alpha^2-3\alpha+1) \times (12\alpha^5-30\alpha^4+33\alpha^3$ $-19\alpha^2+6\alpha-1)(\alpha-1)^{11} x^2 y$ $+\theta(9\alpha^3-13\alpha^2+7\alpha-1)(\alpha-1)^{10} xy^2$ $+\theta^2\alpha(3\alpha^2-3\alpha+1)(\alpha-1)^{17} y^2$	if $n \equiv 10\ (12)$

$$\begin{aligned}
Q_{12} \;=\;& \alpha^2(12\alpha^4-42\alpha^3+58\alpha^2-37\alpha+10)(\alpha-1)^6 x^5 \\
&-\theta(3\alpha^2-3\alpha+1)(30\alpha^5-66\alpha^4+63\alpha^3-31\alpha^2+8\alpha-1)(\alpha-1)^{11} x^4 \\
&+2\alpha^2(3\alpha^2-4\alpha+2)(\alpha-1)^3 x^4 y + \alpha^2 x^3 y^2 \\
&-\alpha^2(12\alpha^6-138\alpha^5+362\alpha^4-457\alpha^3+319\alpha^2-121\alpha+20)(\alpha-1)^8 x^3 y \\
&+\theta(3\alpha^2-3\alpha+1)(28\alpha^5-60\alpha^4+56\alpha^3-27\alpha^2+7\alpha-1)(\alpha-1)^{15} x^2 y \\
&-\theta(12\alpha^3-16\alpha^2+8\alpha-1)(\alpha-1)^{14} xy^2 - \theta^2\alpha(3\alpha^2-3\alpha+1)(\alpha-1)^{21} y^2,
\end{aligned}$$

and

$$\varepsilon = \begin{cases} +1 & \text{if } n \equiv 0,3,8,9,13,14,16,17,18,19,22,23\ (24) \\ -1 & \text{if } n \equiv 1,2,4,5,6,7,10,11,12,15,20,21\ (24), \end{cases}$$

$$p = \begin{cases} 0 & \text{if } n \equiv 0,2\ (12) \\ -1 & \text{if } n \equiv 1\ (12) \\ 3 & \text{if } n \equiv 3,11\ (12) \\ 8 & \text{if } n \equiv 4,10\ (12) \\ 15 & \text{if } n \equiv 5,9\ (12) \\ 12 & \text{if } n \equiv 6\ (12) \\ 23 & \text{if } n \equiv 7\ (12) \\ 24 & \text{if } n \equiv 8\ (12), \end{cases} \quad q = \begin{cases} 0 & \text{if } n \equiv 0\ (12) \\ 59 & \text{if } n \equiv 1,11\ (12) \\ 44 & \text{if } n \equiv 2,10\ (12) \\ 51 & \text{if } n \equiv 3,9\ (12) \\ 80 & \text{if } n \equiv 4\ (12) \\ 35 & \text{if } n \equiv 5,7\ (12) \\ 60 & \text{if } n \equiv 6\ (12) \\ 56 & \text{if } n \equiv 8\ (12), \end{cases}$$

$$r = \begin{cases} 0 & \text{if } n \equiv 0\ (12) \\ 1 & \text{if } n \equiv 1,11\ (12) \\ 4 & \text{if } n \equiv 2,10\ (12) \\ 9 & \text{if } n \equiv 3,9\ (12) \\ 16 & \text{if } n \equiv 4,8\ (12) \\ 25 & \text{if } n \equiv 5,7\ (12) \\ 12 & \text{if } n \equiv 6\ (12), \end{cases} \quad s = \begin{cases} 0 & \text{if } n \equiv 0,8\ (12), \\ 3 & \text{if } n \equiv 1,3,5,7,9,11\ (12) \\ 4 & \text{if } n \equiv 2,6,10\ (12) \\ 8 & \text{if } n \equiv 4\ (12) \end{cases}$$

$$t = \begin{cases} 1 & \text{if } n \equiv 1,2,4,5,7,8,10,11\ (12) \\ 0 & \text{if } n \equiv 0,3,6,9\ (12), \end{cases} \quad m = \begin{cases} k & \text{if } n \equiv k\ (12) \text{ and } k \neq 11 \\ -1 & \text{if } n \equiv k\ (12) \text{ and } k = 11. \end{cases}$$

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