THE CYCLIC HOPF H mod K THEOREM

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The H mod K Theorem gives all possible periodic solutions in a Γ–equivariant dynamical system, based on the group-theoretical aspects. In addition, it classifies the spatio temporal symmetries that are possible. By the contrary, the Equivariant Hopf Theorem guarantees the existence of families of small–amplitude periodic solutions bifurcating from the origin for each $\mathbf{C}$–axial subgroup of $\Gamma \times \mathbb{S}^1$. In this paper we identify which periodic solution types, whose existence is guaranteed by the H mod K Theorem, are obtainable by Hopf bifurcation, when the group Γ is finite cyclic.

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1. INTRODUCTION

In the formalism of equivariant differential equations [2], [3] and [4] have been described two methods for obtaining periodic solutions: the H mod K Theorem and the Equivariant Hopf Theorem. While the H mod K Theorem offers the complete set of possible periodic solutions based exclusively on the structure of the group Γ acting on the differential equation, the Equivariant Hopf Theorem guarantees the existence of families of small-amplitude periodic solutions bifurcating from the origin for all $\mathbf{C}$–axial subgroups of $\Gamma \times \mathbb{S}^1$.

Not always all solutions predicted by the H mod K Theorem can be obtained by the generic Hopf bifurcation [4]. In [1] there are described which periodic solutions, whose existence is guaranteed by the H mod K Theorem are obtainable by the Hopf bifurcation when the group Γ is finite abelian. In this article, we pose a more specific question: what periodic solutions predicted by the H mod K Theorem are obtainable by the Hopf bifurcation when the group Γ is finite cyclic. We will answer this question by finding which additional constraints have to be added to the Abelian Hopf H mod K Theorem [1] so that the periodic solutions predicted by the H mod K Theorem coincide with

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the ones obtained by the Equivariant Hopf Theorem when the group Γ is finite cyclic.

2. THE H mod K THEOREM

We call $(\gamma, \theta) \in \Gamma \times \mathbf{S}^1$ a spatio-temporal symmetry of the solution $x(t)$. A spatio-temporal symmetry of $x(t)$ for which $\theta = 0$ is called a spatial symmetry, since it fixes the point $x(t)$ at every moment of time. The group of all spatio-temporal symmetries of $x(t)$ is denoted

$$\Sigma_{x(t)} \subseteq \Gamma \times \mathbf{S}^1.$$

As shown in [4], the symmetry group $\Sigma_{x(t)}$ can be identified with a pair of subgroups H and K of Γ and a homomorphism $\Theta : H \to \mathbf{S}^1$ with kernel K. Define

$$(1) \qquad \begin{aligned} K &= \{\gamma \in \Gamma : \gamma x(t) = x(t) \quad \forall t\} \\ H &= \{\gamma \in \Gamma : \gamma x(t) = \{x(t)\} \quad \forall t\}. \end{aligned}$$

The subgroup $K \subseteq \Sigma_{x(t)}$ is the group of spatial symmetries of $x(t)$ and the subgroup H consists of those symmetries that preserve the trajectory of $x(t)$, ie. the spatial parts of the spatio-temporal symmetries of $x(t)$. The groups $H \subseteq \Gamma$ and $\Sigma_{x(t)} \subseteq \Gamma \times \mathbf{S}^1$ are isomorphic; the isomorphism is in fact just the restriction to $\Sigma_{x(t)}$ of the projection of $\Gamma \times \mathbf{S}^1$ onto Γ. Therefore the group $\Sigma_{x(t)}$ can be written as

$$\Sigma^{\Theta} = \{(h, \Theta(h)) : h \in H, \Theta(h) \in \mathbf{S}^1\}.$$

Moreover, we call Σ^{Θ} a twisted subgroup of $\Gamma \times \mathbf{S}^1$. In our case Γ is a finite cyclic group and the H mod K Theorem states necessary and sufficient conditions for the existence of a periodic solution to a $\Gamma-$ equivariant system of ODEs with specified spatio-temporal symmetries $K \subset H \subset \Gamma$. Recall that the isotropy subgroup Σ_x of a point $x \in \mathbb{R}^n$ consists of group elements that fix x, that is they satisfy

$$\Sigma_x = \sigma \in \Gamma : \sigma x = x.$$

Let $N(H)$ be the normalizer of H in Γ, satisfying $N(H) = \{\gamma \in \Gamma : \gamma H = H\gamma\}$. Let also $\mathrm{Fix}(K) = \{x \in \mathbb{R}^n : kx = x \ \forall k \in K\}$.

Definition 1. Let $K \subset \Gamma$ be an isotropy subgroup. The variety L_K is defined by

$$L_K = \bigcup_{\gamma \notin K} \mathrm{Fix}(\gamma) \cap \mathrm{Fix}(K).$$

THEOREM 1 (H mod K Theorem [4]). *Let Γ be a finite group acting on $\mathbb{R}^n$. There is a periodic solution to some $\Gamma-$equivariant system of ODEs on $\mathbb{R}^n$ with spatial symmetries K and spatio–temporal symmetries H if and only if the following conditions hold:*

(a) *H/K is cyclic;*

(b) *K is an isotropy subgroup;*

(c) *$dim\ Fix(K) \geqslant 2$. If $dim\ Fix(K) = 2$, then either $H = K$ or $H = N(K)$;*

(d) *H fixes a connected component of $\mathrm{Fix(K)\backslash L_K}$, where L_K appears as in Definition 1 above;*

Moreover, if $(a) - (d)$ hold, the system can be chosen so that the periodic solution is stable.

Definition 2. The pair of subgroups (H, K) is called admissible if the pair satisfies hypotheses $(a) - (d)$ of Theorem 1, that is, if there exist periodic solutions to some $\Gamma-$equivariant system with (H, K) symmetry.

3. HOPF BIFURCATION WITH CYCLIC SYMMETRIES

In the following, we recall two results from [1] needed later for the proof of the Theorem 2. Let $x_0 \in \mathbb{R}^n$. Suppose that V is an $\Sigma_{x_0}-$invariant subspace of $\mathbb{R}^n$. Let $\hat{V} = x_0 + V$, and observe that $\hat{V}$ is also Σ_{x_0}–invariant.

LEMMA 1. *Let g be an $\Sigma_{x_0}-$equivariant map on $\hat{V}$ such that $g(x_0) = 0$. Then g extends to a Γ–equivariant mapping f on $\mathbb{R}^n$ so that the center subspace of $(df)_{x_0}$ equals the center subspace of $(dg)_{x_0}$.*

Proof. See [1]. □

LEMMA 2. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be $\Gamma-$equivariant and let $f(x_0) = 0$. Let V be the center subspace of $(df)_{x_0}$. Then there exists a Γ–equivariant diffeomorphism $\psi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\psi(x_0) = x_0$ and the center manifold of the transformed vector field*

$$\psi_* f(x) \equiv (d\psi)^{-1}_{\psi(x)} f(\psi(x))$$

is $\hat{V}$.

Proof. See [1]. □

In order to state the Cyclic Hopf Theorem, we need first the following lemma.

LEMMA 3. *The group Γ is cyclic if and only it is a homomorphic image of $\mathbb{Z}$.*

Proof. To show that Γ is cyclic if and only if it is a homomorphic image of $\mathbb{Z}$, let $\Gamma = \langle a \rangle$ then the map

$$\mathbb{Z} \to \Gamma, \ n \to a^n$$

is a homomorphism (since $a^{n+m} = a^n a^m$ for all $n, m \in \mathbb{Z}$) whose image is Γ. Conversely, if $f : \mathbb{Z} \to \Gamma$ is an epimorphism then let $a = f(1)$. Every $\gamma \in \Gamma$ takes the form $\gamma = f(n)$ for some $n \in \mathbb{Z}$. If $n \geqslant 0$ then

$$\gamma = f(1 + \ldots + 1) = f(1) \circ_\Gamma \cdots \circ_\Gamma f(1) = (f(1))^n = a^n.$$

The same formula holds if $n < 0$. Thus $\Gamma = \langle a \rangle$. □

THEOREM 2 (Cyclic Hopf Theorem). *In systems with finite cyclic symmetry, generically, Hopf bifurcation at a point x_0 occurs with simple eigenvalues, and there exists a unique branch of small–amplitude periodic solutions emanating from x_0. Moreover the spatio–temporal symmetries of the bifurcating periodic solutions are*

$$H = \Sigma_{x_0}, \tag{2}$$

and

$$K = \ker_V(H), \tag{3}$$

H acts H–simply on V, while Γ and all of its subgroups, in particular H and K are homomorphic images of $\mathbb{Z}$. In addition let $\mathbb{Z}_k$ act on $\mathbb{R}^k$ by a cyclic permutation of coordinates. Let $\mathbb{Z}_q \subseteq \mathbb{Z}_n \subseteq \mathbb{Z}_k$. Then there is a $\mathbb{Z}_n$–simple representation with kernel $\mathbb{Z}_q$ with the single exception when $n = k$ is even and $q = \frac{k}{2}$.

Proof. The proof relies on the proof of the homologous Theorem in [1], with changes concerning the form of the subgroups H and K. However, we will prefer to give the proof entirely, including the parts that coincide with the proof in [1], to easy the lecture of the paper. Since Γ, H and K are homomorphic images of $\mathbb{Z}$, they are cyclic. We begin as in [1], by showing that the equivariant Hopf bifurcation leads to a unique branch of small–amplitude periodic solutions emanating from x_0. From Lemma 1 it follows that the bifurcation point $x_0 = 0$ and therefore $\Gamma = \Sigma_{x_0}$. Moreover, from Lemma 2 it follows that if reducing to the center manifold, we may assume that $\mathbb{R}^n = V$ and therefore from [3] it follows that the center subspace V at the Hopf bifurcation point is Γ–simple. This means that V is either a direct sum of two absolutely irreducible representations or it is itself irreducible but not absolutely irreducible. Since the irreducible representations of abelian groups (and subsequently cyclic groups) are one–dimensional and absolutely irreducible or two–dimensional and non-absolutely irreducible, it follows that V is two–dimensional and therefore the eigenvalues obtained at the linearization about the bifurcation point x_0 are simple. Now the standard Hopf Bifurcation Theorem applies to obtain a unique branch of periodic solutions.

Let $x(t,\lambda)$ be the unique branch of small–amplitude periodic solutions that emanate at the Hopf bifurcation point x_0. For each t,

$$x_0 = \lim_{\lambda \to 0} x(t,\lambda).$$

Let H be the spatio-temporal symmetry subgroup of $x(\cdot,\lambda)$, and let $\Phi : H \to \mathbb{S}^1$ be the homomorphism that associates a symmetry $h \in H$ with a phase shift $\Phi(h) \in \mathbb{S}^1$. To prove that $H \subset \Sigma_{x_0}$ we have

$$\begin{aligned} hx_0 &= \lim_{\lambda \to 0} hx(0,\lambda) && \text{by continuity of } h \\ &= \lim_{\lambda \to 0} x(\Phi(h),\lambda) && \text{by definition of spatio} - \text{temporal symmetries} \\ &= x_0 \end{aligned}$$

and therefore $h \in \Sigma_{x_0}$. In the following we proof that $\Sigma_{x_0} \subset H$. Let $\gamma \in \Sigma_{x_0} \subseteq \Gamma$; therefore $\gamma x(t,\lambda)$ is also a periodic solution. Since the periodic is unique (as shown above), we have

$$\gamma\{x(t,\lambda)\} = \{x(t,\lambda)\},$$

so $\gamma \in H$. Lemma 2 allows us to assume that the center manifold at x_0 is $\hat{V} = v + x_0$, which may be identified with V, and therefore V is H–invariant. Therefore V is H–simple since γ is cyclic (and subsequently abelian). Since Γ is cyclic, all its subgroups are cyclic, in particular H and K.
The proof of the last condition is the proof of Proposition 6.2 in [1]. □

4. CONSTRUCTING SYSTEMS WITH CYCLIC SYMMETRIES NEAR HOPF POINTS

This section consists in recalling the results corresponding section 4 in [1] where the construction of systems with abelian symmetry near Hopf points has been carried out. When Γ is finite cyclic, a key step in constructing H mod K periodic solutions from Hopf bifurcation at x_0 is the construction of a locally Σ_{x_0}–equivariant vector field. We first construct, for finite symmetry groups, a Γ–equivariant vector field that has a stable equilibrium, $x_0 \in \mathbb{R}^n$, with the desired isotropy. We will use

Lemma 4. *For any finite set of distinct points $y_1, \dots, y_l$, vectors $v_1, \dots, v_l$ in $\mathbb{R}^n$ and matrices $A_1, \dots, A_l \in \mathrm{GL}(n)$, there exists a polynomial map $g : \mathbb{R}^n \to \mathbb{R}^n$ such that $g(y_j) = v_j$ and $(dg)_{y_j} = A_j$.*

Proof. See [5]. □

THEOREM 3. *Let Γ be a finite cyclic group acting on $\mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$. Then there exists a Γ–equivariant system of ODEs on $\mathbb{R}^n$ with a stable equilibrium x_0.*

Proof. See [1]. □

In conclusion any point $x_0 \in \mathbb{R}^n$ can be a stable equilibrium for a Γ–equivariant vector field $f : \mathbb{R}^n \to \mathbb{R}^n$. It is clear that $(df)_{x_0}$ must commute with the isotropy subgroup Σ_{x_0} of x_0 [4]. The following result states that the linearization about the equilibrium x_0 can be any linear map that commutes with the isotropy subgroup.

THEOREM 4. *Let $x_0 \in \mathbb{R}^n$ and $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map that commutes with the isotropy subgroup Σ_{x_0} of x_0. Then there exists a polynomial Γ–equivariant vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(x_0) = 0$ and $(df)_{x_0} = A$.*

Proof. See [1]. □

When constructing a Hopf bifurcation at points $x_0 \in \mathbb{R}^n$ we do not necessarily assume full isotropy. Genericity of Σ_{x_0}–simple subspaces at points of Hopf bifurcation is given by Γ–equivariant mappings as follows.

LEMMA 5. *Let Γ act on $\mathbb{R}^n$ and fix $x_0 \in \mathbb{R}^n$. Let V be a Σ_{x_0}–invariant neighborhood of x_0 such that $\gamma\bar{V} \cap \bar{V} = \emptyset$ for any $\gamma \in \Gamma \backslash \Sigma_{x_0}$. Let $g : \bar{V} \times \mathbb{R} \to \mathbb{R}^n$ be a smooth Σ_{x_0}–equivariant vector field. Then there exists an extension of g to a smooth $\Gamma-$equivariant vector field $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$.*

Proof. See [1]. □

5. THE CYCLIC HOPF *H* mod *K* THEOREM

THEOREM 5 (Cyclic Hopf H mod K Theorem). *Let Γ be a finite cyclic group acting on $\mathbb{R}^n$. There is an H mod K periodic solution that arises by a generic Hopf bifurcation if and only if the following seven conditions hold: Theorem 1 (a)$-$(d), Γ and all its subgroups, in particular H and K are homomorphic images of $\mathbb{Z}$, there exists an H–simple subspace V such that $K = \ker_V(H)$, and let $\mathbb{Z}_k$ act on $\mathbb{R}^k$ by a cyclic permutation of coordinates. Let $\mathbb{Z}_q \subseteq \mathbb{Z}_n \subseteq \mathbb{Z}_k$. Then there is a $\mathbb{Z}_n$–simple representation with kernel $\mathbb{Z}_q$ with the single exception when $n = k$ is even and $q = \frac{k}{2}$.*

Proof. Necessity follows from the H mod K Theorem (Theorem 1) and the Cyclic Hopf Theorem (Theorem 2). We'll prove the sufficiency next. The idea of the proof will again, rely heavily on the proof of Abelian Hopf H mod K

Theorem in [1]. Let $x_0 \in \mathbb{R}^n$ and let H be the isotropy subgroup of the point x_0, ie. $H = \Sigma_{x_0}$. Moreover, let W be a H–simple representation. Since Γ is cyclic (in particular, abelian), W is two–dimensional. Now we can define the linear maps $A(\lambda) : W \to W$ by

$$A(\lambda) = \begin{bmatrix} \lambda & -1 \\ 1 & \lambda \end{bmatrix}.$$

Since W is two–dimensional it is easy to prove the commutativity with A. We have

$$A(\lambda) \cdot W = \begin{bmatrix} \lambda & -1 \\ 1 & \lambda \end{bmatrix} \cdot \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} \lambda a - b & -\lambda b - a \\ a + \lambda b & a\lambda - b \end{bmatrix} = W \cdot A(\lambda).$$

Next we can extend Theorem 4 to a bifurcation problem as in Lemma 5. Let $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be a Γ–equivariant polynomial such that for all $\gamma \in \Gamma$, $f(\gamma x_0, \lambda) = 0$ and $(df)_{\gamma x_0, \lambda}|_{\gamma W} = \gamma A(\lambda)\gamma^{-1}$. Moreover, let $g = f|_{W+x_0}$. From the way f has been constructed, g is H–equivariant on $W + x_0$ and $g(x_0) = 0$, hence from Lemma 2 we have that W is the center subspace of $(dg)_{x_0,0}$.

Next consider $(dg)_{x_0,\lambda}|_W$; its eigenvalues are $\sigma(\lambda) \pm i\rho(\lambda)$ with $\sigma(0) = 0$, $\rho(0) = 1$ and $\sigma'(0) \neq 0$. Then the Equivariant Hopf Theorem extended to a point $x_0 \in \mathbb{R}^n$ implies the existence of small–amplitude periodic solutions emanating from x_0 with spatio-temporal symmetries H and spatial symmetries K. $\square$

6. GENERAL CONSIDERATIONS BETWEEN THE DIFFERENCE OF THE RESULTS IN THIS ARTICLE AND [1]

In the first place it must be highlighted that one can start we the methodology used in [1] and add the restrictions presented in this paper to obtain the Cyclic Hopf H mod K Theorem, but not vice–versa. This is obvious, because any cyclic group is abelian but not any abelian group is cyclic.

In this section we use the Cyclic Hopf H mod K Theorem to exhibit symmetry pairs (H, K) that are admissible by the Abelian Hopf H mod K Theorem but not admissible by the Cyclic Hopf H mod K Theorem. Let $\mathbb{Z}_l$ act on $\mathbb{R}^l$ by cyclic permutation of coordinates and $\Gamma = \mathbb{Z}_l \times \mathbb{Z}_k$ act on $\mathbb{R}^l \times \mathbb{R}^k$ by the diagonal action, where $l, k > 1$. We show Abelian Hopf H mod K admissible pairs but not Cyclic Hopf H mod K admissible pairs for this action of Γ by classifying in Theorem 6 all Cyclic Hopf H mod K admissible pairs $K \subseteq H \subseteq \Gamma$ and showing that there are admissible pairs that are not on the list.

Theorem 6. *By applying the Cyclic Hopf H mod K Theorem, the (H, K) Hopf–admissible pairs in Γ are $(\mathbb{Z}_m \times \mathbb{Z}_n,\ \mathbb{Z}_m \times \mathbb{Z}_q)$ where q divides n except when $q = \frac{k}{2}$ and $n = k$, and $(\mathbb{Z}_m \times \mathbb{Z}_n,\ \mathbb{Z}_p \times \mathbb{Z}_n)$ where p divides m except when $p = \frac{l}{2}$ and $m = l$. Moreover, m and n are coprimes, m and q are coprimes and p and n are coprimes.*

Proof. The proof is a restriction to the cases m and n are coprimes, m and q are coprimes and p and n are coprimes, of the proof of Theorem 6.1 in [1]. □

To find an example of a pair (H, K) that is admissible by the Abelian Hopf H mod K Theorem but not by the Cyclic Hopf H mod K Theorem, let's take $(H = \mathbb{Z}_m \times \mathbb{Z}_n,\ K = \mathbb{Z}_m \times \mathbb{Z}_q)$ where q divides n except when $q = \frac{k}{2}$ and $n = k$, and m and n are not coprimes, m and q are not coprimes. They are admissible by the Abelian Hopf H mod K by applying Theorem 6.1 in [1]. However, they are not admissible by the Cyclic Hopf H mod K Theorem because of the application of the the Fundamental Theorem of Finitely Generated Abelian Groups. Indeed, if, for example m and n are not coprimes then they have a common divisor integer $a \in \mathbb{N}$ that is prime, and in this case $m = ab,\ n = ac$ for some integers $b \in \mathbb{N},\ c \in \mathbb{N}$ and the group $\mathbb{Z}_{ab} \times \mathbb{Z}_{ac}$ is not cyclic. A similar case applies for the group $K = \mathbb{Z}_m \times \mathbb{Z}_q$ if m and q are not coprimes, or the group $K = \mathbb{Z}_p \times \mathbb{Z}_n$ if n and p are not coprimes.

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