C_4-SUPERMAGIC LABELINGS OF DISJOINT UNION OF PRISMS

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A simple graph G admits an H-covering if every edge in $E(G)$ belongs to a subgraph of G isomorphic to H. An H-magic labeling of a graph G admitting an H-covering is a bijective function from the vertex set $V(G)$ and the edge set $E(G)$ of the graph G onto the set of integers $\{1, 2, \dots, |V(G)| + |E(G)|\}$ such that for all subgraphs H' isomorphic to H, the sum of labels of all the edges and vertices belonged to H' are the same. Such a labeling is called *H-supermagic* if the smallest possible labels appear on the vertices. In this paper, we will deal with C_4-supermagic labeling for the disjoint union of l isomorphic copies of prism graphs $C_n \times P_m$ for $m \geq 2$ and $n \geq 3$, $n \neq 4$, $l \geq 1$.

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INTRODUCTION

Let $G = (V, E)$ be a finite, simple, connected and undirected graph, where $V(G)$ and $E(G)$ are its vertex-set and edge-set, respectively. A *labeling* (or a *valuation*) of a graph is a map that carries the graph elements to the numbers, usually positive or non-negative integers.

An *edge-covering* of G is a family of subgraphs $H_1, H_2, \dots, H_t$ such that each edge of $E(G)$ belongs to at least one of the subgraphs H_i, $i = 1, 2, \dots, t$. In this case we say that G admits an $(H_1, H_2, \dots, H_t)$-*(edge) covering.* If every subgraph H_i is isomorphic to a given graph H, then the graph G admits an *H-covering.*

An *H-magic labeling* f of a graph G admitting an H-covering is a bijective function from the vertex set and the edge set of the graph G onto the set of integers $\{1, 2, \dots, |V(G)|+|E(G)|\}$ if there exists a positive integer $m(f)$, called the *magic sum*, such that for every subgraph H' of G isomorphic to H, the sum $wt_f(H') = \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e)$ is equal to $m(f)$. The sum $wt_f(H)$ is called the *H-weight.*

If, in addition, the H-magic labeling f has the property that the smallest possible labels appear on the vertices, *i.e.* $\{f(v)\}_{v \epsilon V(G)} = \{1, 2, \dots, |V(G)|\}$, then the labeling is called *H-supermagic.* A graph is called H-(super)magic if it admits a H-(super)magic labeling.

When H is isomorphic to K_2, a K_2-magic labeling is also called an edge-magic total labeling. The notion of H-magic labeling was introduced by Gutiérrez and Lladó [2] as a natural extension of *edge-magic total labeling* defined by Kotzig and Rosa in [5] as *magic valuation.*

In [6] Lladó and Moragas showed the cyclic-magic and cyclic-supermagic behavior of several classes of connected graphs including subdivided wheels and subdivided friendship graphs and prisms. Ngurah *et al.* [8, 9] constructed cycle-supermagic labelings for fans, ladders and books. In [3, 4, 10] Jeyanthi and Selvagopal discussed some cycle-supermagic labelings of families of prism graphs. Maryati *et al.* [7], proved that the disjoint union of k isomorphic copies of a connected graph H is a H-supermagic graph if and only if $|V(H)|+|E(H)|$ is even or k is odd. For further details, the reader is referred to the dynamic survey [1].

In the following section, we will study the C_4-supermagic labelings of the disjoint union of l isomorphic copies of prisms $C_n \times P_m$ for $m \geq 2$ and $n \geq 3$, $n \neq 4$, $l \geq 1$.

MAIN RESULT

Let $n \geq 3$, $m \geq 2$ and $l \geq 1$ be positive integers. The *prism graph*, or simply the *prism*, is a graph isomorphic to the cartesian product $C_n \times P_m$ of a cycle on n vertices with a path on m vertices. Let us consider the disjoint union of l isomorphic copies of the prism graph $C_n \times P_m$, *i.e.* the graph $l(C_n \times P_m)$.

Let the vertex set of $l(C_n \times P_m)$ be

$$V\big(l(C_n \times P_m)\big) = \{v_{i,j}^k : 1 \leq i \leq n; 1 \leq j \leq m; 1 \leq k \leq l\},$$

and the edge set of $C_n \times P_m$ be

$$\begin{aligned} E\big(l(C_n \times P_m)\big) =&\{v_{i,j}^k v_{i+1,j}^k : 1 \leq i \leq n-1; 1 \leq j \leq m; 1 \leq k \leq l\} \\ &\cup \{v_{1,j}^k v_{n,j}^k : 1 \leq j \leq m; 1 \leq k \leq l\} \\ &\cup \{v_{i,j}^k v_{i,j+1}^k : 1 \leq i \leq n, 1 \leq j \leq m-1; 1 \leq k \leq l\}. \end{aligned}$$

The graph $l(C_n \times P_m)$ is of order lnm and of size $ln(2m-1)$. Fig. 1 illustrates the kth copy of $C_n \times P_m$.

In the following theorem we prove that the disjoint union of arbitrary number of isomorphic copies of prisms $C_n \times P_m$, $m \geq 2$ and $n \geq 3$, $n \neq 4$, is C_4-supermagic.

THEOREM 1. *Let m, n, l be positive integers, $m \geq 2$, $n \geq 3$, $n \neq 4$ and $l \geq 1$. Then the graph $l(C_n \times P_m)$ is C_4-supermagic.*

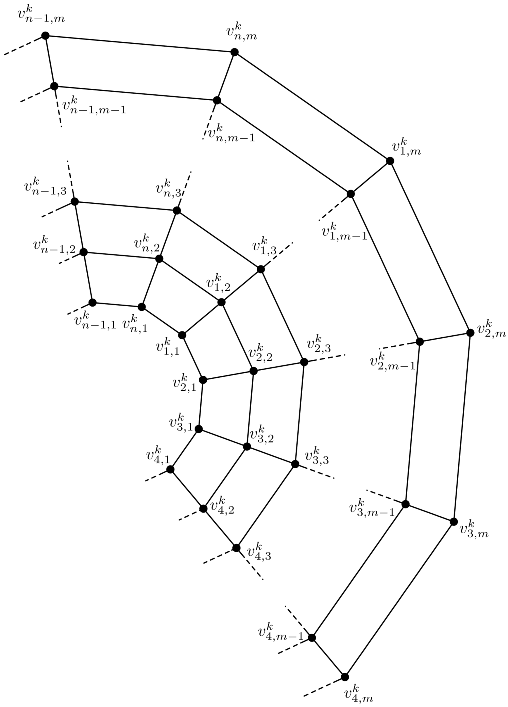

Fig. 1. The kth copy of $C_n \times P_m$.

Proof. Let us consider the labeling f from the vertex set and edge set of $l(C_n \times P_m)$ to the set of integers $\{1, 2, \dots, ln(3m-1)\}$ defined in the following way. For $k = 1, 2, \dots, l$

$$f(v^k_{i,j}) = l(i-1) + ln(j-1) + k \qquad i = 1, 2, \dots, n, \quad j = 1, 2, \dots, m$$

$$f(v^k_{i,j}v^k_{i,j-1}) = 2lnm + 1 - l(i-1) - ln(j-1) - k \qquad i = 1, 2, \dots, n, \quad j = 2, 3, \dots, m$$

$$f(v^k_{i,j}v^k_{i+1,j}) = \begin{cases} 3lnm + 1 - l(i-1) - lnj - k & i = 1, 2, \dots, n-1 \\ & j \equiv 1 \pmod 2,\ j \leq m \\ 3lnm + 1 - li - lnj - k & i = 1, 2, \dots, n-1 \\ & j \equiv 0 \pmod 2,\ j \leq m \end{cases}$$

$$f(v^k_{n,j}v^k_{1,j}) = \begin{cases} 3lnm + l + 1 - ln(j+1) - k & j \equiv 1 \pmod 2,\ j \leq m \\ 3lnm + 1 - lnj - k & j \equiv 0 \pmod 2,\ j \leq m. \end{cases}$$

It is easy to see that every number from the set $\{1, 2, \dots, ln(3m-1)\}$ is used exactly once as a label, thus f is a bijection. Moreover, the vertices are labeled with the smallest possible numbers $\{1, 2, \dots, lnm\}$, thus f is super.

Every cycle C_4 in $l(C_n \times P_m)$ is either of the form

$$C_4(i,j,k) = v_{i,j}^k v_{i+1,j}^k v_{i+1,j-1}^k v_{i,j-1}^k v_{i,j}^k,$$

where $i = 1, 2, \dots, n-1$, $j = 2, 3, \dots, m$, $k = 1, 2, \dots, l$,

or of the form

$$C_4(n,j,k) = v_{n,j}^k v_{1,j}^k v_{1,j-1}^k v_{n,j-1}^k v_{n,j}^k,$$

where $j = 2, 3, \dots, m$, $k = 1, 2, \dots, l$.

For the C_4-weight of the cycle $C_4(i,j,k)$, $i = 1, 2, \dots, n-1$, $j = 2, 3, \dots, m$, $k = 1, 2, \dots, l$, we get

$$\begin{aligned} wt_f(C_4(i,j,k)) = & \sum_{v \in V(C_4(i,j,k))} f(v) + \sum_{e \in E(C_4(i,j,k))} f(e) \\ = & \big(f(v_{i,j}^k) + f(v_{i+1,j}^k) + f(v_{i+1,j-1}^k) + f(v_{i,j-1}^k)\big) \\ & + \big(f(v_{i,j}^k v_{i+1,j}^k) + f(v_{i+1,j}^k v_{i+1,j-1}^k) \\ & + f(v_{i+1,j-1}^k v_{i,j-1}) + f(v_{i,j-1}^k v_{i,j}^k)\big) \\ = & \big(l(i-1) + ln(j-1) + k\big) + \big(li + ln(j-1) + k\big) \\ & + \big(li + ln(j-2) + k\big) + \big(l(i-1) + ln(j-2) + k\big) \\ & + \big(2lnm + 1 - li - ln(j-1) - k\big) \\ & + \big(2lnm + 1 - l(i-1) - ln(j-1) - k\big) \\ & + f(v_{i,j}^k v_{i+1,j}^k) + f(v_{i+1,j-1}^k v_{i,j-1}) \\ = & 4lnm + 2 + l(2i-1) + 2ln(j-2) + 2k \\ & + f(v_{i,j}^k v_{i+1,j}^k) + f(v_{i,j-1} v_{i+1,j-1}^k). \end{aligned}$$

Now we will distinguish two subcases according to the parity of j.

For j odd, $j \le m$, we get

$$\begin{aligned} wt_f(C_4(i,j,k)) = & 4lnm + 2 + l(2i-1) + 2ln(j-2) + 2k \\ & + \big(3lnm + 1 - l(i-1) - lnj - k\big) \\ & + \big(3lnm + 1 - li - ln(j-1) - k\big) \\ = & 10lnm - 3ln + 4. \end{aligned}$$

For j even, $j \le m$, we get

$$\begin{aligned} wt_f(C_4(i,j,k)) = & 4lnm + 2 + l(2i-1) + 2ln(j-2) + 2k \\ & + \big(3lnm + 1 - li - lnj - k\big) \end{aligned}$$

$$
\begin{aligned}
&+ \big(3lnm + 1 - l(i-1) - ln(j-1) - k\big)\\
=&10lnm - 3ln + 4.
\end{aligned}
$$

Now we will calculate the C_4-weight of the cycle $C_4(n,j,k)$, $j = 2,3,\dots,m$, $k = 1,2,\dots,l$.

$$
\begin{aligned}
wt_f(C_4(n,j,k)) =& \sum_{v\in V(C_4(n,j,k))} f(v) + \sum_{e\in E(C_4(n,j,k))} f(e)\\
=&\big(f(v^k_{n,j}) + f(v^k_{1,j}) + f(v^k_{1,j-1}) + f(v^k_{n,j-1})\big)\\
&+ \big(f(v^k_{n,j}v^k_{1,j}) + f(v^k_{1,j}v^k_{1,j-1})\\
&+ f(v^k_{1,j-1}v_{n,j-1}) + f(v^k_{n,j-1}v^k_{n,j})\big)\\
=&\big(l(n-1) + ln(j-1) + k\big) + \big(ln(j-1) + k\big)\\
&+ \big(ln(j-2) + k\big) + \big(l(n-1) + ln(j-2) + k\big)\\
&+ \big(2lnm + 1 - ln(j-1) - k\big)\\
&+ \big(2lnm + 1 - l(n-1) - ln(j-1) - k\big)\\
&+ f(v^k_{n,j}v^k_{1,j}) + f(v^k_{1,j-1}v_{n,j-1})\\
=&4lnm + 2 + l(n-1) + 2ln(j-2) + 2k\\
&+ f(v^k_{n,j}v^k_{1,j}) + f(v_{n,j-1}v^k_{1,j-1}).
\end{aligned}
$$

Again we will consider two subcases.

For j odd, $j \le m$, we have

$$
\begin{aligned}
wt_f(C_4(n,j,k)) =&4lnm + 2 + l(n-1) + 2ln(j-2) + 2k\\
&+ \big(3lnm + l + 1 - ln(j+1) - k\big)\\
&+ \big(3lnm + 1 - ln(j-1) - k\big)\\
=&10lnm - 3ln + 4.
\end{aligned}
$$

For j even, $j \le m$, it holds

$$
\begin{aligned}
wt_f(C_4(n,j,k)) =&4lnm + 2 + l(n-1) + 2ln(j-2) + 2k\\
&+ \big(3lnm + 1 - lnj - k\big)\\
&+ \big(3lnm + l + 1 - lnj - k\big)\\
=&10lnm - 3ln + 4.
\end{aligned}
$$

In all cases, the C_4-weights are the same and are equal to the number $10lnm - 3ln + 4$. It means, that f is the C_4-supermagic labeling of $l(C_n \times P_m)$ for $m \ge 2$, $n \ge 3$, $n \ne 4$ and $l \ge 1$.

CONCLUSION

In this paper, we have shown that the disjoint union of l isomorphic copies of prisms $C_n \times P_m$ for $m \geq 2$ and $n \geq 3$, $n \neq 4$, $l \geq 1$ is C_4-supermagic. However, the construction described in Theorem 1 does not solve the case when $n = 4$. For further investigation we state the following open problem.

OPEN PROBLEM. Find, if there exists, a C_4-supermagic labeling of the graph $l(C_4 \times P_m)$ for $m \geq 2$ and $l \geq 1$.

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