ON POSITIVE DEFINITE QUADRATIC FORMS
AND THE EXTENDED HECKE GROUP $\mathcal{H}(\sqrt{2})$

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In this work, our main goal is to determine the connection between positive quadratic forms $F = (a, b, c)$ in $\mathcal{H}(\sqrt{2})$ whose base points $z = z(F)$ lie on the line $x = -\sqrt{2}/m$ and the elements of extended Hecke group $\mathcal{H}(\sqrt{2})$.

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1. INTRODUCTION

A real binary quadratic form (or just a form) $F$ is polynomial in two variables $x$ and $y$ of the type

$$F = F(x, y) = ax^2 + bxy + cy^2$$

with real coefficients $a, b, c$. We denote $F$ briefly by $F = (a, b, c)$. The discriminant of $F$ is defined by the formula $b^2 - 4ac$ and is denoted by $\Delta = \Delta(F)$. A binary quadratic form $F$ is called

(1) integral form if and only if $a, b, c \in \mathbb{Z}$
(2) positive definite if and only if $\Delta(F) < 0$ and $a, c > 0$
(3) indefinite if and only if $\Delta(F) > 0$.

Most properties of quadratic forms can be given by the aid of extended modular group $\overline{\Gamma}$. Gauss (1777–1855) defined the group action of $\overline{\Gamma}$ on the set of forms as follows:

$$gF(x, y) = (ar^2 + brs + cs^2)x^2 + (2art + bru + bts + 2csu)xy + (at^2 + btu + cu^2)y^2$$

for $g = \begin{bmatrix} r & s \\ t & u \end{bmatrix} \in \overline{\Gamma}$, that is, $gF$ is obtained from $F$ by making the substitution $x \rightarrow rx + tu$ and $y \rightarrow sx + uy$, that is,

$$gF = (rx + ty, sx + uy).$$

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Moreover, $\Delta (F) = \Delta (gF)$ for all $g \in \Gamma$, that is, the action of $\Gamma$ on forms leaves the discriminant invariant. If $F$ is indefinite or integral, then so is $gF$ for all $g \in \Gamma$. Let $F$ and $G$ be two forms. If there exists a $g \in \Gamma$ such that $gF = G$, then $F$ and $G$ are called equivalent. If $\det g = 1$, then $F$ and $G$ are called properly equivalent, and if $\det g = -1$, then $F$ and $G$ are called improperly equivalent (for further details on binary quadratic forms see [1,2,4,7]).

2. POSITIVE DEFINITE FORMS

AND THE EXTENDED HECKE GROUP $\mathcal{H}(\sqrt{2})$

In this section, we deal with the connection between positive definite forms and the extended Hecke groups. For this reason, we first give some preliminary results on Hecke groups.

In [5], Hecke introduced the groups $H(\lambda)$ generated by two linear fractional transformations

$$T(z) = -\frac{1}{z} \quad \text{and} \quad U(z) = z + \lambda$$

Hecke showed that $H(\lambda)$ is discrete if and only if $\lambda = \lambda_q = 2 \cos(\frac{\pi}{q})$, $q \in \mathbb{N}$, $q \geq 3$ or $\lambda \geq 2$. These groups have come to be known as the Hecke groups, and we will denote them by $H(\lambda_q)$ for $q \geq 3$. The Hecke group $H(\lambda_q)$ is isomorphic to the free product of two finite cyclic groups of orders 2 and $q$, and it has the presentation

$$H(\lambda_q) = \langle T, S \mid T^2 = S^q = I \rangle \cong C_2 * C_q.$$ 

For $q = 4$, we get a presentation of $H(\sqrt{2})$ as

$$H(\lambda_4) = H(\sqrt{2}) = \langle T, S \mid T^2 = S^4 = I \rangle \cong C_2 * C_4.$$

Adding the reflection $R(z) = \frac{1}{z}$ to Hecke groups, we then obtain the extended Hecke group $\overline{H}(\lambda)$. The extended Hecke groups $\overline{H}(\lambda_q)$ are isomorphic to the free product of two dihedral groups of orders 4 and $2q$ with amalgamation $C_2$. We know that the extended Hecke group $\overline{H}(\sqrt{2})$ is isomorphic to $D_2 *_{C_2} D_4$ (where $D_n$ denote the Dihedral group) and has a presentation

$$\overline{H}(\sqrt{2}) = \langle T, S, R \mid T^2 = R^2 = S^4 = (TR)^2 = (RS)^2 = I \rangle.$$

The fundamental region of $\overline{H}(\sqrt{2})$ is

$$\text{Reg}(\overline{H}(\sqrt{2})) = \{ z \in \mathbb{U}: -\frac{\sqrt{2}}{2} \leq \text{Re}(z) \leq 0 \text{ and } |z| \geq 1 \},$$

where $\mathbb{U}$ denotes the upper half–plane (for further details see [3,6,8,9]).

It is known that given any positive definite form $F = (a, b, c)$, there exists a complex number $z$ in $\mathbb{U}$ such that

$$F(x, y) = a(x + zy)(x + \overline{z}y).$$
In this case $z$ is called the base point of $F$ and is denoted by $z = z(F)$. Taking $z = u + iv$ in (2.1), we get

$$F(x, y) = ax^2 + 2auxy + a|z|^2 y^2.$$ 

So we obtain $2au = b$ and $a|z|^2 = c$. Therefore, we get $u = \frac{b}{2a}$ and $v = \frac{\sqrt{-\Delta(F)}}{2a}$. Since $v$ is positive, we get

$$z = b + i\frac{\sqrt{-\Delta(F)}}{2a} \in \mathbb{U}.$$ 

Conversely for a given point $z \in \mathbb{U}$, Tekcan and Bizim proved in [10] that there exists a positive definite quadratic form $F = (a, b, c)$ whose base point is $z$. They showed that given any complex number $z = x + iy$ in $\mathbb{U}$, there exists a positive definite form $F$ of the form

$$(2.2) \quad F = (a, b, c) = \left( \frac{1}{|z|^2}, \frac{2x}{|z|^2}, 1 \right)$$

of discriminant $\Delta(F) = -\frac{4y^2}{|z|^4} < 0$ whose base point is $z$. So there is a one–to–one correspondence between positive definite forms and points in $\mathbb{U}$.

Tekcan and Bizim considered the positive definite forms whose base points lying on the line $x = -\frac{1}{m}$ for some integer $m \geq 2$ and proved the following result.

**Lemma 2.1 ([10], Theorem 2.1).** For $m \geq 2$ consider the line $x = -\frac{1}{m}$. Then there exists a positive definite quadratic form $F = (a, b, c)$ of discriminant $\Delta(F) = -D$, where $0 < D < m^2$, whose base point $z(F)$ lies on the line $x = -\frac{1}{m}$.

Later they determined the number of integral positive definite forms whose base points lying on the line $x = -\frac{1}{m}$ and proved

**Lemma 2.2 ([10], Corollary 2.2).** If $m$ is odd, say $m = 2k + 1$, for $k \in \mathbb{Z}^+$ then there exist $k$ positive definite integral forms

$$F_j = (mj, -2j, 1), \quad 1 \leq j \leq k$$

of discriminant $\Delta(F_j) = -4j(m - j)$ whose base points $z(F_j)$ lie on the line $x = -\frac{1}{m}$. If $m$ is even, say $m = 2k$, for $k \in \mathbb{Z}^+$, then there exist $m - 1$ positive definite integral forms

$$F_j = (kj, -j, 1), \quad 1 \leq j \leq m - 1$$

of discriminant $\Delta(F_j) = -j(2m - j)$ whose base points $z(F_j)$ lie on the line $x = -\frac{1}{m}$.

In the present paper, we consider the same problem by considering the Hecke group $\overline{H}(\sqrt{2})$ instead of $\Gamma$. Now let

$$F = \mathbb{Z}[\sqrt{2}]F = \mathbb{Z}[\sqrt{2}](a, b, c)$$

be a positive definite form such that $a, b, c \in \mathbb{Z}[\sqrt{2}]$. Then we can give the following theorem.
Theorem 2.3. Let \( m \geq 3 \) be an integer and let \( 0 < 2D < m^2 \). Then there exists a positive definite form \( \mathbb{Z}[\sqrt{2}] F \) of discriminant \(-D\) whose base point lies on the line \( x = -\frac{\sqrt{2}}{m} \).

Proof. Let \( m \geq 3 \) be an integer, and let \( x = -\frac{\sqrt{2}}{m} \). Then from (2.2), we get the positive definite form

\[
F = \left( \frac{mD}{2(m + \sqrt{m^2 - 2D})}, \frac{-D\sqrt{2}}{m + \sqrt{m^2 - 2D}}, 1 \right)
\]

in \( \mathbb{Z}[\sqrt{2}] \). Notice that this form is not integral. To do it integral, we have two cases:

Case 1: Let \( m \) be odd, say \( m = 2k + 1 \) for some \( k \in \mathbb{Z}^+ \). Then \( F \) is an integral positive definite form, that is, \( F \in \mathbb{Z}[\sqrt{2}] \) if and only if \( 2D = m^2 - (2l - 1)^2 \) for \( \lvert l \rvert \leq k \). Indeed, let \( F \in \mathbb{Z}[\sqrt{2}] \). Since \( m \) is odd, \( D \) must be even and so \( \sqrt{m^2 - 2D} \) is odd, say \( \sqrt{m^2 - 2D} = \lvert 2l - 1 \rvert \) for some \( l \in \mathbb{Z}^+ \). Then clearly, \( 2D = m^2 - (2l - 1)^2 \). Since \( D \) must be positive, we get

\[
D > 0 \iff m^2 - (2l - 1)^2 > 0 \iff (m - 2l + 1)(m + 2l - 1) > 0.
\]

Note that \( m = 2k + 1 \). So we have to

\[
(k + l)(k - l + 1) > 0.
\]

In this case either \( l > 0 \) or \( l < 0 \). If \( l > 0 \), then \( k + l > 0 \) since \( k \) is positive. Therefore \( k - l + 1 \) must be positive and hence \( k > l - 1 \). Let \( l < 0 \). If \( k + l < 0 \), then we get \( k - l + 1 < 0 \) and so \( k + 1 < l \), which contradicts with \( k \in \mathbb{Z}^+ \). Therefore, \( k + 1 > 0 \). Thus, \( k - l + 1 > 0 \). Which means that \( -k \leq l \). From these two conditions, we obtain \( \lvert l \rvert \leq k \).

Conversely let \( \lvert l \rvert \leq k \) for \( 2D = m^2 - (2l - 1)^2 \). In this case since \( m - (2l - 1) \) is even, we get

\[
a = \frac{mD}{2(m + \sqrt{m^2 - 2D})} = \frac{(2k + 1)(k - l + 1)}{2} \in \mathbb{Z}
\]

since \( (k - l) \) is odd and \( \lvert l \rvert \leq k \). Similarly we easily deduce that

\[
b = \frac{-D}{m + \sqrt{m^2 - 2D}} = l - k - 1 \in \mathbb{Z}.
\]

Case 2: Let \( m \) be even, say \( m = 2k \) for some \( k \in \mathbb{Z}^+ \). Then \( F \) is an integral positive definite form, that is, \( F \in \mathbb{Z}[\sqrt{2}] \) if and only if \( 2D = m^2 - t^2 \) for \( \lvert t \rvert \leq m - 1 \), where \( t \neq 0 \) is an integer. Let \( F \in \mathbb{Z}[\sqrt{2}] \). Then \( 0 < 2D < m^2 \implies \sqrt{m^2 - 2D} = \lvert t \rvert \) for \( t \in \mathbb{Z} \). So \( 2D = m^2 - t^2 \). Since \( D \) is positive, \( m^2 - t^2 \) must be positive, that is,

\[
(m - t)(m + t) > 0.
\]
Therefore it can be easily seen $|t| \leq m - 1$.

Conversely, let $2D = m^2 - t^2$ for $|t| \leq m - 1$. Then since $m$ is even, we get

$$a = \frac{mD}{2(m + \sqrt{m^2 - 2D})} = \frac{k(m - t)}{2} \in \mathbb{Z}$$

and

$$b = \frac{-D}{m + \sqrt{m^2 - 2D}} = -\frac{(m - t)}{2} \in \mathbb{Z}.$$

This completes the proof. □

From the above theorem, we can give the following result.

**Corollary 2.4.** Let $F = \mathbb{Z}[\sqrt{2}](a, b, c)$ be the positive definite form obtained in Theorem 2.3.

1. **If** $m$ **is odd**, say $m = 2k + 1$ for $k \in \mathbb{Z}^+$, **then there are** $k$ **positive definite forms** $F_j = (mj, -2\sqrt{2}j, 1)$ **for** $1 \leq j \leq k$ **in** $\mathbb{Z}[\sqrt{2}]$ **of discriminant** $\Delta(F_j) = 4j(4j - m)$ **whose base points lying on the line** $x = -\frac{\sqrt{2}}{m}$.

2. **If** $m$ **is even**, say $m = 2k$ for $k \in \mathbb{Z}^+$, **then there are** $m - 1$ **positive definite forms** $F_j = (kj, -\sqrt{2}j, 1)$ **for** $1 \leq j \leq m - 1$ **in** $\mathbb{Z}[\sqrt{2}]$ **of discriminant** $\Delta(F_j) = 2j(j - 2k)$ **whose base points lying on the line** $x = -\frac{\sqrt{2}}{m}$.

**Example 2.5.** Let $m = 3$. Then there is one positive definite form $F = (3, -2\sqrt{2}, 1)$ of discriminant $-4$ whose base point lying on the line $x = -\frac{\sqrt{2}}{3}$, and for $m = 4$, there are three positive definite quadratic forms $F_1 = (2, -\sqrt{2}, 1)$, $F_2 = (4, -2\sqrt{2}, 1)$ and $F_3 = (6, -3\sqrt{2}, 1)$ whose base points lying on the line $x = -\frac{\sqrt{2}}{4}$.

A positive definite form $F = (a, b, c)$ is called reduced if $|b| \leq a \leq c$. If a positive definite form $F$ of discriminant $\Delta$ is not reduced, than it can be transferred into a reduced form

$$F_R = \begin{cases} (1, 0, \Delta) & \text{if } \Delta \equiv 0(\text{mod } 4) \\ (1, 1, \frac{1-\Delta}{4}) & \text{if } \Delta \equiv 1(\text{mod } 4) \end{cases}$$

of discriminant $\Delta$ by an element of $\Gamma$, that is, there exists a $g \in \Gamma$ such that $gF = F_R$. Similarly the form $F = \mathbb{Z}[\sqrt{2}](a, b, c)$ is called reduced if $|b| \leq a \leq c$ and if a positive definite form $F$ of discriminant $\Delta$ is not reduced, than it can be transferred into a reduced form

$$F_R = \begin{cases} (1, 0, -\Delta) & \text{if } \Delta \equiv 0(\text{mod } 4) \\ (1, -\sqrt{2}, \frac{2-\Delta}{4}) & \text{if } \Delta \equiv 2(\text{mod } 4) \end{cases}$$

of discriminant $\Delta$ by an element of $\mathcal{H}((\sqrt{2})$, that is, there exists a $g \in \mathcal{H}((\sqrt{2})$ such that $gF = F_R$. 
The positive definite forms obtained in Corollary 2.4 are not reduced. But we can transfer them into the reduced forms as follows:

**Theorem 2.6.** Let $F = \mathbb{Z}[\sqrt{2}](a, b, c) \in \overline{H}((\sqrt{2}))$ be a non-reduced positive definite form obtained in Corollary 2.4. Then there exists a $g \in \overline{H}((\sqrt{2}))$ such that $gF = F_R$, where $F_R$ is defined in (2.3).

**Proof.** Let $m$ be odd. Then positive definite forms are $F_j = (mj, -2\sqrt{2}j, 1)$ for $1 \leq j \leq k$. The discriminant of $F_j$ is

$$\Delta(F_j) = (-2\sqrt{2}j)^2 - 4mj \equiv 0 \pmod{4}.$$ 

So the reduced form is $F_{Rj} = (1, 0, mj - 2j^2)$ by (2.3). Let $g_j = \begin{bmatrix} r & s \\ t & u \end{bmatrix} \in \overline{H}((\sqrt{2}))$. Then from (1.1), we have the system of equations:

$$mjr^2 - 2\sqrt{2}jrs + s^2 = 1$$
$$2mjrt - 2\sqrt{2}jru - 2\sqrt{2}jts + 2su = 0$$
$$mj^2 - 2\sqrt{2}jtu + u^2 = mj - 2j^2.$$

This system of equations has a solution for $r = 0$, $s = 1$, $t = 1$ and $u = \sqrt{2}j$, that is, $g_jF_j = F_{Rj}$ for $g_j = \begin{bmatrix} 0 & 1 \\ 1 & \sqrt{2}j \end{bmatrix} \in \overline{H}((\sqrt{2}))$.

Let $m$ be even. Then positive definite forms are $F_j = (kj, -\sqrt{2}j, 1)$ for $1 \leq j \leq m - 1$. The discriminant of $F_j$ is $\Delta(F_j) = 2j^2 - 4kj$. Here we have two cases.

(i) If $j$ is even, say $j = 2h$, for some $h \in \mathbb{Z}^+$, then the discriminant of $F_j$ is

$$\Delta(F_j) = 4(2h^2 - kh) \equiv 0 \pmod{4}.$$ 

So the reduced form is $F_{Rj} = (1, 0, kj - \frac{j^2}{2})$. The system of equations

$$kj^2 - \sqrt{2}jrs + s^2 = 1$$
$$2kjrt - \sqrt{2}jru - \sqrt{2}jts + 2su = 0$$
$$kjt^2 - \sqrt{2}jtu + u^2 = kj - \frac{j^2}{2}$$

has a solution for $r = 0$, $s = 1$, $t = 1$ and $u = \frac{\sqrt{2}j}{2}$, that is, $g_jF_j = F_{Rj}$ for $g_j = \begin{bmatrix} 0 & 1 \\ 1 & \frac{\sqrt{2}j}{2} \end{bmatrix} \in \overline{H}((\sqrt{2}))$.

(ii) If $j$ is odd, say $j = 2e - 1$ for some $e \in \mathbb{Z}^+$, then $F_j = (k(2e - 1), -\sqrt{2}(2e - 1), 1)$ of discriminant

$$\Delta = 8e^2 - 8e + 2 - 8ke + 4k \equiv 2 \pmod{4}.$$
So the reduced form is $F_{R_j} = (1, -\sqrt{2}, -2e^2 + 2e + 2ke - k)$. The system of equations

\[
k(2e - 1)r^2 - \sqrt{2}(2e - 1)rs + s^2 = 1
\]
\[
2k(2e - 1)rt - \sqrt{2}(2e - 1)ru - \sqrt{2}(2e - 1)ts + 2su = -\sqrt{2}
\]
\[
k(2e - 1)t^2 - \sqrt{2}(2e - 1)tu + u^2 = -2e^2 + 2e + 2ke - k
\]

has a solution for $r = 0$, $s = -1$, $t = 1$ and $u = e\sqrt{2}$, that is, $g_jF_j = F_{R_j}$ for $g_j = \begin{bmatrix} 0 & -1 \\ 1 & (i+1)/2 \end{bmatrix} \in \mathcal{H}(\sqrt{2})$. □

REFERENCES


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