$\mathfrak{a}$-MINIMAX MODULES AND EXTENSION FUNCTORS OF LOCAL COHOMOLOGY

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Let R be a commutative Noetherian ring and $\mathfrak{a}$ an ideal of R. The concepts of $\mathfrak{a}$-minimax and $\mathfrak{a}$-cominimax modules were introduced by Azami, Naghipour and Vakili in [1] as generalization of minimax and $\mathfrak{a}$-cofinite modules, respectively. The finiteness of extension functors of local cohomology modules was viewed by Dibaei and Yassemi in [4]. In this paper, we discuss the $\mathfrak{a}$-minimaxness of extension functors of local cohomology modules, in several cases. Let M be an $\mathfrak{a}$-minimax R-module, and let t be a non-negative integer such that $\mathrm{H}^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cominimax for all $i < t$. In [1, Theorem 4.1] we have shown that for any $\mathfrak{a}$-minimax submodule N of $\mathrm{H}^t_{\mathfrak{a}}(M)$ and for any finitely generated R-module L with $\operatorname{Supp} L \subseteq V(\mathfrak{a})$, the R-module $\operatorname{Hom}_R(L, \mathrm{H}^t_{\mathfrak{a}}(M)/N)$ is $\mathfrak{a}$-minimax. In this paper, it is shown that $\operatorname{Ext}^i_R(L, \mathrm{H}^t_{\mathfrak{a}}(M)/N)$ is $\mathfrak{a}$-minimax for $i = 0, 1$; in addition, if $\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^{t+1}_{\mathfrak{a}}(M))$ is $\mathfrak{a}$-minimax, then $\operatorname{Ext}^i_R(L, \mathrm{H}^t_{\mathfrak{a}}(M)/N)$ is $\mathfrak{a}$-minimax for $i = 0$, 1, 2.

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1. INTRODUCTION

Throughout this paper, R will always be a commutative Noetherian ring with non-zero identity, and $\mathfrak{a}$ will be an ideal of R. Let M be an R-module. The $\mathfrak{a}$-*torsion submodule* of M is defined as $\Gamma_{\mathfrak{a}}(M) = \bigcup_{n\geq 1}(0 :_M \mathfrak{a}^n)$. The i^{th} *local cohomology functor* $\mathrm{H}^i_{\mathfrak{a}}(.)$ is defined as the i^{th} right derived functor $\Gamma_{\mathfrak{a}}(.)$. It is known that for each $i \geq 0$ there is a natural isomorphism of R-modules

$$\mathrm{H}^i_{\mathfrak{a}}(M) \cong \varinjlim_{n\geq 1} \operatorname{Ext}^i_R(R/\mathfrak{a}^n, M).$$

We refer the reader to [6] or [3] for the basic properties of local cohomology.

An R-module M is said to have *finite Goldie dimension* (written $G\dim M < \infty$), if the injective hull $E(M)$ of M decomposes as a finite direct sum of indecomposable (injective) submodules. It is clear by the definition of the Goldie dimension that

$$G\dim M = \sum_{\mathfrak{p}\in \operatorname{Ass}_R M} \mu^0(\mathfrak{p}, M),$$

where $\mu^0(\mathfrak{p}, M)$ is the 0-th Bass number of M with respect to prime ideal $\mathfrak{p}$ and $\operatorname{Ass}_R M$ is the set of associated prime ideals of R. The $\mathfrak{a}$-*relative Goldie dimension of* M has been studied by Divaani-Aazar and Esmkhani in [5] and it is defined as

$$G\dim_{\mathfrak{a}} M = \sum_{\mathfrak{p}\in V(\mathfrak{a})} \mu^0(\mathfrak{p}, M).$$

In particular, they proved that $G\dim_{\mathfrak{a}} M = G\dim \Gamma_{\mathfrak{a}}(M)$. An R-module M is said to have *finite* $\mathfrak{a}$-*relative Goldie dimension* if the Goldie dimension of the $\mathfrak{a}$-torsion submodule $\Gamma_{\mathfrak{a}}(M)$ of M is finite.

In [7], Zöschinger introduced the interesting class of minimax modules. An R-module M is said to be a *minimax* module, if there is a finitely generated submodule N of M, such that M/N is Artinian. Zöschinger, has given in [7] and [8] many equivalent conditions for a module to be minimax. In particular, when R is a Noetherian ring, a module is minimax if and only if each of its quotients has finite Goldie dimension. The concepts of $\mathfrak{a}$-minimax and $\mathfrak{a}$-cominimax modules were introduced by Azami, Naghipour and Vakili in [1] as generalization of minimax and $\mathfrak{a}$-cofinite modules. An R-module M is called $\mathfrak{a}$-*minimax* if the $\mathfrak{a}$-relative Goldie dimension of any quotient module of M is finite. It is clear that if M is $\mathfrak{a}$-torsion then M is $\mathfrak{a}$-minimax if and only if M is minimax. Also, we say that an R-module M is $\mathfrak{a}$-*cominimax* if the support of M is contained in $V(\mathfrak{a})$ and $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ is $\mathfrak{a}$-minimax for all $i \geq 0$.

The finiteness of extension functors of local cohomology modules was viewed by Dibaei and Yassemi in [4]. In this paper, we discuss the $\mathfrak{a}$-minimaxness of extension functors of local cohomology modules, in several cases. Let M be an $\mathfrak{a}$-minimax R-module, and let t be a non-negative integer such that $\mathrm{H}^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cominimax for all $i < t$. In [1, Theorem 4.1] we have shown that for any $\mathfrak{a}$-minimax submodule N of $\mathrm{H}^t_{\mathfrak{a}}(M)$ and for any finitely generated R-module L with $\operatorname{Supp} L \subseteq V(\mathfrak{a})$, the R-module $\operatorname{Hom}_R(L, \mathrm{H}^t_{\mathfrak{a}}(M)/N)$ is $\mathfrak{a}$-minimax. This generalizes the main result of Brodmann and Lashgari [2]. In this paper, it is shown that both $\operatorname{Hom}_R(L, \mathrm{H}^t_{\mathfrak{a}}(M)/N)$ and $\operatorname{Ext}^1_R(L, \mathrm{H}^t_{\mathfrak{a}}(M)/N)$ are $\mathfrak{a}$-minimax; in addition, if $\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^{t+1}_{\mathfrak{a}}(M))$ is $\mathfrak{a}$-minimax, then $\operatorname{Hom}_R(L, \mathrm{H}^t_{\mathfrak{a}}(M)/N)$, $\operatorname{Ext}^1_R(L, \mathrm{H}^t_{\mathfrak{a}}(M)/N)$ and $\operatorname{Ext}^2_R(L, \mathrm{H}^t_{\mathfrak{a}}(M)/N)$ are all $\mathfrak{a}$-minimax.

2. THE RESULTS

To prove the main results of this paper, we first bring the following lemma.

LEMMA 2.1. *Let M be an R-module and $L = E(M/\Gamma_{\mathfrak{a}}(M))/M/\Gamma_{\mathfrak{a}}(M)$. Then*

(i) $\mathrm{H}^i_{\mathfrak{a}}(L) \simeq \mathrm{H}^{i+1}_{\mathfrak{a}}(M)$ and $\mathrm{Ext}^i_R(R/\mathfrak{a}, L) \simeq \mathrm{Ext}^{i+1}_R(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M))$ *for all* $i \geq 0$.

(ii) $\mathrm{Ext}^1_R(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M)) \simeq \mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^1_{\mathfrak{a}}(M))$.

Proof. For (i) See [4, Remark 2.1], and for (ii) See to the proof of [4, Theorem B]. □

Proposition 2.2. *Let* M *be an* R*-module. If* $\mathrm{Ext}^1_R(R/\mathfrak{a}, M)$ *and* $\mathrm{Ext}^2_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M))$ *are* $\mathfrak{a}$*-minimax, then* $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^1_{\mathfrak{a}}(M))$ *is* $\mathfrak{a}$*-minimax.*

Proof. Consider the exact sequence

$$\mathrm{Ext}^1_R(R/\mathfrak{a}, M) \to \mathrm{Ext}^1_R(R/\mathfrak{a}, M//\Gamma_{\mathfrak{a}}(M)) \to \mathrm{Ext}^2_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M)).$$

Therefore, it follows from [1, Proposition 2.3] and assumption that $\mathrm{Ext}^1_R(R/\mathfrak{a}, M//\Gamma_{\mathfrak{a}}(M))$ is $\mathfrak{a}$-minimax, and so by previous lemma $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^1_{\mathfrak{a}}(M))$ is $\mathfrak{a}$-minimax. □

Now, we are ready to present some sufficient conditions for the $\mathfrak{a}$-minimaxness of extension functors of local cohomology modules.

Theorem 2.3. *Let* M *be an* R*-module. Let* t *be a non-negative integer such that the local cohomology modules* $\mathrm{H}^i_{\mathfrak{a}}(M)$ *are* $\mathfrak{a}$*-cominimax for all* $i < t$.

(i) *If* $\mathrm{Ext}^{t+1}_R(R/\mathfrak{a}, M)$ *is* $\mathfrak{a}$*-minimax, then* $\mathrm{Ext}^1_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M)/N)$ *is* $\mathfrak{a}$*-minimax for any* $\mathfrak{a}$*-minimax submodule* N *of* $\mathrm{H}^t_{\mathfrak{a}}(M)$.

(ii) *If* $\mathrm{Ext}^t_R(R/\mathfrak{a}, M)$ *and* $\mathrm{Ext}^{t+1}_R(R/\mathfrak{a}, M)$ *are* $\mathfrak{a}$*-minimax, then for any* $\mathfrak{a}$*-minimax submodule* N *of* $\mathrm{H}^t_{\mathfrak{a}}(M)$ *and for any finitely generated* R*-module* L *with* $\mathrm{Supp}\, L \subseteq V(\mathfrak{a})$*, the* R*-module* $\mathrm{Ext}^i_R(L, \mathrm{H}^t_{\mathfrak{a}}(M)/N)$ *is* $\mathfrak{a}$*-minimax for all* $i \leq 1$.

(iii) *If both* $\mathrm{Ext}^{t+2}_R(R/\mathfrak{a}, M)$ *and* $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^{t+1}_{\mathfrak{a}}(M))$ *are* $\mathfrak{a}$*-minimax, then the* R*-module* $\mathrm{Ext}^2_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M)/N)$ *is* $\mathfrak{a}$*-minimax for any* $\mathfrak{a}$*-minimax submodule* N *of* $\mathrm{H}^t_{\mathfrak{a}}(M)$.

(iv) *If the* R*-modules* $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^{t+1}_{\mathfrak{a}}(M))$ *and* $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ *for* $i = t, t+1, t+2$ *are* $\mathfrak{a}$*-minimax, then for any* $\mathfrak{a}$*-minimax submodule* N *of* $\mathrm{H}^t_{\mathfrak{a}}(M)$ *and for any finitely generated* R*-module* L *with* $\mathrm{Supp}\, L \subseteq V(\mathfrak{a})$*, the* R*-module* $\mathrm{Ext}^i_R(L, \mathrm{H}^t_{\mathfrak{a}}(M)/N)$ *is* $\mathfrak{a}$*-minimax for all* $i \leq 2$.

(v) *If* $\mathrm{Ext}^{t+1}_R(R/\mathfrak{a}, M)$ *and* $\mathrm{Ext}^2_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M))$ *are both* $\mathfrak{a}$*-minimax, then the* R*-module* $\mathrm{Hom}_R(L, \mathrm{H}^{t+1}_{\mathfrak{a}}(M)/N)$ *is* $\mathfrak{a}$*-minimax for any* $\mathfrak{a}$*-minimax submodule* N *of* $\mathrm{H}^{t+1}_{\mathfrak{a}}(M)$ *and for any finitely generated* R*-module* L *with* $\mathrm{Supp}\, L \subseteq V(\mathfrak{a})$.

Proof. (i) The exact sequence

$$0 \to N \to \mathrm{H}^t_{\mathfrak{a}}(M) \to \mathrm{H}^t_{\mathfrak{a}}(M)/N \to 0$$

induces the following exact sequence,

$$\operatorname{Ext}^1_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M)) \to \operatorname{Ext}^1_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M)/N) \to \operatorname{Ext}^2_R(R/\mathfrak{a}, N).$$

Since $\operatorname{Ext}^2_R(R/\mathfrak{a}, N)$ is $\mathfrak{a}$-minimax, so in view of [1, Proposition 2.3], it is sufficient to show that $\operatorname{Ext}^1_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M))$ is $\mathfrak{a}$-minimax. We use induction on t. When $t = 0$, then by assumption $\operatorname{Ext}^1_R(R/\mathfrak{a}, M)$ is $\mathfrak{a}$-minimax, and so the exact sequence

$$0 = \operatorname{Hom}_R(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M)) \to \operatorname{Ext}^1_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M)) \to \operatorname{Ext}^1_R(R/\mathfrak{a}, M)$$

implies that $\operatorname{Ext}^1_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M))$ is $\mathfrak{a}$-minimax. Now suppose, inductively, that $t > 0$ and suppose that the result has been proved for $t-1$. As $\Gamma_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cominimax, it follows that $\operatorname{Ext}^i_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M))$ is $\mathfrak{a}$-minimax for all $i \geq 0$. Hence by the exact sequence

$$\operatorname{Ext}^{t+1}_R(R/\mathfrak{a}, M) \to \operatorname{Ext}^{t+1}_R(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M)) \to \operatorname{Ext}^{t+2}_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M))$$

and assumption, $\operatorname{Ext}^{t+1}_R(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M))$ is $\mathfrak{a}$-minimax. Also, as $H^0_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M)) = \Gamma_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M)) = 0$ and $H^i_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M)) \simeq H^i_{\mathfrak{a}}(M)$ for all $i > 0$, it follows by assumption that $H^i_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M))$ is $\mathfrak{a}$-cominimax for all $i < t$. Therefore, we may assume that M is $\mathfrak{a}$-torsion free. Let E be an injective envelope of M and put $M_1 = E/M$. Then by Lemma 2.1, $\operatorname{Ext}^i_R(R/\mathfrak{a}, M_1) \simeq \operatorname{Ext}^{i+1}_R(R/\mathfrak{a}, M)$ and $H^i_{\mathfrak{a}}(M_1) \simeq H^{i+1}_{\mathfrak{a}}(M)$ for all $i \geq 0$. The induction hypothesis applied to M_1, yields that $\operatorname{Ext}^1_R(R/\mathfrak{a}, H^{t-1}_{\mathfrak{a}}(M_1))$ is $\mathfrak{a}$-minimax. Hence again by Lemma 2.1, $\operatorname{Ext}^1_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M))$ is $\mathfrak{a}$-minimax.

(ii) Let N be an $\mathfrak{a}$-minimax submodule of $\mathrm{H}^t_{\mathfrak{a}}(M)$ and let L be a finitely generated R-module with $\operatorname{Supp} L \subseteq V(\mathfrak{a})$. Hence by part (i) and [1, Theorem 4.2], the R-module $\operatorname{Ext}^i_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M)/N)$ is $\mathfrak{a}$-minimax for all $i \leq 1$. Now, in view of [1, Theorem 2.7], the R-module $\operatorname{Ext}^i_R(L, \mathrm{H}^t_{\mathfrak{a}}(M)/N)$ is $\mathfrak{a}$-minimax for all $i \leq 1$.

(iii) By the exact sequence

$$\operatorname{Ext}^2_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M)) \to \operatorname{Ext}^2_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M)/N) \to \operatorname{Ext}^3_R(R/\mathfrak{a}, N)$$

and the fact that $\operatorname{Ext}^3_R(R/\mathfrak{a}, N)$ is $\mathfrak{a}$-minimax, it is sufficient to show that the R-module $\operatorname{Ext}^2_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M))$ is $\mathfrak{a}$-minimax. To do this, we use induction on t. When $t = 0$, by assumption $\operatorname{Ext}^2_R(R/\mathfrak{a}, M)$ and $\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^1_{\mathfrak{a}}(M))$ are both $\mathfrak{a}$-minimax. Therefore, by Proposition 2.2, the R-module $\operatorname{Ext}^1_R(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M))$ is $\mathfrak{a}$-minimax. Thus the exact sequence

$$\operatorname{Ext}^1_R(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M)) \to \operatorname{Ext}^2_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M)) \to \operatorname{Ext}^2_R(R/\mathfrak{a}, M),$$

implies that $\operatorname{Ext}^2_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M))$ is $\mathfrak{a}$-minimax. Now suppose, inductively, that $t > 0$ and suppose that the result has been proved for $t-1$. As $\Gamma_{\mathfrak{a}}(M)$ is

$\mathfrak{a}$-cominimax, it follows from the exact sequence

$$\mathrm{Ext}_R^{t+2}(R/\mathfrak{a}, M) \to \mathrm{Ext}_R^{t+2}(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M)) \to \mathrm{Ext}_R^{t+3}(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M))$$

that $\mathrm{Ext}_R^{t+2}(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M))$ is $\mathfrak{a}$-minimax. Therefore we may assume that M is $\mathfrak{a}$-torsion free. Let E be an injective envelope of M and put $M_1 = E/M$. By using a similar proof as in the proof of part (i), the induction hypothesis applied to M_1, yields that $\mathrm{Ext}_R^2(R/\mathfrak{a}, H_{\mathfrak{a}}^{t-1}(M_1))$ is $\mathfrak{a}$-minimax, and so $\mathrm{Ext}_R^2(R/\mathfrak{a}, H_{\mathfrak{a}}^t(M))$ is $\mathfrak{a}$-minimax.

(iv) Let N be an $\mathfrak{a}$-minimax submodule of $\mathrm{H}_{\mathfrak{a}}^t(M)$ and let L be a finitely generated R-module with $\mathrm{Supp}\, L \subseteq V(\mathfrak{a})$. Hence by parts (ii) and (iii), the R-module $\mathrm{Ext}_R^i(R/\mathfrak{a}, \mathrm{H}_{\mathfrak{a}}^t(M)/N)$ is $\mathfrak{a}$-minimax for all $i \leq 2$. Now, in view of [1, Theorem 2.7], the R-module $\mathrm{Ext}_R^i(L, \mathrm{H}_{\mathfrak{a}}^t(M)/N)$ is $\mathfrak{a}$-minimax for all $i \leq 2$.

(v) In view of [1, Theorem 2.7] it is enough to show that $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}_{\mathfrak{a}}^{t+1}(M)/N)$ is $\mathfrak{a}$-minimax. To this end, by the exact sequence

$$\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}_{\mathfrak{a}}^{t+1}(M)) \to \mathrm{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^{t+1}(M)/N) \to \mathrm{Ext}_R^1(R/\mathfrak{a}, N)$$

and that $\mathrm{Ext}_R^1(R/\mathfrak{a}, N)$ is $\mathfrak{a}$-minimax, it is sufficient to show that $\mathrm{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^{t+1}(M))$ is $\mathfrak{a}$-minimax. To do this, we use induction on t. When $t = 0$, by assumption $\mathrm{Ext}_R^1(R/\mathfrak{a}, M)$ and $\mathrm{Ext}_R^2(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M))$ are $\mathfrak{a}$-minimax, and so in view of Proposition 2.2 the R-module $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}_{\mathfrak{a}}^1(M))$ is $\mathfrak{a}$-minimax. Now suppose, inductively, that $t > 0$ and suppose that the result has been proved for $t-1$. Since $\Gamma_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cominimax, it follows from the exact sequence

$$\mathrm{Ext}_R^{t+1}(R/\mathfrak{a}, M) \to \mathrm{Ext}_R^{t+1}(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M)) \to \mathrm{Ext}_R^{t+2}(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M))$$

that $\mathrm{Ext}_R^{t+1}(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M))$ is $\mathfrak{a}$-minimax. Therefore we may assume that M is $\mathfrak{a}$-torsion free. Let E be an injective envelope of M and put $M_1 = E/M$. Again, by using a similar proof as in the proof of part (i), the induction hypothesis applied to M_1, yields that $\mathrm{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^t(M_1))$ is $\mathfrak{a}$-minimax, and so $\mathrm{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^{t+1}(M))$ is $\mathfrak{a}$-minimax. □

By using the above result, we can deduce the following corollary, which is the main result of this paper.

Corollary 2.4. *Let M be an $\mathfrak{a}$-minimax R-module, and let t be a non-negative integer such that $\mathrm{H}_{\mathfrak{a}}^i(M)$ is $\mathfrak{a}$-cominimax for all $i < t$. Then the following hold:*

(i) *For any $\mathfrak{a}$-minimax submodule N of $\mathrm{H}_{\mathfrak{a}}^t(M)$ and for any finitely generated R-module L with $\mathrm{Supp}\, L \subseteq V(\mathfrak{a})$, the R-modules $\mathrm{Hom}_R(L, \mathrm{H}_{\mathfrak{a}}^t(M)/N)$ and $\mathrm{Ext}_R^1(L, \mathrm{H}_{\mathfrak{a}}^t(M)/N)$ are $\mathfrak{a}$-minimax.*

(ii) *If the R-modules $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}_{\mathfrak{a}}^{t+1}(M))$ is $\mathfrak{a}$-minimax, then for any $\mathfrak{a}$-minimax submodule N of $\mathrm{H}_{\mathfrak{a}}^t(M)$ and for any finitely generated R-module L*

with $\operatorname{Supp} L \subseteq V(\mathfrak{a})$*, the R-modules* $\operatorname{Hom}_R(L, \mathrm{H}^t_{\mathfrak{a}}(M)/N)$*,* $\operatorname{Ext}^1_R(L, \mathrm{H}^t_{\mathfrak{a}}(M)/N)$ *and* $\operatorname{Ext}^2_R(L, \mathrm{H}^t_{\mathfrak{a}}(M)/N)$ *are all* $\mathfrak{a}$*-minimax.*

Proof. Since M is $\mathfrak{a}$-minimax, so $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ is $\mathfrak{a}$-minimax for all i. Thus the result is followed by Theorem 2.3. □

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