THE PERIOD, RANK AND ORDER OF THE SEQUENCE OF BALANCING NUMBERS MODULO m

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Communicated by Alexandru Zaharescu

The goal of this article is to investigate some of the arithmetical properties of the sequence of balancing numbers using the elementary matrix algebra. The purpose is to establish some important relations between period, rank and order of the sequence of balancing numbers such as the period is equal to the product of the rank and the order.

AMS 2010 Subject Classification: 11B37, 11B39.

Key words: balancing numbers, balancers, balancing matrix, periodicity of balancing numbers.

1. INTRODUCTION

The balancing numbers and the balancers were introduced by Behera *et al.* (1999), which were obtained from a simple diophantine equation [1]. The balancing numbers are defined recursively by $B_{n+1} = 6B_n - B_{n-1}$ with initials $B_0 = 0$ and $B_1 = 1$, where B_n denotes the n^{th} balancing number, for any natural number n. There is another way to represent balancing numbers through matrices. In [7], Ray introduced balancing Q_B matrix as a second order matrix whose entries are the first three balancing numbers 0, 1 and 6 and defined by $Q_B = \begin{bmatrix} 6 & -1 \\ 1 & 0 \end{bmatrix}$. The n^{th} power of the Q_B matrix is given by $Q_B^n = \begin{bmatrix} B_{n+1} & -B_n \\ B_n & -B_{n-1} \end{bmatrix}$ [7]. He has also shown that the sequence of balancing matrices satisfies the same recurrence relation as that of balancing numbers, that is, $Q_B^{n+1} = 6Q_B^n - Q_B^{n-1}$ with initials $Q_B^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $Q_B^1 = \begin{bmatrix} 6 & -1 \\ 1 & 0 \end{bmatrix}$. Many interesting and important properties of balancing numbers, balancing matrices and their related sequences were established by several authors. The interested readers may refer to [2–10] for a detailed review.

In [11], Wall (1960) studied the periodicity of Fibonacci numbers modulo arbitrary natural numbers and established some interesting results about the periodicity of Fibonacci numbers modulo primes [11]. In [6], Panda *et al.*

(2014) studied the periodicity of balancing numbers modulo and any integer which help to explore divisibility properties of these numbers. Some interesting results about the periodicity of balancing numbers modulo primes and modulo terms of certain sequences were also studied in [6]. They defined that a natural number n is called the period of the balancing sequence modulo m, denoted by $\pi(m)$, if $(B_n, B_{n+1}) \equiv (0, 1)\ (mod\ m)$, where m is any positive integer greater than or equal to 2.

In this article, the authors define the rank $r(m)$ and the order $o(m)$ of the sequence of balancing numbers modulo m and prove that the period $\pi(m)$ is equal to the product of the rank $r(m)$ and the order $o(m)$. Also, the authors prove similar divisibility results for the rank as for the period and show that the period is either equal to the rank or is twice the rank and characterize when each happens.

The rank and the order of the sequence of balancing numbers are defined as follows:

Definition 1.1. The rank of the sequence of balancing numbers modulo m, denoted by $r(m)$, is the smallest positive integer k such that $m|B_k$.

By virtue of Definition 1.1, $mn|B_{r(mn)}$ and therefore $m|B_{r(mn)}$, yielding $r(m)|r(mn)$. From the relationship $(B_{r(m)}, B_{r(m)+1}) \equiv t(0, 1)\ (mod\ m)$, it is observed that for integers a and b, the terms of balancing numbers starting with $0, t, at, (a^2 + b)t, \ldots$ are exactly the initial terms of balancing numbers multiplied by a factor of some integer t. Further, the multiplier of the sequence of balancing numbers modulo m, denoted by $t(m)$, is the least residue of $B_{r(m)+1}$ modulo m.

Definition 1.2. The order of the sequence of balancing numbers modulo m, denoted by $o(m)$ is defined by the order of the multiplier $t(m)$ modulo m.

Keeping in mind that the determinant of the Q_B matrix is 1, we now present the balancing matrix Q_B in a different way by interchanging the diagonal elements. That is, $Q_B = \begin{bmatrix} 0 & 1 \\ -1 & 6 \end{bmatrix}$. Then, for every positive integer n, Q_B^n will be $Q_B^n = \begin{bmatrix} -B_{n-1} & B_n \\ -B_n & B_{n+1} \end{bmatrix}$. It is important to note that, the matrix Q_B^n is so formed that its determinant is invariant without loss of generality to the Cassini formula for balancing numbers, that is, $B_n^2 - B_{n+1}B_{n-1} = 1$ [5].

2. RELATIONS AMONG PERIOD, RANK AND ORDER OF THE SEQUENCE OF BALANCING NUMBERS MODULO m

In this section, we establish some relations between the period, rank and order of the sequence of balancing numbers modulo a positive integer

m. We use balancing Q_B matrix to prove most of the results concerning these relations.

It can be easily shown that, $(B_n, B_{n+1}) \equiv (0,1)\ (mod\ m)$ if and only if $Q_B^n \equiv I\ (mod\ m)$. Therefore, the study of the period of the sequence of balancing numbers modulo m is equivalent to the study of the period of the sequence $I, Q_B, Q_B^2, ..., Q_B^n$, reduced modulo m. Also the identity $Q_B^{l+n} \equiv Q_B^l \equiv I\ (mod\ m)$ is valid, where $l, n \in Z$ and $l + n > l \geq 0$, as there are only a finite number of matrices in the sequence of balancing numbers. Since the determinant of $Q_B = 1$, there exist a least positive integer n such that $Q_B^n \equiv I\ (mod\ m)$ if and only if $\pi(m)|n$. Similarly, $(B_n, B_{n+1}) \equiv t(0,1)\ (mod\ m)$ if and only if the exponents n for which Q_B^n is congruent to a scalar multiple of I modulo m. It follows that, $Q_B^n \equiv tI\ (mod\ m)$ if and only if $r(m)|n$.

The above discussion proves the following result.

THEOREM 2.1. *$m|B_n$ if and only if $r(m)|n$ and $m|B_n$, $m|(B_{n+1} - 1)$ if and only if $\pi(m)|n$.*

The following result shows an important relation among the period, rank and order of the sequence of balancing numbers.

THEOREM 2.2. *The period of the sequence of balancing numbers is equal to the product of the rank of apparition and the order of the sequence of balancing numbers.*

Proof. The zeros of the sequence of balancing numbers modulo m are evenly spaced because of the identities $B_{k+l} = B_k B_{l+1} - B_{k-1} B_l$ and $B_{k-l} = B_{k-1} B_l - B_k B_{l-1}$, for all integers k, l with the fact that, if B_k and B_l are congruent to 0 modulo m, then so are B_{k+l} and B_{k-l}. Which implies that $r(m)k_1 = \pi(m)$ for some positive integer k_1, where $r(m)$ is the index of the first zero. Representing through matrices, it is observed that, $Q_B^{r(m)} \equiv t(m)I\ (mod\ m)$, where $t(m)$ is the least residue of $B_{r(m)+1}$ modulo m. Then,

$$Q_B^{r(m)o(m)} \equiv t(m)^{o(m)} I \equiv I\ (mod\ m),$$

yielding $\pi(m)|r(m)o(m)$. Conversely, it is observed that $r(m)|\pi(m)$, by similar idea that $o(m)|\pi(m)/r(m)$ and the result follows. □

In [6], Panda *et al.* has shown the following result about the period of the sequence of balancing numbers modulo m.

THEOREM 2.3. *If $\pi(p^2) \neq \pi(p)$, then $\pi(p^e) = p^{e-1}\pi(p)$. Further, if l is the largest integer such that $\pi(p^l) = \pi(p)$ and $e > l$, then $\pi(p^e) = p^{e-l}\pi(p)$.*

Similar property satisfy for the rank of apparition of the sequence of balancing numbers. We prove this result by using elementary matrix algebra.

THEOREM 2.4. *If $r(p^2) \neq r(p)$, then $r(p^e) = p^{e-1}r(p)$. Further, if l is the largest integer such that $r(p^l) = r(p)$ and $e > l$, then $r(p^e) = p^{e-l}r(p)$.*

Proof. Clearly, $Q_B^{r(p^e)} \equiv tI \ (mod\ p^e)$, where p be any prime and e be any positive integer. Since $Q_B^{r(p^e)} = tI + p^e A$ for some matrix A,

$$Q_B^{p\ r(p^e)} \equiv (tI + p^e A)^p \equiv t^p I \ (mod\ p^{e+1}),$$

yielding $r(p^{e+1})|p\ r(p^e)$. Since $r(p^e)|r(p^{e+1})$, we conclude that $r(p^{e+1}) = r(p^e)$ or $r(p^{e+1}) = p\ r(p^e)$. For $e = 1$, $r(p^2) \neq r(p)$, then $r(p^2) = p\ r(p)$. So, if $r(p^2) \neq r(p)$, then $r(p^2) = p\ r(p)$. Further, if l is the largest integer such that $r(p^l) = r(p)$, then $r(p^{l+c}) = p\ r(p^{l+c-1}) = ... = p^c\ r(p^l) = p^c\ r(p)$ for each natural number c. □

The following is an interesting result for the order of the sequence of balancing numbers for any prime p.

THEOREM 2.5. *For any prime p, $o(p^e) = o(p)$ where e is any positive integer.*

Proof. By virtue of THEOREM 2.4 and THEOREM 2.5, $\frac{\pi(p^e)}{\pi(p)} = p^i$ and $\frac{r(p^e)}{r(p)} = p^j$, where i and j are any positive integers. Clearly,

$$\frac{r(p^e)}{r(p)}\frac{\pi(p^e)}{r(p^e)} = \frac{\pi(p)}{r(p)}\frac{\pi(p^e)}{\pi(p)} .$$

By THEOREM 2.2, $\frac{\pi(p^e)}{r(p^e)} = o(p^e)$ and $\frac{\pi(p)}{r(p)} = o(p)$ and therefore $p^i(1\ or\ 2) = (1\ or\ 2)p^j$. Hence the result follows. □

LEMMA 2.6. *Let p be any odd prime and let e be any positive integer. Then*

$$(i) \quad o(p^e) = 1 \ \ if \ \ 2 \nmid \pi(p^e),$$
$$(ii) \quad o(p^e) = 2 \ \ if \ \ 2|\pi(p^e),$$

Proof. To prove (i), suppose $m = p^e$. Therefore by THEOREM 2.2, $o(p^e)r(p^e) = \pi(p^e)$. Since, $p^e > 2$, $o(p^e)$ is either 1 or 2. Observe that if $o(p^e)$ is 1, then $r(p^e)$ is even, for which $2 \nmid \pi(p^e)$. If $o(p^e)$ is 2, then $2 \mid \pi(p^e)$. □

The following are some arithmetical properties involving the period, rank and order of the sequence of balancing numbers.

THEOREM 2.7. *Let m be a positive integer, then $\pi(m) = (2, o(m))\ [r(m), 1]$, where (a, b) and $[a, b]$ denote the greatest common divisor and the least common multiple of the integers a and b, respectively.*

Proof. It is known that $Q_B^{r(m)} \equiv t(m)I$ modulo m. Comparing the determinants, we get $1^{r(m)} \equiv [t(m)]^2 \ (mod\ m)$. So, we note that $[t(m)]^2$ and $1^{r(m)}$ have the same order modulo m. Specifically:

$$\frac{o(m)}{(2, o(m))} = \frac{1}{(r(m), 1)}.$$

Therefore,

$$\pi(m) = r(m)o(m) = r(m)\frac{(2, o(m))}{(r(m), 1)} = (2, o(m))[r(m), 1)],$$

which ends the proof. □

The following result is an immediate consequence of THEOREM 2.7.

COROLLARY 2.8. *For $m > 2$, $o(m) = 1$ or 2.*

Proof. By virtue of THEOREM 2.7, we have

$$\begin{aligned} \pi(m) &= (2, o(m)).[r(m), 1] \\ &= (1 \text{ or } 2).r(m) \\ &= r(m) \text{ or } 2.r(m), \end{aligned}$$

and the result follows from THEOREM 2.2. □

THEOREM 2.9. *Let m and n are positive integers, then $r([m, n]) = [r(m), r(n)]$ and $\pi([m, n]) = [\pi(m), \pi(n)]$.*

Proof. Since $B_{r([m,n])} \equiv 0 \ (mod\ [m, n])$, we have $B_{r([m,n])} \equiv 0 \ (mod\ m)$ and $B_{r([m,n])} \equiv 0 \ (mod\ n)$. Therefore, both $r(m)$ and $r(n)$ divide $r([m, n])$, that is $[r(m), r(n)]|\ r([m, n])$. On the other hand, since $B_{[r(m),r(n)]} \equiv 0 \ (mod\ m)$ and $B_{[r(m),r(n)]} \equiv 0 \ (mod\ n)$, $B_{[r(m),r(n)]} \equiv 0 \ (mod\ [m, n])$. It follows that $r([m, n])|\ [r(m), r(n)]$, which completes the proof of the first part.

In order to prove the second part, we use the balancing matrix as follows. Since $Q_B^{\pi([m,n])} \equiv I \ (mod\ [m, n])$, we have $Q_B^{\pi([m,n])} \equiv I \ (mod\ m)$ and $Q_B^{\pi([m,n])} \equiv I \ (mod\ n)$. Therefore, both $\pi(m)$ and $\pi(n)$ divide $\pi([m, n])$, that is $[\pi(m), \pi(n)]|\ \pi([m, n])$. On the other hand, suppose $\pi(m)$ and $\pi(n)$ both divide $[\pi(m), \pi(n)]$. As $Q_B^{[\pi(m),\pi(n)]} \equiv I \ (mod\ m)$ and $Q_B^{[\pi(m),\pi(n)]} \equiv I \ (mod\ n)$, we have $Q_B^{[\pi(m),\pi(n)]} \equiv I \ (mod\ [m, n])$. Therefore $\pi([m, n])|\ [\pi(m), \pi(n)]$ and the result follows. □

The following result is useful while proving the subsequent results.

LEMMA 2.10. *For any integers m and n, $r(m)|n$ if and only if $m|B_n$.*

Proof. Suppose $r(m)$ divides n. Then for the least positive integer n,

$B_n \equiv 0 \ (mod\ m)$, which implies $m|B_n$. Conversely, for a least positive integer n, suppose $m|B_n$. Then $B_n \equiv 0(mod\ m)$ and the result follows. □

THEOREM 2.11. *If m has the prime factorization $m = p_1^{e_1}p_2^{e_2}...p_n^{e_n}$, then $r(m)$ is the $lcm\{r(p_i^{e_i})\}$ and $\pi(m)$ is the $lcm\{\pi(p_i^{e_i})\}$ for $i = 1, 2, 3, ..., n$ and e be any positive integer.*

Proof. Since $p_i^{e_i}$ are pairwise relatively prime, $m|B_n$ is equivalent to $p_i^{e_i}|B_n$. By virtue of Lemma 2.10, which is equivalent to $r(p_i^{e_i})|n$. The least positive integer n which satisfies these conditions is $n = lcm\{r(p_i^{e_i})\}$ for all i, which completes the proof of the first part. Since $p_i^{e_i}|m$ for all i, then $\pi(p_i^{e_i})|\pi(m)$ and hence $lcm\{\pi(p_i^{e_i})\}|\pi(m)$. On the other hand, since $\pi(p_i^{e_i})|\ lcm\{\pi(p_i^{e_i})\}$, $Q_B^{lcm\{\pi(p_i^{e_i})\}} \equiv I \ (mod\ p_i^{e_i})$, by Chinese remainder theorem, $Q_B^{lcm\{\pi(p_i^{e_i})\}} \equiv I \ (mod\ m)$. Hence $\pi(m)|\ lcm\{\pi(p_i^{e_i})\}$, which follows the second part. □

THEOREM 2.12. *If p be any prime. Then*

$$\begin{array}{ll} (i) & \pi(p)|p-1 \ \ if \ \ p \equiv \pm 1 \ (mod\ \ 8), \\ (ii) & \pi(p)|p+1 \ \ if \ \ p \equiv \pm 3 \ (mod\ \ 8), \\ (iii) & r(p)|p-1 \ \ if \ \ p \equiv \pm 1 \ (mod\ \ 8), \\ (iv) & r(p)|p+1 \ \ if \ \ p \equiv \pm 3 \ (mod\ \ 8). \end{array}$$

Proof. In [6], Panda *et al.* show that, if p is a prime of the form $8x \pm 1$, then $B_{p-1} \equiv 0(mod\ p)$, $B_p \equiv 1(mod\ p)$. Again, for $p = 8x \pm 3$, $B_p \equiv -1(mod\ p)$, $B_{p+1} \equiv 0(mod\ p)$, which follows (i) and (ii). By virtue of Lemma 2.10, $B_{p-1} \equiv 0 \ (mod\ p)$ if and only if $r(p)|p-1$ and $B_p \equiv 1 \ (mod\ p)$ if and only if $r(p)|p$, that is $\pi(p)|B_{p-1}$ if and only if $r(p)|p-1$, which follows (iii) and (iv). □

Acknowledgments. The authors are grateful to the anonymous reviewers for their useful and helpful remarks and suggestions to improve this article.

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Received 27 January 2015

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