ON POSITIVE DEFINITE QUADRATIC FORMS AND THE EXTENDED HECKE GROUP $\overline{H}(\sqrt{2})$

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In this work, our main goal is to determine the connection between positive quadratic forms $F = (a, b, c)$ in $\overline{H}(\sqrt{2})$ whose base points $z = z(F)$ lie on the line $x = \frac{-\sqrt{2}}{m}$ and the elements of extended Hecke group $\overline{H}(\sqrt{2})$.

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1. INTRODUCTION

A real binary quadratic form (or just a form) F is polynomial in two variables x and y of the type

$$F = F(x, y) = ax^2 + bxy + cy^2$$

with real coefficients a, b, c. We denote F briefly by $F = (a, b, c)$. The discriminant of F is defined by the formula $b^2 - 4ac$ and is denoted by $\Delta = \Delta(F)$. A binary quadratic form F is called

(1) integral form if and only if $a, b, c \in \mathbb{Z}$
(2) positive definite if and only if$\Delta(F) < 0$ and $a, c > 0$
(3) indefinite if and only if $\Delta(F) > 0$.

Most properties of quadratic forms can be given by the aid of extended modular group $\overline{\Gamma}$. Gauss (1777–1855) defined the group action of $\overline{\Gamma}$ on the set of forms as follows:

$$gF(x, y) = (ar^2 + brs + cs^2)x^2 + (2art + bru + bts + 2csu)xy + (at^2 + btu + cu^2)y^2 \tag{1.1}$$

for $g = \begin{bmatrix} r & s \\ t & u \end{bmatrix} \in \overline{\Gamma}$, that is, gF is obtained from F by making the substitution $x \to rx + tu$ and $y \to sx + uy$, that is,

$$gF = (rx + ty, sx + uy).$$

Moreover, $\Delta(F) = \Delta(gF)$ for all $g \in \overline{\Gamma}$, that is, the action of $\overline{\Gamma}$ on forms leaves the discriminant invariant. If F is indefinite or integral, then so is gF for all $g \in \overline{\Gamma}$. Let F and G be two forms. If there exists a $g \in \overline{\Gamma}$ such that $gF = G$, then F and G are called equivalent. If $\det g = 1$, then F and G are called properly equivalent, and if $\det g = -1$, then F and G are called improperly equivalent (for further details on binary quadratic forms see [1, 2, 4, 7]).

2. POSITIVE DEFINITE FORMS AND THE EXTENDED HECKE GROUP $\overline{H}(\sqrt{2})$

In this section, we deal with the connection between positive definite forms and the extended Hecke groups. For this reason, we first give some preliminary results on Hecke groups.

In [5], Hecke introduced the groups $H(\lambda)$ generated by two linear fractional transformations

$$T(z) = -\frac{1}{z} \quad \text{and} \quad U(z) = z + \lambda$$

Hecke showed that $H(\lambda)$ is discrete if and only if $\lambda = \lambda_q = 2\cos(\frac{\pi}{q})$, $q \in \mathbb{N}$, $q \geq 3$ or $\lambda \geq 2$. These groups have come to be known as the Hecke groups, and we will denote them by $H(\lambda_q)$ for $q \geq 3$. The Hecke group $H(\lambda_q)$ is isomorphic to the free product of two finite cyclic groups of orders 2 and q, and it has the presentation

$$H(\lambda_q) = \langle T, S \mid T^2 = S^q = I \rangle \cong C_2 * C_q.$$

For $q = 4$, we get a presentation of $H(\sqrt{2})$ as

$$H(\lambda_4) = H(\sqrt{2}) = \langle T, S \mid T^2 = S^4 = I \rangle \cong C_2 * C_4.$$

Adding the reflection $R(z) = \frac{1}{\overline{z}}$ to Hecke groups, we then obtain the extended Hecke group $\overline{H}(\lambda)$. The extended Hecke groups $\overline{H}(\lambda_q)$ are isomorphic to the free product of two dihedral groups of orders 4 and $2q$ with amalgamation C_2. We know that the extended Hecke group $\overline{H}(\sqrt{2})$ is isomorphic to $D_2 *_{C_2} D_4$ (where D_n denote the Dihedral group) and has a presentation

$$\overline{H}(\sqrt{2}) = \langle T, S, R \mid T^2 = R^2 = S^4 = (TR)^2 = (RS)^2 = I \rangle.$$

The fundamental region of $\overline{H}(\sqrt{2})$ is

$$\text{Reg}(\overline{H}(\sqrt{2})) = \{z \in \mathbb{U}\colon -\frac{\sqrt{2}}{2} \leq \text{Re}(z) \leq 0 \text{ and } |z| \geq 1\},$$

where $\mathbb{U}$ denotes the upper half–plane (for further details see [3, 6, 8, 9]).

It is known that given any positive definite form $F = (a, b, c)$, there exists a complex number z in $\mathbb{U}$ such that

$$F(x, y) = a(x + zy)(x + \overline{z}y). \tag{2.1}$$

In this case z is called the base point of F and is denoted by $z = z(F)$. Taking $z = u + iv$ in (2.1), we get

$$F(x,y) = ax^2 + 2auxy + a\left|z\right|^2 y^2.$$

So we obtain $2au = b$ and $a\left|z\right|^2 = c$. Therefore, we get $u = \frac{b}{2a}$ and $v = \frac{\sqrt{-\Delta(F)}}{2a}$. Since v is positive, we get

$$z = \frac{b + i\sqrt{-\Delta(F)}}{2a} \in \mathbb{U}.$$

Conversely for a given point $z \in \mathbb{U}$, Tekcan and Bizim proved in [10] that there exists a positive definite quadratic form $F = (a,b,c)$ whose base point is z. They showed that given any complex number $z = x + iy$ in $\mathbb{U}$, there exists a positive definite form F of the form

$$F = (a,b,c) = \left(\frac{1}{\left|z\right|^2}, \frac{2x}{\left|z\right|^2}, 1\right) \tag{2.2}$$

of discriminant $\Delta(F) = \frac{-4y^2}{|z|^4} < 0$ whose base point is z. So there is a one–to–one correspondence between positive definite forms and points in $\mathbb{U}$.

Tekcan and Bizim considered the positive definite forms whose base points lying on the line $x = \frac{-1}{m}$ for some integer $m \geq 2$ and proved the following result.

Lemma 2.1 ([10], Theorem 2.1). *For $m \geq 2$ consider the line $x = \frac{-1}{m}$. Then there exists a positive definite quadratic form $F = (a,b,c)$ of discriminant $\Delta(F) = -D$, where $0 < D < m^2$, whose base point $z(F)$ lies on the line $x = \frac{-1}{m}$.*

Later they determined the number of integral positive definite forms whose base points lying on the line $x = \frac{-1}{m}$ and proved

Lemma 2.2 ([10], Corollary 2.2). *If m is odd, say $m = 2k+1$, for $k \in \mathbb{Z}^+$ then there exist k positive definite integral forms*

$$F_j = (mj,\ -2j,\ 1),\ 1 \leq j \leq k$$

of discriminant $\Delta(F_j) = -4j(m-j)$ whose base points $z(F_j)$ lie on the line $x = \frac{-1}{m}$. If m is even, say $m = 2k$, for $k \in \mathbb{Z}^+$, then there exist $m-1$ positive definite integral forms

$$F_j = (kj,\ -j,\ 1),\ 1 \leq j \leq m-1$$

of discriminant $\Delta(F_j) = -j(2m-j)$ whose base points $z(F_j)$ lie on the line $x = \frac{-1}{m}$.

In the present paper, we consider the same problem by considering the Hecke group $\overline{H}(\sqrt{2})$ instead of $\overline{\Gamma}$. Now let

$$F = \mathbb{Z}[\sqrt{2}]F = \mathbb{Z}[\sqrt{2}](a,b,c)$$

be a positive definite form such that $a,b,c \in \mathbb{Z}[\sqrt{2}]$. Then we can give the following theorem.

THEOREM 2.3. *Let $m \geq 3$ be an integer and let $0 < 2D < m^2$. Then there exists a positive definite form $\mathbb{Z}[\sqrt{2}]F$ of discriminant $-D$ whose base point lies on the line $x = \frac{-\sqrt{2}}{m}$.*

Proof. Let $m \geq 3$ be an integer, and let $x = \frac{-\sqrt{2}}{m}$. Then from (2.2), we get the positive definite form

$$F = \left(\frac{mD}{2(m + \sqrt{m^2 - 2D})}, \frac{-D\sqrt{2}}{m + \sqrt{m^2 - 2D}}, 1 \right)$$

in $\mathbb{Z}[\sqrt{2}]$. Notice that this form is not integral. To do it integral, we have two cases:

Case 1: Let m be odd, say $m = 2k + 1$ for some $k \in \mathbb{Z}^+$. Then F is an integral positive definite form, that is, $F \in \mathbb{Z}[\sqrt{2}]$ if and only if $2D = m^2 - (2l - 1)^2$ for $|l| \leq k$. Indeed, let $F \in \mathbb{Z}[\sqrt{2}]$. Since m is odd, D must be even and so $\sqrt{m^2 - 2D}$ is odd, say $\sqrt{m^2 - 2D} = |2l - 1|$ for some $l \in \mathbb{Z}^+$. Then clearly, $2D = m^2 - (2l - 1)^2$. Since D must be positive, we get

$$D > 0 \Leftrightarrow m^2 - (2l - 1)^2 > 0 \Leftrightarrow (m - 2l + 1)(m + 2l - 1) > 0.$$

Note that $m = 2k + 1$. So we have to

$$(k + l)(k - l + 1) > 0.$$

In this case either $l > 0$ or $l < 0$. If $l > 0$, then $k + l > 0$ since k is positive. Therefore $k - l + 1$ must be positive and hence $k > l - 1$. Let $l < 0$. If $k + l < 0$, then we get $k - l + 1 < 0$ and so $k + 1 < l$, which contradicts with $k \in \mathbb{Z}^+$. Therefore, $k + 1 > 0$. Thus, $k - l + 1 > 0$. Which means that $-k \leq l$. From these two conditions, we obtain $|l| \leq k$.

Conversely let $|l| \leq k$ for $2D = m^2 - (2l - 1)^2$. In this case since $m - (2l - 1)$ is even, we get

$$a = \frac{mD}{2(m + \sqrt{m^2 - 2D})} = \frac{(2k + 1)(k - l + 1)}{2} \in \mathbb{Z}$$

since $(k - l)$ is odd and $|l| \leq k$. Similarly we easily deduce that

$$b = \frac{-D}{m + \sqrt{m^2 - 2D}} = l - k - 1 \in \mathbb{Z}.$$

Case 2: Let m be even, say $m = 2k$ for some $k \in \mathbb{Z}^+$. Then F is an integral positive definite form, that is, $F \in \mathbb{Z}[\sqrt{2}]$ if and only if $2D = m^2 - t^2$ for $|t| \leq m - 1$, where $t \neq 0$ is an integer. Let $F \in \mathbb{Z}[\sqrt{2}]$. Then $0 < 2D < m^2 \implies \sqrt{m^2 - 2D} = |t|$ for $t \in \mathbb{Z}$. So $2D = m^2 - t^2$. Since D is positive, $m^2 - t^2$ must be positive, that is,

$$(m - t)(m + t) > 0.$$

Therefore it can be easily seen $|t| \leq m-1$.

Conversely, let $2D = m^2 - t^2$ for $|t| \leq m-1$. Then since m is even, we get

$$a = \frac{mD}{2(m+\sqrt{m^2-2D})} = \frac{k(m-t)}{2} \in \mathbb{Z}$$

and

$$b = \frac{-D}{m+\sqrt{m^2-2D}} = -\frac{(m-t)}{2} \in \mathbb{Z}.$$

This completes the proof. □

From the above theorem, we can give the following result.

Corollary 2.4. *Let $F = \mathbb{Z}[\sqrt{2}](a,b,c)$ be the positive definite form obtained in Theorem 2.3.*

(1) *If m is odd, say $m = 2k+1$ for $k \in \mathbb{Z}^+$, then there are k positive definite forms $F_j = (mj, -2\sqrt{2}j, 1)$ for $1 \leq j \leq k$ in $\mathbb{Z}[\sqrt{2}]$ of discriminant $\Delta(F_j) = 4j(4j-m)$ whose base points lying on the line $x = -\frac{\sqrt{2}}{m}$.*

(2) *If m is even, say $m = 2k$ for $k \in \mathbb{Z}^+$, then there are $m-1$ positive definite forms $F_j = (kj, -\sqrt{2}j, 1)$ for $1 \leq j \leq m-1$ in $\mathbb{Z}[\sqrt{2}]$ of discriminant $\Delta(F_j) = 2j(j-2k)$ whose base points lying on the line $x = -\frac{\sqrt{2}}{m}$.*

Example 2.5. Let $m = 3$. Then there is one positive definite form $F = (3, -2\sqrt{2}, 1)$ of discriminant -4 whose base point lying on the line $x = -\frac{\sqrt{2}}{3}$, and for $m = 4$, there are three positive definite quadratic forms $F_1 = (2, -\sqrt{2}, 1)$, $F_2 = (4, -2\sqrt{2}, 1)$ and $F_3 = (6, -3\sqrt{2}, 1)$ whose base points lying on the line $x = -\frac{\sqrt{2}}{4}$.

A positive definite form $F = (a,b,c)$ is called reduced if $|b| \leq a \leq c$. If a positive definite form F of discriminant Δ is not reduced, than it can be transferred into a reduced form

$$F_R = \begin{cases} (1, 0, \frac{\Delta}{4}) & \text{if } \Delta \equiv 0 (\text{mod } 4) \\ (1, 1, \frac{1-\Delta}{4}) & \text{if } \Delta \equiv 1 (\text{mod } 4) \end{cases}$$

of discriminant Δ by an element of $\overline{\Gamma}$, that is, there exists a $g \in \overline{\Gamma}$ such that $gF = F_R$. Similarly the form $F = \mathbb{Z}[\sqrt{2}](a,b,c)$ is called reduced if $|b| \leq a \leq c$ and if a positive definite form F of discriminant Δ is not reduced, than it can be transferred into a reduced form

$$F_R = \begin{cases} (1, 0, -\frac{\Delta}{4}) & \text{if } \Delta \equiv 0 (\text{mod } 4) \\ (1, -\sqrt{2}, \frac{2-\Delta}{4}) & \text{if } \Delta \equiv 2 (\text{mod } 4) \end{cases} \tag{2.3}$$

of discriminant Δ by an element of $\overline{H}(\sqrt{2})$, that is, there exists a $g \in \overline{H}(\sqrt{2})$ such that $gF = F_R$.

The positive definite forms obtained in Corollary 2.4 are not reduced. But we can transfer them into the reduced forms as follows:

THEOREM 2.6. *Let* $F = \mathbb{Z}[\sqrt{2}](a,b,c) \in \overline{H}(\sqrt{2})$ *be a non–reduced positive definite form obtained in Corollary 2.4. Then there exists a* $g \in \overline{H}(\sqrt{2})$ *such that* $gF = F_R$, *where* F_R *is defined in* (2.3).

Proof. Let m be odd. Then positive definite forms are $F_j = (mj, -2\sqrt{2}j, 1)$ for $1 \leq j \leq k$. The discriminant of F_j is

$$\Delta(F_j) = (-2\sqrt{2}j)^2 - 4mj \equiv 0(\text{mod } 4).$$

So the reduced form is $F_{Rj} = (1, 0, mj - 2j^2)$ by (2.3). Let $g_j = \begin{bmatrix} r & s \\ t & u \end{bmatrix} \in \overline{H}(\sqrt{2})$. Then from (1.1), we have the system of equations:

$$\begin{aligned} mjr^2 - 2\sqrt{2}jrs + s^2 &= 1 \\ 2mjrt - 2\sqrt{2}jru - 2\sqrt{2}jts + 2su &= 0 \\ mjt^2 - 2\sqrt{2}jtu + u^2 &= mj - 2j^2. \end{aligned}$$

This system of equations has a solution for $r = 0$, $s = 1$, $t = 1$ and $u = \sqrt{2}j$, that is, $g_jF_j = F_{R_j}$ for $g_j = \begin{bmatrix} 0 & 1 \\ 1 & \sqrt{2}j \end{bmatrix} \in \overline{H}(\sqrt{2})$.

Let m be even. Then positive definite forms are $F_j = (kj, -\sqrt{2}j, 1)$ for $1 \leq j \leq m-1$. The discriminant of F_j is $\Delta(F_j) = 2j^2 - 4kj$. Here we have two cases.

(i) If j is even, say $j = 2h$, for some $h \in \mathbb{Z}^+$, then the discriminant of F_j is

$$\Delta(F_j) = 4(2h^2 - kh) \equiv 0(\text{mod } 4).$$

So the reduced form is $F_{R_j} = (1, 0, kj - \frac{j^2}{2})$. The system of equations

$$\begin{aligned} kjr^2 - \sqrt{2}jrs + s^2 &= 1 \\ 2kjrt - \sqrt{2}jru - \sqrt{2}jts + 2su &= 0 \\ kjt^2 - \sqrt{2}jtu + u^2 &= kj - \frac{j^2}{2} \end{aligned}$$

has a solution for $r = 0$, $s = 1$, $t = 1$ and $u = \frac{\sqrt{2}j}{2}$, that is, $g_jF_j = F_{R_j}$ for $g_j = \begin{bmatrix} 0 & 1 \\ 1 & \frac{\sqrt{2}j}{2} \end{bmatrix} \in \overline{H}(\sqrt{2})$.

(ii) If j is odd, say $j = 2e - 1$ for some $e \in \mathbb{Z}^+$, then $F_j = (k(2e-1), -\sqrt{2}(2e-1), 1)$ of discriminant

$$\Delta = 8e^2 - 8e + 2 - 8ke + 4k \equiv 2(\text{mod } 4).$$

So the reduced form is $F_{R_j} = (1, -\sqrt{2}, -2e^2 + 2e + 2ke - k)$. The system of equations

$$k(2e-1)r^2 - \sqrt{2}(2e-1)rs + s^2 = 1$$
$$2k(2e-1)rt - \sqrt{2}(2e-1)ru - \sqrt{2}(2e-1)ts + 2su = -\sqrt{2}$$
$$k(2e-1)t^2 - \sqrt{2}(2e-1)tu + u^2 = -2e^2 + 2e + 2ke - k$$

has a solution for $r = 0$, $s = -1$, $t = 1$ and $u = e\sqrt{2}$, that is, $g_jF_j = F_{R_j}$ for $g_j = \begin{bmatrix} 0 & -1 \\ 1 & (\frac{j+1}{2})\sqrt{2} \end{bmatrix} \in \overline{H}(\sqrt{2})$. □

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