WILLMORE LAGRANGIAN SUBMANIFOLDS IN COMPLEX HYPERBOLIC SPACE CH^n

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In [15] and [16], the second author and Liu obtained some integral inequalities of Simons'type and rigidity theorems for n-dimensional compact Willmore Lagrangian submanifolds in the complex projective space CP^n and complex Euclidean space C^n. In this paper, we continue to study the interesting topic of Willmore Lagrangian submanifold in the complex hyperbolic space CH^n. Let M be an n-dimensional compact Willmore Lagrangian submanifold in the complex hyperbolic space CH^n. Denote by $\rho^2 = S - nH^2$ the non-negative function on M, where S and H are the square of the length of the second fundamental form and the mean curvature of M. If K, Q is the function which assigns to each point of M the infimum of the sectional curvature, Ricci curvature at the point, we prove some integral inequalities of Simons'type and rigidity theorems for n-dimensional compact Willmore Lagrangian submanifolds in CH^n in terms of ρ^2, K, Q, H.

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1. INTRODUCTION

Let N^{n+p} be an oriented smooth Riemannian manifold of dimension $n+p$ and let $\varphi : M \to N^{n+p}$ be an n-dimensional compact submanifold of N^{n+p}. Denote by $h^{\alpha}_{ij}, S, \vec{H}$ and H the second fundamental form, the square of the length of the second fundamental form, the mean curvature vector and the mean curvature of M. We define the following non-negative function on M by $\rho^2 = S - nH^2$, which vanishes exactly at the umbilical points of M. The Willmore functional is the non-negative functional (see [3, 14, 17]) $W(\varphi) = \int_M (S - nH^2)^{\frac{n}{2}} \mathrm{d}v$, where $\mathrm{d}v$ is the volume element of M. From [3, 14] and [17], we know that $W(\varphi)$ is an invariant under Moebius (or conformal) transformations of N^{n+p}. The Willmore submanifold was defined by Li [11] and Hu-Li [9, 10], that is, a submanifold is called a Willmore submanifold if it is an extremal submanifold to the Willmore functional. When $n = 2$, the functional essentially coincides with the well-known Willmore functional and its critical points are the Willmore surfaces. In [11] (also see [8, 14]), Li obtained an Euler-Lagrange equation of Willmore functional in terms of Euclidean geometry, which is very important to the study of rigidity and geometry of Willmore submanifold in N^{n+p}.

Let CH^n be the Complex hyperbolic space of constant holomorphic sectional curvature -4,J the standard complex structure on CH^n. Let $\varphi : M \to CH^n$ be an immersion of an n-dimensional manifold M in CH^n. φ is called Lagrangian if $\varphi^*\Omega \equiv 0$, this means that the complex structure J of CH^n carries each tangent space of M into its corresponding normal space. The typical examples of Lagrangian submanifolds of CH^n are the Whitney spheres:

Example 1.1 ([1, 2, 4]). Whitney spheres in CH^n. They are a one-parameter family of Lagrangian spheres in CH^n, given by $\Phi_\theta : S^n \to CH^n$, $\theta > 0$,

$$\Phi_\theta(x_1, \cdots, x_{n+1}) = \Pi \circ \Big(\frac{(x_1, \cdots, x_n)}{s_\theta + ic_\theta x_{n+1}}; \frac{s_\theta c_\theta(1 + x_{n+1}^2) - ix_{n+1}}{s_\theta^2 + c_\theta^2 x_{n+1}^2}\Big),$$

where $c_\theta = \cosh\theta, s_\theta = \sinh\theta, \Pi : H_1^{2n+1} \to CH^n$ is the Hopf projection. Φ_θ are also embedding except in double points.

If the mean curvature vector of the immersion $\varphi : M \to CH^n$ vanishes identically, φ is called minimal. Minimality means that the submanifolds is critical for compact supported variations of the volume functional. We notice that in recent years, due to their backgrounds in mathematical physics, special Lagrangian submanifolds have been extensively studied (see [1, 2, 10] and [12]). In [10] Hu-Li obtained the following:

THEOREM 1.2 ([10]). *A Lagrangian submanifold $\varphi : M \to CH^n$ is Willmore submanifold if and only if for $n+1 \le m^*, l^* \le 2n$*

$$\begin{aligned}
(1.1)\qquad &\rho^{n-2}\Big\{\sum_{i,j,k,l^*} h_{ij}^{l^*}h_{ik}^{l^*}h_{kj}^{m^*} - \sum_{i,j,l^*} H^{l^*}h_{ij}^{l^*}h_{ij}^{m^*} - \rho^2 H^{m^*} - 3(n-1)H^{m^*}\Big\}\\
&\quad + (n-1)\rho^{n-2}\Delta^\perp H^{m^*} + 2(n-1)\sum_i(\rho^{n-2})_i H_{,i}^{m^*}\\
&\quad + (n-1)H^{m^*}\Delta(\rho^{n-2}) - \Box^{m^*}(\rho^{n-2}) = 0,
\end{aligned}$$

where $\Delta(\rho^{n-2}) = \sum_i(\rho^{n-2})_{i,i}$, $\Box^{m^*}(\rho^{n-2}) = \sum_{i,j}(\rho^{n-2})_{i,j}(nH^{m^*}\delta_{ij} - h_{ij}^{m^*})$, $\Delta^\perp H^{m^*} = \sum_i H_{,ii}^{m^*}$, and $(\rho^{n-2})_{i,j}$ is the Hessian of ρ^{n-2} with respect to the induced metric $dx\cdot dx$, $H_{,i}^{m^*}$ and $H_{,ij}^{m^*}$ are the components of the first and second covariant derivative of the mean curvature vector field $\vec{H}$.

Remark 1.3. Fix the index m^* with $n+1 \leq m^* \leq 2n$, define $\Box^{m^*} : M \to R$ by

$$\Box^{m^*} f = \sum_{i,j} (nH^{m^*}\delta_{ij} - h_{ij}^{m^*}) f_{i,j}, \tag{1.2}$$

where f is any smooth function on M. We know that $\Box^{m^*}$ is a self-adjoint operator (see Cheng-Yau [6]). We can see that this operator naturally appears in the Willmore equation (1.1) and will play an important role in the proofs of our theorems.

From [13] and [10], the following results are well known:

Proposition 1.4 ([13]). *Every minimal Lagrangian surface $\varphi : M \to CH^2$ in a complex hyperbolic space CH^2 is Willmore Lagrangian surface.*

Proposition 1.5 ([10]). *Every minimal and Einstein Lagrangian submanifold $\varphi : M \to CH^n$ in a complex hyperbolic space CH^n is Willmore Lagrangian submanifold.*

From the above Propositions, we know that every minimal Lagrangian surface, every minimal and Einstein Lagrangian submanifold is Willmore. But we do not know whether every Willmore Lagrangian submanifold is minimal or not.

We notice that in [15] and [16], the second author and Liu obtained some integral inequalities of Simons'type and rigidity theorems for n-dimensional compact Willmore Lagrangian submanifolds in the complex projective space CP^n and complex Euclidean space C^n. In this paper, we shall prove some integral inequalities of Simons type and rigidity theorems for n-dimensional compact Willmore Lagrangian submanifolds in the Complex hyperbolic space CH^n in terms of the scalar curvatures, sectional curvatures, Ricci curvatures and mean curvatures of the submanifolds. More precisely, we obtain the following:

Theorem 1.6. *Let $\varphi : M \to CH^n$ be an $n(n \geq 2)$-dimensional compact Willmore Lagrangian submanifold in CH^n. Then there holds the following*

$$\int_M \rho^{n-2}\{(\frac{1}{n} - 2)\rho^4 - (n+1)\rho^2 + 4n(n-1)H^2\}\mathrm{d}v \leq 0. \tag{1.3}$$

In particular, if $(\frac{1}{n} - 2)\rho^4 - (n+1)\rho^2 + 4n(n-1)H^2 \geq 0$, then $\varphi : M \to CH^n$ is totally umbilical.

Theorem 1.7. *Let $\varphi : M \to CH^n$ be an $n(n \geq 2)$-dimensional compact Willmore Lagrangian submanifold in CH^n. Then there holds the following*

$$\int_M \rho^n\{(2n-1)(K - \frac{n-2}{\sqrt{n(n-1)}}H\rho - H^2) + (n-2)\}\mathrm{d}v \leq 0. \tag{1.4}$$

In particular, if $K \geq H^2 + \frac{n-2}{\sqrt{n(n-1)}}H\rho - \frac{n-2}{2n-1}$*, then* $\varphi: M \to CH^n$ *is totally umbilical, or* $\varphi: M \to CH^n$ *is a minimal Lagrangian submanifold in* CH^n *with parallel second fundamental form.*

THEOREM 1.8. *Let* $\varphi: M \to CH^n$ *be an* $n(n \geq 2)$*-dimensional compact Willmore Lagrangian submanifold in* CH^n*. Then there holds the following*

$$\int_M \rho^n\{(\frac{4}{n}-1)\rho^2 - 4(-\frac{3n-5}{4} + (n-2)H\rho + H^2 - Q)\}\mathrm{d}v \leq 0. \tag{1.5}$$

In particular, if $Q \geq \frac{n-4}{4n}\rho^2 + (n-2)H\rho + H^2 - \frac{3n-5}{4}$*, then* $\varphi: M \to CH^n$ *is totally umbilical.*

2. BASIC FORMULAS AND LEMMAS

In this paper, we will agree with the following convention on the range of indices: $A, B, C, \ldots = 1, \ldots, n, 1^*, \ldots, n^*$; $1^* = n+1, \ldots, n^* = 2n$; $i, j, k, \ldots = 1, \ldots, n$. Let $\varphi: M \to CH^n$ be an n-dimensional Lagrangian submanifold. We choose a local field of orthonormal frames $e_1, \ldots, e_n, e_{1^*} = Je_1, \ldots, e_{n^*} = Je_n$ in CH^n, such that, restricted to M, the vectors $e_1, \ldots, e_n$ are tangent to M, where J is the complex structure of CH^n. Let $\omega_1, \ldots, \omega_{2n}$ is the field of dual frames, θ_A, θ_{AB} be the restriction of ω_A, ω_{AB} to M. Then $\theta_{i^*} = 0$, taking its exterior derivative and making use of the structure equations of CH^n and the Cartan lemma we get

$$\theta_{ik^*} = \sum_j h_{ij}^{k^*}\theta_j, \quad h_{ij}^{k^*} = h_{ji}^{k^*}, \tag{2.1}$$

from which we can define the second fundamental form $II = \sum_{i,j,k^*} h_{ij}^{k^*}\omega_i \bigotimes \omega_j e_{k^*}$ and the mean curvature vector $\vec{H}$ of $\varphi: M \to CH^n$ as follows: $S = \sum_{i,j,k^*}(h_{ij}^{k^*})^2$, $\vec{H} = \sum_{k^*} H^{k^*}e_{k^*}$, $H^{k^*} = \frac{1}{n}\sum_i h_{ii}^{k^*}$, $H = |\vec{H}|$. Since $\varphi: M \to C^n$ is Lagrangian, we have for any i, j

$$\langle Je_i, e_j\rangle = 0, \quad \langle e_{i^*}, Je_j\rangle = \delta_{ij}. \tag{2.2}$$

Taking exterior derivative of (2.2), we get for any i, j, k

$$h_{ij}^{k^*} = h_{jk}^{i^*} = h_{ik}^{j^*}, \tag{2.3}$$

$$\theta_{i^*j^*} = \theta_{ij}. \tag{2.4}$$

Denote by R_{ijkl} the Riemannian curvature tensor of M, we get the Gauss equations

$$R_{ijkl} = -(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_{m^*}(h_{ik}^{m^*}h_{jl}^{m^*} - h_{il}^{m^*}h_{jk}^{m^*}), \tag{2.5}$$

$$R_{ik} = -(n-1)\delta_{ik} + n\sum_{m^*} H^{m^*} h^{m^*}_{ik} - \sum_{j,m^*} h^{m^*}_{ij} h^{m^*}_{jk}, \tag{2.6}$$

$$n(n-1)R = -n(n-1) + n^2H^2 - S, \tag{2.7}$$

where R is the normalized scalar curvature of M.

The first covariant derivative $\{h^{m^*}_{ijk}\}$ and the second covariant derivative $\{h^{m^*}_{ijkl}\}$ of $h^{m^*}_{ij}$ are defined by

$$\sum_k h^{m^*}_{ijk}\theta_k = dh^{m^*}_{ij} + \sum_k h^{m^*}_{kj}\theta_{ki} + \sum_k h^{m^*}_{ik}\theta_{kj} + \sum_{k^*} h^{k^*}_{ij}\theta_{k^*m^*}, \tag{2.8}$$

$$\sum_l h^{m^*}_{ijkl}\theta_l = dh^{m^*}_{ijk} + \sum_l h^{m^*}_{ljk}\theta_{li} + \sum_l h^{m^*}_{ilk}\theta_{lj} + \sum_l h^{m^*}_{ijl}\theta_{lk} + \sum_{l^*} h^{l^*}_{ijk}\theta_{\beta m^*}. \tag{2.9}$$

The Codazzi equations and the Ricci identities

$$h^{m^*}_{ijk} = h^{m^*}_{ikj}, \tag{2.10}$$

$$h^{m^*}_{ijkl} - h^{m^*}_{ijlk} = \sum_m h^{m^*}_{mj}R_{mikl} + \sum_m h^{m^*}_{im}R_{mjkl} + \sum_{k^*} h^{k^*}_{ij}R_{k^*m^*kl}. \tag{2.11}$$

The Ricci equations are

$$R_{i^*j^*kl} = -(\delta_{jl}\delta_{ik} - \delta_{jk}\delta_{il}) + \sum_m (h^{i^*}_{km}h^{j^*}_{lm} - h^{j^*}_{km}h^{i^*}_{lm}). \tag{2.12}$$

Define the first, second covariant derivatives and Laplacian of the mean curvature vector field $\vec{H} = \sum_{m^*} H^{m^*} e_{m^*}$ in the normal bundle $N(M)$ as follows

$$\sum_i H^{m^*}_{,i}\theta_i = dH^{m^*} + \sum_{k^*} H^{k^*}\theta_{k^*m^*}, \tag{2.13}$$

$$\sum_j H^{m^*}_{,ij}\theta_j = dH^{m^*}_{,i} + \sum_j H^{m^*}_{,j}\theta_{ji} + \sum_{k^*} H^{k^*}_{,i}\theta_{k^*m^*}, \tag{2.14}$$

$$\Delta^\perp H^{m^*} = \sum_i H^{m^*}_{,ii}, \quad H^{m^*} = \frac{1}{n}\sum_k h^{m^*}_{kk}. \tag{2.15}$$

Let f be a smooth function on M. The first, second covariant derivatives $f_i, f_{i,j}$ and Laplacian of f are defined by

$$df = \sum_i f_i\theta_i, \quad \sum_j f_{i,j}\theta_j = df_i + \sum_j f_j\theta_{ji}, \quad \Delta f = \sum_i f_{i,i}. \tag{2.16}$$

For the fix index $m^*(n+1 \le m^* \le 2n)$, we introduce an operator $\square^{m^*}$ due to Cheng-Yau [6] by

$$\square^{m^*} f = \sum_{i,j} (nH^{m^*}\delta_{ij} - h^{m^*}_{ij})f_{i,j}. \tag{2.17}$$

Since M is compact, the operator $\Box^{m^*}$ is self-adjoint (see[6]) if and only if

$$\int_M (\Box^{m^*} f) g \mathrm{d}v = \int_M f(\Box^{m^*} g)\mathrm{d}v, \tag{2.18}$$

where f and g are any smooth functions on M.

In general, for a matrix $A = (a_{ij})$ we denote by $N(A)$ the square of the norm of A, that is,

$$N(A) = \mathrm{trace}(A \cdot A^t) = \sum_{i,j} (a_{ij})^2.$$

Clearly, $N(A) = N(T^t AT)$ for any orthogonal matrix T.

We need the following Lemmas due to Chern-Do Carmo-Kobayashi [7], Li [12] and Cheng [5].

Lemma 2.1 ([7]). *Let A and B be symmetric $(n \times n)$-matrices. Then*

$$N(AB - BA) \leq 2N(A)N(B), \tag{2.19}$$

and the equality holds for nonzero matrices A and B if and only if A and B can be transformed simultaneously by on orthogonal matrix into multiples of $\tilde{A}$ and $\tilde{B}$ respectively, where

$$\tilde{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \qquad \tilde{B} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Moreover, if A_1, A_2 and A_3 are $(n \times n)$-symmetric matrices and if

$$N(A_\alpha A_\beta - A_\beta A_\alpha) = 2N(A_\alpha)N(A_\beta), 1 \leq \alpha, \beta \leq 3$$

then at least one of the matrices A_α must be zero.

Lemma 2.2 ([12]). *Let $\varphi : M \to CH^n$ be an n-dimensional $(n \geq 2)$ Lagrangian submanifold. Then we have*

$$|\nabla h|^2 \geq \frac{3n^2}{n+2} |\nabla^\perp \vec{H}|^2, \tag{2.20}$$

where $|\nabla h|^2 = \sum_{i,j,k,m^*} (h_{ijk}^{m^*})^2$, $|\nabla^\perp \vec{H}|^2 = \sum_{i,m^*} (H_{,i}^{m^*})^2$.

LEMMA 2.3 ([5]). *Let b_i for $i = 1, \cdots, n$ be real numbers satisfying $\sum_{i=1}^{n} b_i = 0$ and $\sum_{i=1}^{n} b_i^2 = B$. Then*

$$\sum_{i=1}^{n} b_i^4 - \frac{B^2}{n} \le \frac{(n-2)^2}{n(n-1)} B^2. \tag{2.21}$$

LEMMA 2.4 ([5]). *Let a_i and b_i for $i = 1, \cdots, n$ be real numbers satisfying $\sum_{i=1}^{n} a_i = 0$ and $\sum_{i=1}^{n} a_i^2 = a$. Then*

$$\left| \sum_{i=1}^{n} a_i b_i^2 \right| \le \sqrt{\sum_{i=1}^{n} b_i^4 - \frac{(\sum_{i=1}^{n} b_i^2)^2}{n}} \sqrt{a}. \tag{2.22}$$

3. INTEGRAL EQUALITIES AND PROPOSITIONS

In this section, we shall obtain some integral equalities of Willmore Lagrangian submanifolds $\varphi : M \to CH^n$. Defining tensors

$$\tilde{h}_{ij}^{m^*} = h_{ij}^{m^*} - H^{m^*} \delta_{ij}, \tag{3.1}$$

$$\tilde{\sigma}_{m^* l^*} = \sum_{i,j} \tilde{h}_{ij}^{m^*} \tilde{h}_{ij}^{l^*}, \qquad \sigma_{m^* l^*} = \sum_{i,j} h_{ij}^{m^*} h_{ij}^{l^*}, \tag{3.2}$$

we see that the $(n \times n)$-matrix $(\tilde{\sigma}_{\alpha\beta})$ is symmetric and can be assumed to be diagonized for a suitable choice of $e_{1^*}, \ldots, e_{n^*}$ and we set

$$\tilde{\sigma}_{m^* l^*} = \tilde{\sigma}_{m^*} \delta_{m^* l^*}. \tag{3.3}$$

By a direct calculation, we have

$$\sum_k \tilde{h}_{kk}^{m^*} = 0, \quad \tilde{\sigma}_{m^* l^*} = \sigma_{m^* l^*} - n H^{m^*} H^{l^*}, \quad \rho^2 = \sum_{m^*} \tilde{\sigma}_{m^*} = S - nH^2, \tag{3.4}$$

$$\sum_{i,j,k,m^*} h_{kj}^{l^*} h_{ij}^{m^*} h_{ik}^{m^*} = \sum_{i,j,k,m^*} \tilde{h}_{kj}^{l^*} \tilde{h}_{ij}^{m^*} \tilde{h}_{ik}^{m^*} + 2 \sum_{i,j,m^*} H^{m^*} \tilde{h}_{ij}^{m^*} \tilde{h}_{ij}^{l^*} + H^{l^*} \rho^2 + nH^2 H^{l^*}. \tag{3.5}$$

From (3.1), (3.4) and (3.5), the Euler-Lagrange equation (1.1) can be rewritten as follows:

PROPOSITION 3.1. *A Lagrangian submanifold $\varphi : M \to CH^n$ is Willmore submanifold if and only if for $n+1 \le m^*, l^* \le 2n$*

$$\begin{aligned}\Box^{m^*}(\rho^{n-2}) =&(n-1)\rho^{n-2}\Delta^{\perp}H^{m^*} + 2(n-1)\sum_i(\rho^{n-2})_i H^{m^*}_{,i} \\ &+ (n-1)H^{m^*}\Delta(\rho^{n-2}) - 3(n-1)\rho^{n-2}H^{m^*} \\ &+ \rho^{n-2}\Big(\sum_{l^*} H^{l^*}\tilde{\sigma}_{m^*l^*} + \sum_{i,j,k,l^*}\tilde{h}^{m^*}_{ij}\tilde{h}^{l^*}_{ik}\tilde{h}^{l^*}_{kj}\Big).\end{aligned} \tag{3.6}$$

Setting $f = nH^{m^*}$ in (2.17), we have

$$\begin{aligned}\Box^{m^*}(nH^{m^*}) =&\sum_{i,j}(nH^{m^*}\delta_{ij} - h^{m^*}_{ij})(nH^{m^*})_{i,j} \\ =&\sum_i(nH^{m^*})(nH^{m^*})_{i,i} - \sum_{i,j}h^{m^*}_{ij}(nH^{m^*})_{i,j}.\end{aligned} \tag{3.7}$$

We also have

$$\begin{aligned}\frac{1}{2}\Delta(nH)^2 =&\frac{1}{2}\Delta\sum_{m^*}(nH^{m^*})^2 = \frac{1}{2}\sum_{m^*}\Delta(nH^{m^*})^2 \\ =&\frac{1}{2}\sum_{m^*,i}[(nH^{m^*})^2]_{i,i} = \sum_{m^*,i}[(nH^{m^*})_{,i}]^2 + \sum_{m^*,i}(nH^{m^*})(nH^{m^*})_{i,i} \\ =&n^2|\nabla^{\perp}\vec{H}|^2 + \sum_{m^*,i}(nH^{m^*})(nH^{m^*})_{i,i}.\end{aligned} \tag{3.8}$$

Therefore, from (3.7) and (3.8), we get

$$\begin{aligned}\sum_{m^*}\Box^{m^*}(nH^{m^*}) =& \frac{1}{2}\Delta(nH)^2 - n^2|\nabla^{\perp}\vec{H}|^2 - \sum_{i,j,m^*}h^{m^*}_{ij}(nH^{m^*})_{i,j} \\ =&\frac{1}{2}\Delta(n(n-1)H^2 - \rho^2 + S) - n^2|\nabla^{\perp}\vec{H}|^2 - \sum_{i,j,m^*}h^{m^*}_{ij}(nH^{m^*})_{i,j} \\ =& \frac{1}{2}\Delta S + \frac{1}{2}n(n-1)\Delta H^2 - \frac{1}{2}\Delta\rho^2 - n^2|\nabla^{\perp}\vec{H}|^2 - \sum_{i,j,m^*}h^{m^*}_{ij}(nH^{m^*})_{i,j}.\end{aligned} \tag{3.9}$$

On the other hand, we have

$$\begin{aligned}\frac{1}{2}\Delta S =& \sum_{i,j,k,m^*}(h^{m^*}_{ijk})^2 + \sum_{i,j,m^*}h^{m^*}_{ij}\Delta h^{m^*}_{ij} \\ =&|\nabla h|^2 + \sum_{i,j,m^*}h^{m^*}_{ij}(nH^{m^*})_{i,j} + \sum_{m^*}\sum_{i,j,k,l}h^{m^*}_{ij}(h^{m^*}_{kl}R_{lijk} + h^{m^*}_{li}R_{lkjk}) \\ &+ \sum_{m^*,l^*}\sum_{i,j,k}h^{m^*}_{ij}h^{l^*}_{ki}R_{l^*m^*jk}.\end{aligned} \tag{3.10}$$

Putting (3.10) into (3.9), we have

$$\begin{aligned}(3.11)\quad \sum_{m^*}\Box^{m^*}(nH^{m^*}) =& |\nabla h|^2 - n^2|\nabla^{\perp}\vec{H}|^2 + \frac{1}{2}n(n-1)\Delta H^2 - \frac{1}{2}\Delta\rho^2\\ &+\sum_{m^*}\sum_{i,j,k,l} h_{ij}^{m^*}(h_{kl}^{m^*}R_{lijk}+h_{li}^{m^*}R_{lkjk}) + \sum_{m^*,l^*}\sum_{i,j,k} h_{ij}^{m^*}h_{ki}^{l^*}R_{l^*m^*jk}.\end{aligned}$$

Multiplying (3.11) by ρ^{n-2} and taking integration, using (2.18), we have

$$\begin{aligned}(3.12)\quad \sum_{m^*}\int_M(nH^{m^*})\Box^{m^*}(\rho^{n-2})\mathrm{d}v =& \int_M \rho^{n-2}(|\nabla h|^2 - n^2|\nabla^{\perp}\vec{H}|^2)\mathrm{d}v\\ &+ \frac{1}{2}n(n-1)\int_M \rho^{n-2}\Delta H^2\mathrm{d}v - \frac{1}{2}\int_M \rho^{n-2}\Delta\rho^2\mathrm{d}v\\ &+ \int_M \rho^{n-2}\sum_{m^*}\sum_{i,j,k,l} h_{ij}^{m^*}(h_{kl}^{m^*}R_{lijk} + h_{li}^{m^*}R_{lkjk})\mathrm{d}v\\ &+ \int_M \rho^{n-2}\sum_{m^*,l^*}\sum_{i,j,k} h_{ij}^{m^*}h_{ki}^{l^*}R_{l^*m^*jk}\mathrm{d}v.\end{aligned}$$

Taking the Euler-Lagrange equation (3.6) into (3.12) and making use of the following

$$\begin{aligned}\int_M \rho^{n-2}\sum_{m^*}H^{m^*}\triangle^{\perp}H^{m^*}\mathrm{d}v =&\frac{1}{2}\int_M \rho^{n-2}\sum_{m^*}\Delta^{\perp}(H^{m^*})^2\mathrm{d}v\\ &-\int_M \rho^{n-2}\sum_{i,m^*}(H_{,i}^{m^*})^2\mathrm{d}v\\ =&\frac{1}{2}\int_M \rho^{n-2}\Delta H^2\mathrm{d}v - \int_M \rho^{n-2}|\nabla\vec{H}|^2\mathrm{d}v,\end{aligned}$$

$$\begin{aligned}\int_M H^2\Delta(\rho^{n-2})\mathrm{d}v =& \int_M \sum_{m^*}(H^{m^*})^2\sum_i(\rho^{n-2})_{i,i}\mathrm{d}v\\ =&\sum_{m^*,i}\int_M (H^{m^*})^2(\rho^{n-2})_{i,i}\mathrm{d}v\\ =&-\sum_{m^*,i}\int_M (\rho^{n-2})_i((H^{m^*})^2)_{,i}\mathrm{d}v\\ =&-2\int_M\sum_{m^*}H^{m^*}\sum_i(\rho^{n-2})_iH_{,i}^{m^*}\mathrm{d}v,\end{aligned}$$

$$-\frac{1}{2}\int_M \rho^{n-2}\Delta\rho^2\mathrm{d}v = -\frac{1}{2}\sum_i\int_M \rho^{n-2}(\rho^2)_{i,i}\mathrm{d}v$$

$$=\frac{1}{2}\sum_i\int_M(\rho^2)_i(\rho^{n-2})_i\mathrm{d}v=(n-2)\int_M\rho^{n-2}|\nabla\rho|^2\mathrm{d}v,$$

we have the following:

Proposition 3.2. *For any n-dimensional compact Willmore Lagrangian submanifold $\varphi: M\to CH^n$, there holds the following integral equality*

$$\begin{aligned}\int_M\rho^{n-2}(|\nabla h|^2-n|\nabla^\perp\vec{H}|^2)\mathrm{d}v+(n-2)\int_M\rho^{n-2}|\nabla\rho|^2\mathrm{d}v\\ +3n(n-1)\int_M\rho^{n-2}H^2\mathrm{d}v\\ -\int_M\rho^{n-2}\sum_{m^*,l^*}nH^{m^*}(H^{l^*}\tilde{\sigma}_{m^*l^*}+\sum_{i,j,k}\tilde{h}^{m^*}_{ij}\tilde{h}^{l^*}_{ik}\tilde{h}^{l^*}_{kj})\mathrm{d}v\\ +\int_M\rho^{n-2}\sum_{m^*}\sum_{i,j,k,l}h^{m^*}_{ij}(h^{m^*}_{kl}R_{lijk}+h^{m^*}_{li}R_{lkjk})\mathrm{d}v\\ +\int_M\rho^{n-2}\sum_{m^*,l^*}\sum_{i,j,k}h^{m^*}_{ij}h^{l^*}_{ki}R_{l^*m^*jk}\mathrm{d}v=0.\end{aligned}\tag{3.13}$$

From (2.3), (2.12) and (3.1), we have

$$\begin{aligned}\sum_{m^*,l^*}\sum_{i,j,k}h^{m^*}_{ij}h^{l^*}_{ki}R_{l^*m^*jk}=&-\sum_{m^*,l^*}\sum_{i,j,k}h^{m^*}_{ij}h^{l^*}_{ki}(\delta_{lj}\delta_{mk}-\delta_{lk}\delta_{mj})\\ &+\sum_{m^*,l^*}\sum_{i,j,k,p}h^{m^*}_{ij}h^{l^*}_{ki}(h^{l^*}_{jp}h^{m^*}_{pk}-h^{l^*}_{kp}h^{m^*}_{pj})\\ =&-\rho^2+n(n-1)H^2-\frac{1}{2}\sum_{m^*,l^*,j,k}(\sum_p h^{l^*}_{jp}h^{m^*}_{pk}-\sum_p h^{m^*}_{jp}h^{l^*}_{pk})^2\\ =&-\rho^2+n(n-1)H^2-\frac{1}{2}\sum_{m^*,l^*,j,k}(\sum_p\tilde{h}^{l^*}_{jp}\tilde{h}^{m^*}_{pk}-\sum_p\tilde{h}^{m^*}_{jp}\tilde{h}^{l^*}_{pk})^2\\ =&-\rho^2+n(n-1)H^2-\frac{1}{2}\sum_{m^*,l^*}N(\tilde{A}_{m^*}\tilde{A}_{l^*}-\tilde{A}_{l^*}\tilde{A}_{m^*}),\end{aligned}\tag{3.14}$$

where $\tilde{A}_{m^*}:=(\tilde{h}^{m^*}_{ij})=(h^{m^*}_{ij}-H^{m^*}\delta_{ij})$. By use of (2.3), (2.5), (3.2), (3.4), (3.5) and (3.14) we have

$$\begin{aligned}\sum_{m^*}\sum_{i,j,k,l}h^{m^*}_{ij}(h^{m^*}_{kl}R_{lijk}+h^{m^*}_{li}R_{lkjk})=&-n\rho^2-\sum_{m^*,l^*}\sum_{i,j,k,l}h^{m^*}_{ij}h^{l^*}_{ij}h^{m^*}_{lk}h^{l^*}_{lk}\\ &+n\sum_{m^*,l^*}\sum_{i,j,k}H^{l^*}h^{l^*}_{kj}h^{m^*}_{ij}h^{m^*}_{ik}+\sum_{m^*,l^*}\sum_{i,j,k,l}h^{m^*}_{ij}h^{l^*}_{ki}(h^{l^*}_{jl}h^{m^*}_{lk}-h^{l^*}_{kl}h^{m^*}_{lj})\end{aligned}\tag{3.15}$$

$$
\begin{aligned}
=& - n\rho^2 - \sum_{m^*,l^*} \sigma^2_{m^*l^*} + n \sum_{m^*,l^*} \sum_{i,j,k} H^{l^*} \tilde{h}^{l^*}_{kj} \tilde{h}^{m^*}_{ij} \tilde{h}^{m^*}_{ik} \\
&+ 2n \sum_{m^*,l^*} \sum_{i,j} H^{m^*} H^{l^*} \tilde{h}^{m^*}_{ij} \tilde{h}^{l^*}_{ij} \\
&+ n \sum_{l^*} (H^{l^*})^2 \rho^2 + n^2 H^2 \sum_{l^*} (H^{l^*})^2 - \frac{1}{2} \sum_{m^*,l^*} N(\tilde{A}_{m^*}\tilde{A}_{l^*} - \tilde{A}_{l^*}\tilde{A}_{m^*}) \\
=& - n\rho^2 - \sum_{m^*,l^*} \tilde{\sigma}^2_{m^*l^*} + nH^2\rho^2 + n \sum_{m^*,l^*} \sum_{i,j,k} H^{l^*} \tilde{h}^{l^*}_{kj} \tilde{h}^{m^*}_{ij} \tilde{h}^{m^*}_{ik} \\
&- \frac{1}{2} \sum_{m^*,l^*} N(\tilde{A}_{m^*}\tilde{A}_{l^*} - \tilde{A}_{l^*}\tilde{A}_{m^*}).
\end{aligned}
$$

Putting (3.14) and (3.15) into (3.13), we have the following:

PROPOSITION 3.3. *For any n-dimensional compact Willmore Lagrangian submanifold $\varphi : M \to CH^n$, there holds the following integral equality*

(3.16)

$$
\begin{aligned}
&\int_M \rho^{n-2}(|\nabla h|^2 - n|\nabla^\perp \vec{H}|^2)\mathrm{d}v + (n-2)\int_M \rho^{n-2}|\nabla\rho|^2 \mathrm{d}v \\
&\quad + 4n(n-1)\int_M \rho^{n-2}H^2\mathrm{d}v + n\int_M \rho^{n-2}(H^2\rho^2 - \sum_{m^*,l^*} H^{m^*}H^{l^*}\tilde{\sigma}_{m^*l^*})\mathrm{d}v \\
&\quad - (n+1)\int_M \rho^n \mathrm{d}v - \int_M \rho^{n-2} \sum_{m^*,l^*} (N(\tilde{A}_{m^*}\tilde{A}_{l^*} - \tilde{A}_{l^*}\tilde{A}_{m^*}) + \tilde{\sigma}^2_{m^*l^*})\mathrm{d}v = 0.
\end{aligned}
$$

4. PROOFS OF THEOREMS

In this section, we shall give the proofs of Theorem 1.6–1.8.

Proof of Theorem 1.6. From Lemma 2.1, (3.2) and (3.3), we have

$$
\begin{aligned}
(4.1)\qquad & - \sum_{m^*,l^*} N(\tilde{A}_{m^*}\tilde{A}_{l^*} - \tilde{A}_{l^*}\tilde{A}_{m^*}) - \sum_{m^*,l^*} \tilde{\sigma}^2_{m^*l^*} \\
&\geq - \sum_{m^*} \tilde{\sigma}^2_{m^*} - 2 \sum_{m^* \neq l^*} \tilde{\sigma}_{m^*}\tilde{\sigma}_{l^*} = -2(\sum_{m^*} \tilde{\sigma}_{m^*})^2 + \sum_{m^*} \tilde{\sigma}^2_{m^*} \\
&\geq -2\rho^4 + \frac{1}{n}(\sum_{m^*} \tilde{\sigma}_{m^*})^2 = -(2 - \frac{1}{n})\rho^4,
\end{aligned}
$$

where, we used

$$
(4.2)\qquad \sum_{m^*} \tilde{\sigma}^2_{m^*} \geq \frac{1}{n}(\sum_{m^*} \tilde{\sigma}_{m^*})^2.
$$

We also have

$$\sum_{m^*,l^*} H^{m^*}H^{l^*}\tilde{\sigma}_{m^*l^*} = \sum_{m^*}(H^{m^*})^2\tilde{\sigma}_{m^*} \le \sum_{m^*}(H^{m^*})^2\sum_{l^*}\tilde{\sigma}_{l^*} = H^2\rho^2. \tag{4.3}$$

By making use of Lemma 2.2, (3.16), (4.1) and (4.3), we have

(4.4)

$$\begin{aligned}
0 \ge& \int_M \rho^{n-2}(|\nabla h|^2 - \frac{3n^2}{n+2}|\nabla^\perp \vec{H}|^2)\mathrm{d}v + \int_M \rho^{n-2}(\frac{3n^2}{n+2} - n)|\nabla^\perp \vec{H}|^2\mathrm{d}v\\
&+ 4n(n-1)\int_M \rho^{n-2}H^2\mathrm{d}v - (n+1)\int_M \rho^n\mathrm{d}v - \int_M \rho^{n-2}(2-\frac{1}{n})\rho^4\mathrm{d}v\\
\ge& \int_M \rho^{n-2}\{(\frac{1}{n}-2)\rho^4 - (n+1)\rho^2 + 4n(n-1)H^2\}\mathrm{d}v.
\end{aligned}$$

(i) If $n=2$, from $-\frac{3}{2}\rho^4 - 3\rho^2 + 8H^2 \ge 0$ and (4.4), we have $\frac{3}{2}\rho^4 + 3\rho^2 - 8H^2 = 0$ on M. If $\rho^2 = 0$ on M, then M is totally umbilical. If $\rho^2 \ne 0$ on M, from $\frac{3}{2}\rho^4 + 3\rho^2 - 8H^2 = 0$ we know that the equality in (4.4) holds. Therefore, we have

$$\begin{aligned}
N(\tilde{A}_3\tilde{A}_4 - \tilde{A}_4\tilde{A}_3) &= 2N(\tilde{A}_3)N(\tilde{A}_4),\\
2(\tilde{\sigma}_3^2 + \tilde{\sigma}_4^2) &= (\tilde{\sigma}_3 + \tilde{\sigma}_4)^2,
\end{aligned} \tag{4.5}$$

that is

$$\tilde{\sigma}_3 = \tilde{\sigma}_4. \tag{4.6}$$

We also have for $m^*, l^* = 3, 4$,

$$\sum_{m^*,l^*} H^{m^*}H^{l^*}\tilde{\sigma}_{m^*l^*} = H^2\rho^2. \tag{4.7}$$

From Lemma 2.1, we know that at most two of $\tilde{A}_{m^*} = (\tilde{h}_{ij}^{m^*}), m^* = 3, 4$, are different from zero. If all of $\tilde{A}_{m^*} = (\tilde{h}_{ij}^{m^*})$ are zero, which is a contradiction with M, then it is not totally umbilical. If only one of them, say $\tilde{A}_{m^*}$, is different from zero, it is a contradiction with (4.6). Therefore, we may assume that

$$\tilde{A}_3 = \lambda\tilde{A}, \quad \tilde{A}_4 = \mu\tilde{B}, \quad \lambda, \mu \ne 0,$$

where $\tilde{A}$ and $\tilde{B}$ are defined in Lemma 2.1.

From (4.7), we have

$$\lambda^2(H^3)^2 + \mu^2(H^4)^2 = (\lambda^2 + \mu^2)((H^3)^2 + (H^4)^2).$$

Since $\lambda, \mu \ne 0$, we infer that $H^3 = H^4 = 0$, that is, $\vec{H} = 0$, *i.e.*, $\varphi : M \to CH^2$ is a minimal Lagrangian submanifold in CH^2. Therefore we know that $\frac{3}{2}\rho^4 + 3\rho^2 = 0$ on M and this is a contradiction with $\rho^2 \ne 0$.

(ii) If $n>2$, from $(\frac{1}{n}-2)\rho^4-(n+1)\rho^2+4n(n-1)H^2\geq 0$ and (4.4), we have $\rho=0$ on M, that is, M is totally umbilical, or $(\frac{1}{n}-2)\rho^4-(n+1)\rho^2+4n(n-1)H^2=0$. In the latter case, if $\rho^2=0$ on M, we have M is totally umbilical. If $\rho^2\neq 0$ on M, we know that the equality in (4.4) holds. Therefore, we have

$$\begin{aligned}&\nabla^{\perp}\vec{H}=0,\quad \nabla h=0,\\&N(\tilde{A}_{m^*}\tilde{A}_{l^*}-\tilde{A}_{l^*}\tilde{A}_{m^*})=2N(\tilde{A}_{m^*})N(\tilde{A}_{l^*}),\quad m^*\neq l^*,\\&n\sum_{m^*}\tilde{\sigma}_{m^*}^2=(\sum_{m^*}\tilde{\sigma}_{m^*})^2,\end{aligned}\tag{4.8}$$

that is

$$\tilde{\sigma}_{n+1}=\cdots=\tilde{\sigma}_{2n}.\tag{4.9}$$

We also have

$$\sum_{m^*,l^*}H^{m^*}H^{l^*}\tilde{\sigma}_{m^*l^*}=H^2\rho^2.\tag{4.10}$$

From Lemma 2.1, we know that at most two of $\tilde{A}_{m^*}=(\tilde{h}_{ij}^{m^*}), m^*=n+1,\cdots,2n$, are different from zero. If all of $\tilde{A}_{m^*}=(\tilde{h}_{ij}^{m^*})$ are zero, which is a contradiction with M, then it is not totally umbilical. If only one of them, say $\tilde{A}_{m^*}$, is different from zero, it is a contradiction with (4.9). Therefore, we may assume that

$$\begin{aligned}\tilde{A}_{n+1}=\lambda\tilde{A},\quad \tilde{A}_{n+2}=\mu\tilde{B},\quad \lambda,\mu\neq 0,\\ \tilde{A}_{m^*}=0,\qquad m^*\geq n+3,\end{aligned}$$

where $\tilde{A}$ and $\tilde{B}$ are defined in Lemma 2.1.

From (4.10), we have

$$\lambda^2(H^{n+1})^2+\mu^2(H^{n+2})^2=(\lambda^2+\mu^2)\sum_{m^*}(H^{m^*})^2.$$

Since $\lambda,\mu\neq 0$, we infer that $H^{m^*}=0, n+1\leq m^*\leq 2n$, that is, $\vec{H}=0$, *i.e.*, $\varphi: M\to CH^n$ is a minimal Lagrangian submanifold in CH^n and $(2-\frac{1}{n})\rho^4+(n+1)\rho^2=0$ on M, a contradiction with $\rho^2\neq 0$. This completes the proof of Theorem 1.6. □

Proof of Theorem 1.7. From (3.13), (3.14), (3.15) and (3.16), we know that for any real number a, the following integral equality holds

$$\begin{aligned}&\int_M\rho^{n-2}(|\nabla h|^2-n|\nabla^{\perp}\vec{H}|^2)\mathrm{d}v+(n-2)\int_M\rho^{n-2}|\nabla\rho|^2\mathrm{d}v\\&\quad+4n(n-1)\int_M\rho^{n-2}H^2\mathrm{d}v\end{aligned}\tag{4.11}$$

$$+ n\int_M \rho^{n-2}(H^2\rho^2 - \sum_{m^*,l^*} H^{m^*}H^{l^*}\tilde{\sigma}_{m^*l^*})\mathrm{d}v - (a+1)n\int_M H^2\rho^n \mathrm{d}v$$

$$+ (1+a)\int_M \rho^{n-2}\sum_{m^*}\sum_{i,j,k,l} h_{ij}^{m^*}(h_{kl}^{m^*}R_{lijk} + h_{li}^{m^*}R_{lkjk})\mathrm{d}v$$

$$- (1+a)n\int_M \rho^{n-2}\sum_{m^*,l^*}\sum_{i,j,k} H^{m^*}\tilde{h}_{ij}^{m^*}\tilde{h}_{ik}^{l^*}\tilde{h}_{kj}^{l^*}\mathrm{d}v + (an-1)\int_M \rho^n \mathrm{d}v$$

$$+ a\int_M \rho^{n-2}\sum_{m^*,l^*}\tilde{\sigma}_{m^*l^*}^2 \mathrm{d}v - \frac{1-a}{2}\int_M \rho^{n-2}\sum_{m^*,l^*} N(\tilde{A}_{m^*}\tilde{A}_{l^*} - \tilde{A}_{l^*}\tilde{A}_{m^*})\mathrm{d}v = 0.$$

For a fixed $m^*, n+1 \le m^* \le 2n$, we can take a local orthonormal frame field $\{e_1, \cdots, e_n\}$ such that $h_{ij}^{m^*} = \lambda_i^{m^*}\delta_{ij}$, then, $\tilde{h}_{ij}^{m^*} = \mu_i^{m^*}\delta_{ij}$ with $\mu_i^{m^*} = \lambda_i^{m^*} - H^{m^*}$, $\sum_i \mu_i^{m^*} = 0$. Thus

(4.12)
$$\begin{aligned}\sum_{m^*,i,j,k,l} h_{ij}^{m^*}(h_{kl}^{m^*}R_{lijk} + h_{li}^{m^*}R_{lkjk}) =& \frac{1}{2}\sum_{m^*,i,j}(\lambda_i^{m^*} - \lambda_j^{m^*})^2 R_{ijij} \\ =& \frac{1}{2}\sum_{m^*,i,j}(\mu_i^{m^*} - \mu_j^{m^*})^2 R_{ijij} \ge nK\rho^2,\end{aligned}$$

where K is the function which assigns to each point of M the infimum of the sectional curvature at that point and the equality in (4.12) holds if and only if $R_{ijij} = K$ for any $i \ne j$.

Let $\sum_i (\tilde{h}_{ii}^{l^*})^2 = \tau_{l^*}$. Then $\tau_{l^*} \le \sum_{i,j}(\tilde{h}_{ij}^{l^*})^2 = \tilde{\sigma}_{l^*}$. Since $\sum_i \tilde{h}_{ii}^{l^*} = 0$, $\sum_i \mu_i^{m^*} = 0$ and $\sum_i (\mu_i^{m^*})^2 = \tilde{\sigma}_{m^*}$. From Lemma 2.3 and Lemma 2.4, we have

$$\begin{aligned}(4.13)\qquad \sum_{m^*,l^*}\sum_{i,j,k} H^{m^*}\tilde{h}_{ij}^{m^*}\tilde{h}_{kj}^{l^*}\tilde{h}_{ik}^{l^*} =& \sum_{l^*,m^*}\sum_{i,j,k} H^{l^*}\tilde{h}_{ij}^{l^*}\tilde{h}_{kj}^{m^*}\tilde{h}_{ik}^{m^*} \\ =& \sum_{m^*,l^*} H^{l^*}\sum_i \tilde{h}_{ii}^{l^*}(\mu_i^{m^*})^2 \le \frac{n-2}{\sqrt{n(n-1)}}\sum_{m^*,l^*}|H^{l^*}|\tilde{\sigma}_{m^*}\sqrt{\tau_{l^*}} \\ \le& \frac{n-2}{\sqrt{n(n-1)}}\sum_{m^*}\tilde{\sigma}_{m^*}\sum_{l^*}|H^{l^*}|\sqrt{\tilde{\sigma}_{l^*}} \\ \le& \frac{n-2}{\sqrt{n(n-1)}}\rho^2\sqrt{\sum_{l^*}(H^{l^*})^2\sum_{l^*}\tilde{\sigma}_{l^*}} = \frac{n-2}{\sqrt{n(n-1)}}H\rho^3.\end{aligned}$$

From (3.3), we get

$$(4.14)\qquad \sum_{m^*,l^*}\tilde{\sigma}_{m^*l^*}^2 = \sum_{m^*}\tilde{\sigma}_{m^*}^2 \ge \frac{1}{n}(\sum_{m^*}\tilde{\sigma}_{m^*})^2 = \frac{1}{n}\rho^4.$$

From Lemma 2.1, (3.2) and (3.3), we have

$$\begin{aligned}
(4.15)\quad \sum_{m^*,l^*} N(\tilde{A}_{m^*}\tilde{A}_{l^*} - \tilde{A}_{l^*}\tilde{A}_{m^*}) &\le 2\sum_{m^*\neq l^*} \tilde{\sigma}_{m^*}\tilde{\sigma}_{l^*} = 2(\sum_{m^*}\tilde{\sigma}_{m^*})^2 - 2\sum_{m^*}\tilde{\sigma}_{m^*}^2 \\
&\le 2\rho^4 - 2\frac{1}{n}(\sum_{m^*}\tilde{\sigma}_{m^*})^2 = 2\frac{n-1}{n}\rho^4.
\end{aligned}$$

Thus, from (4.3), (4.11), Lemma 2.2, (4.12)–(4.15), we obtain for $0 \le a \le 1$

$$\begin{aligned}
0 \ge& \int_M \rho^{n-2}(|\nabla h|^2 - \frac{3n^2}{n+2}|\nabla^\perp \vec{H}|^2)\mathrm{d}v + \int_M \rho^{n-2}(\frac{3n^2}{n+2} - n)|\nabla^\perp \vec{H}|^2\mathrm{d}v \\
(4.16)\quad & + (n-2)\int_M \rho^{n-2}|\nabla\rho|^2\mathrm{d}v + 4n(n-1)\int_M \rho^{n-2}H^2\mathrm{d}v \\
& + n\int_M \rho^{n-2}(H^2\rho^2 - \sum_{m^*,l^*} H^{m^*}H^{l^*}\tilde{\sigma}_{m^*l^*})\mathrm{d}v \\
& - (1+a)n\int_M H^2\rho^n\mathrm{d}v + (1+a)\int_M \rho^{n-2}nK\rho^2\mathrm{d}v \\
& - (1+a)n\int_M \rho^{n-2}\frac{n-2}{\sqrt{n(n-1)}}H\rho^3\mathrm{d}v + (an-1)\int_M \rho^n\mathrm{d}v \\
& + a\int_M \rho^{n-2}\frac{1}{n}\rho^4\mathrm{d}v - (1-a)\int_M \rho^{n-2}\frac{n-1}{n}\rho^4\mathrm{d}v \\
\ge& (1+a)n\int_M \rho^n(K - \frac{n-2}{\sqrt{n(n-1)}}H\rho - H^2)\mathrm{d}v \\
& + (an-1)\int_M \rho^n\mathrm{d}v + [\frac{a}{n} - (1-a)\frac{n-1}{n}]\int_M \rho^{n+2}\mathrm{d}v.
\end{aligned}$$

Putting $a = \frac{n-1}{n}$, we have

$$(4.17)\qquad 0 \ge \int_M \rho^n\{(2n-1)(K - \frac{n-2}{\sqrt{n(n-1)}}H\rho - H^2) + (n-2)\}\mathrm{d}v.$$

From $K \ge H^2 + \frac{n-2}{\sqrt{n(n-1)}}H\rho - \frac{n-2}{2n-1}$ and (4.17), we have $\rho = 0$, that is, M is totally umbilical, or $K = H^2 + \frac{n-2}{\sqrt{n(n-1)}}H\rho - \frac{n-2}{2n-1}$. In the latter case, if $\rho^2 = 0$ on M, we have M is totally umbilical. If $\rho^2 \neq 0$ on M, then the equality in (4.17) holds. Therefore, we know that the equalities in (4.16) hold, this implies that

$$(4.18)\qquad \nabla^\perp \vec{H} = 0, \quad \nabla h = 0, \quad H = 0.$$

Therefore we see that $\varphi : M \to CH^n$ is a minimal Lagrangian submanifold in CH^n with parallel second fundamental form. This completes the proof of Theorem 1.7. $\square$

In order to prove Theorem 1.8, we first prove the following:

LEMMA 4.1. *For any n-dimensional Lagrangian submanifold in CH^n, there holds the following*

$$\sum_{m^*,l^*} N(\tilde{A}_{m^*}\tilde{A}_{l^*}-\tilde{A}_{l^*}\tilde{A}_{m^*}) \leq 4\{-(n-1)+(n-2)H\rho+H^2-Q\}\rho^2-\frac{4}{n}\rho^4, \tag{4.19}$$

where Q is the function which assigns to each point of M the infimum of the Ricci curvature at that point.

Proof. From Gauss equation (2.6) and (3.1), we have

$$R_{ik} = -(n-1)\delta_{ik} + (n-2)\sum_{m^*} H^{m^*}\tilde{h}_{ik}^{m^*} + (n-1)H^2\delta_{ik} - \sum_{m^*,j}\tilde{h}_{ij}^{m^*}\tilde{h}_{jk}^{m^*}.$$

Thus, we get

$$R_{ii} = -(n-1) + (n-2)\sum_{m^*} H^{m^*}h_{ii}^{m^*} + H^2 - \sum_{m^*,j}(\tilde{h}_{ij}^{m^*})^2. \tag{4.20}$$

By Cauchy-Schwarz inequality, we have

$$\sum_{m^*} H^{m^*}h_{ii}^{m^*} \leq \sqrt{\sum_{m^*}(H^{m^*})^2}\sqrt{\sum_{m^*}(h_{ii}^{m^*})^2} \leq H\rho. \tag{4.21}$$

From (4.20) and (4.21), we infer that

$$Q \leq -(n-1) + (n-2)H\rho + H^2 - \sum_{m^*,j}(\tilde{h}_{ij}^{m^*})^2. \tag{4.22}$$

Therefore, we have

$$\sum_{m^*\neq l^*,i}(\tilde{h}_{il}^{m^*})^2 \leq -(n-1) + (n-2)H\rho + H^2 - Q - (\tilde{h}_{il}^{m^*})^2. \tag{4.23}$$

From (4.23) and $\tilde{h}_{ij}^{m^*} = \mu_i^{m^*}\delta_{ij}$, it is easy to see

(4.24)

$$\begin{aligned}
&\sum_{l^*} N(\tilde{A}_{m^*}\tilde{A}_{l^*}-\tilde{A}_{l^*}\tilde{A}_{m^*}) \\
&\quad= \sum_{l^*\neq m^*,i,l}(\tilde{h}_{il}^{l^*})^2(\mu_i^{m^*}-\mu_l^{m^*})^2 \leq 4\sum_{l^*\neq m^*,i,l}(\tilde{h}_{il}^{l^*})^2(\mu_l^{m^*})^2 \\
&\quad\leq 4\sum_{l}\{-(n-1)+(n-2)H\rho+H^2-Q-(\mu_l^{m^*})^2\}(\mu_l^{m^*})^2 \\
&\quad= 4\{-(n-1)+(n-2)H\rho+H^2-Q\}\sum_{l}(\mu_l^{m^*})^2 - 4\sum_{l}(\mu_l^{m^*})^4
\end{aligned}$$

$$\le 4\{-(n-1)+(n-2)H\rho+H^2-Q\}\sum_l(\mu_l^{m^*})^2-\frac{4}{n}(\sum_l(\mu_l^{m^*})^2)^2.$$

Therefore, we know that (4.19) holds. This completes the proof of Lemma 4.1. □

Proof of Theorem 1.8. From (3.16), Lemma 2.2, (3.3), (4.3) and Lemma 4.1, we have

(4.25)

$$\begin{aligned}
0 \ge\, & 4n(n-1)\int_M \rho^{n-2}H^2\mathrm{d}v-(n+1)\int_M\rho^n\mathrm{d}v \\
& -\int_M \rho^{n-2}\{4(-(n-1)+(n-2)H\rho+H^2-Q)\rho^2-\frac{4}{n}\rho^4\}\mathrm{d}v-\int_M\rho^{n-2}\rho^4\mathrm{d}v \\
& =\int_M\rho^n\{(\frac{4}{n}-1)\rho^2-4(-\frac{3n-5}{4}+(n-2)H\rho+H^2-Q)\}\mathrm{d}v,
\end{aligned}$$

where we used

$$\sum_{m^*,l^*}\tilde{\sigma}_{m^*l^*}^2=\sum_{m^*}\tilde{\sigma}_{m^*}^2\le(\sum_{m^*}\tilde{\sigma}_{m^*})^2=\rho^4. \tag{4.26}$$

From $Q\ge\frac{n-4}{4n}\rho^2+(n-2)H\rho+H^2-\frac{3n-5}{4}$ and (4.25), we conclude $\rho=0$, that is M is totally umbilical, or

$$Q=\frac{n-4}{4n}\rho^2+(n-2)H\rho+H^2-\frac{3n-5}{4}.$$

In the latter case, if $\rho^2=0$, then M is totally umbilical; if $\rho^2\neq 0$, we have the equalities in (4.25) and (4.26) hold. From $\sum\limits_{m^*}\tilde{\sigma}_{m^*}^2=(\sum\limits_{m^*}\tilde{\sigma}_{m^*})^2$, we have $\sum\limits_{m^*\neq l^*}\tilde{\sigma}_{m^*}\tilde{\sigma}_{l^*}=0$, this implies that $(n-1)$ of the $\tilde{\sigma}_{m^*}$ must be zero. Since $\rho^2=\sum\limits_{m^*,i,j}(\tilde{h}_{ij}^{m^*})^2\neq 0$ and $\tilde{\sigma}_{m^*}=\sum\limits_{i,j}(\tilde{h}_{ij}^{m^*})^2$, we infer that $(n-1)$ of the $\tilde{A}_{m^*}=(\tilde{h}_{ij}^{m^*})$ must be zero so that $n=1$, a contradiction with the assumption $n\ge 2$. This completes the proof of Theorem 1.8. □

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