COMPLETE CONTROLLABILITY OF GENERAL STOCHASTIC INTEGRODIFFERENTIAL SYSTEMS

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Communicated by Dan Crişan

In this paper, the notion of complete controllability for nonlinear stochastic neutral impulsive integrodifferential systems in finite dimensional spaces is introduced. Sufficient conditions ensuring the complete controllability of the nonlinear stochastic impulsive system are established. An example is provided to illustrate the result.

AMS 2010 Subject Classification: 93B05, 93E03, 93E20.

Key words: Banach fixed point theorem, complete controllability, Holder's inequality, stochastic neutral impulsive systems.

1. INTRODUCTION

The notion of controllability has played a central role throughout the history of modern control theory. The problem of controllability is to show the existence of a control function, which steers dynamical control systems from its initial state to the final state, where the initial and final states may vary over the entire space.

The problem of controllability of linear deterministic system is well documented. The controllability of nonlinear deterministic systems in a finite dimensional space has been extensively studied [2, 9]. The problem of controllability of the linear stochastic system of the form

$$\begin{cases} \mathrm{d}x(t) = [A(t)x(t) + B(t)u(t)]\,\mathrm{d}t + \sigma(t)dw(t), \quad t \in [0, T]\,, \\ x(0) = x_0, \end{cases} \tag{1}$$

has been studied by various authors [10, 17, 19] where $\sigma : [0, T] \to R^{n\times n}$. In Mahmudov [17], it is shown that complete controllability and approximate controllability of the system (1) coincide. He establishes the equivalence between complete controllability of the linear stochastic system on $[0, T]$ and the corresponding deterministic system on every $[s, T]$, $0 \leq s \leq T$.

Controllability of non-linear stochastic systems in finite-dimensional spaces has been investigated by many authors. Klamka and Socha [10] derived sufficient conditions for the stochastic controllability of linear and nonlinear systems using a Lyapunov technique. Mahmudov and Zorlu [16] derived sufficient conditions for complete and approximate controllability of semilinear stochastic systems with non-Lipschitz coefficients via Picard-type iterations. Balachandran *et al.* [3, 4] studied the controllability of semilinear stochastic integrodifferential systems using the Banach fixed point theorem.

In recent years, many systems in physics and biology exhibit impulsive dynamical behavior due to sudden jumps at certain instants in the evolution process. Differential equations involving impulsive effects occur in many applications [1,6,12,14]. The presence of impulses implies that the trajectories of the system do not necessarily preserve the basic properties of the non-impulsive dynamical systems. Yang, Xu and Xiang [21] established the exponential stability of non-linear impulsive stochastic differential equations with delays. Liu and Liao [13] studied the existence, uniqueness and stability of stochastic impulsive systems using Lyapunov-like functions. For the basic theory of impulsive differential equations the reader can refer to [11].

A natural generalization of impulsive ordinary differential equations are the impulsive neutral functional differential equations. Impulsive neutral functional differential equations describe models of real processes and phenomena where both dependence on the past and momentary disturbances are observed. In recent years, the interest in impulsive neutral systems has been growing rapidly due to their successful applications in practical fields such as circuit theory, bioengineering, chemical technology, etc. [18, 20].

In this article, we investigate the controllability problem of a class of nonlinear stochastic neutral impulsive systems. Further, we show the complete controllability of nonlinear stochastic system under the natural assumption that the associated linear impulsive stochastic control system is completely controllable. Motivation for these kind of equations can be found in [3, 8].

The paper is organized as follows. In Section 2, some basic notations and preliminary facts are recalled. In Section 3, we obtain the sufficient controllability conditions *via* one of the fixed point methods, in Section 4, we have given an example to illustrate the result.

2. PROBLEM FORMULATION

In this paper, we consider the following nonlinear stochastic integrodifferential impulsive systems of the form

$$(2)\quad \begin{cases} \mathrm{d}\left[x(t) - G(t, x(t), g(\eta x(t))\right] = A(t)\left[x(t) + \int_0^t H(t,s)x(s)\mathrm{d}s\right]\mathrm{d}t \\ \;+\left\{B(t)u(t) + F_1\left(t, x(t), f_{1,1}(\eta x(t)), f_{1,2}(\delta x(t)), f_{1,3}(\xi x(t))\right)\right\}\mathrm{d}t \\ \;+F_2\left(t, x(t), f_{2,1}(\eta x(t)), f_{2,2}(\delta x(t)), f_{2,3}(\xi x(t))\right)\mathrm{d}w(t),\ t\in[0,T],\ t\neq t_k, \\ \Delta x(t_k) = I_k(x(t_k^-)), \quad t = t_k, \quad k = 1, 2, ..., r, \\ x(0) = x_0 \in R^n, \end{cases}$$

where, for $i = \overline{1,2}$:

$$f_{i,1}(\eta x(t)) = \int_0^t f_{i,1}(t,s,x(s))\mathrm{d}s,\ g(\eta x(t)) = \int_0^t g(t,s,x(s))\mathrm{d}s,$$
$$f_{i,2}(\delta x(t)) = \int_0^T f_{i,2}(t,s,x(s))\mathrm{d}s,\ f_{i,3}(\xi x(t)) = \int_0^t f_{i,3}(t,s,x(s))\mathrm{d}w(s),$$

and $A(t)$, $B(t)$ are continuous matrices of dimensions $n\times n$, $n\times m$ respectively,

$$F_1 : [0,T]\times R^n\times R^n\times R^n\times R^n \to R^n,\ f_{i,1} : [0,T]\times[0,T]\times R^n \to R^n,$$
$$F_2 : [0,T]\times R^n\times R^n\times R^n\times R^n \to R^{n\times n},\ f_{i,2} : [0,T]\times[0,T]\times R^n \to R^n,$$
$$G : [0,T]\times R^n\times R^n \to R^n,\ f_{i,3} : [0,T]\times[0,T]\times R^n \to R^{n\times n},$$
$$H : [0,T]\times[0,T] \to R^{n\times n},\ g : [0,T]\times[0,T]\times R^n \to R^n,$$

$I_k \in C(R^n, R^n)$, $u(t)$ is a feedback control and w is an n-dimensional standard Brownian motion. Furthermore, $0 = t_0 < t_1 < \dots < t_r < t_{r+1} = T$, $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively. Also $\Delta x(t_k) = x(t_k^+) - x(t_k^-) = x(t_k) - x(t_k^-)$, represents the jump in the state x at time t_k with I_k determining the size of the jump, the initial value x_0 is $\mathcal{F}_0$-measurable with $E\left\|x_0\right\|^2 < \infty$.

This type of equations occur in population models [5,7] where the integral term specifies how much weight is attached to the population at various past times, in order to arrive at their present effect on the resources availability. The complete controllability of such equations is of quite fundamental importance biologically when the parameters are subject to some random disturbances, like environmental factors, since it concerns the long time survival of species. The study of this phenomenon has become an essential part of the qualitative theory of stochastic differential equations.

We define the operator Ψ_1 from $\mathbb{H}_2$ to $\mathbb{H}_2$ as follows:

$$(3)\quad \begin{aligned}(\Psi_1 x)(t) &= (\widehat{G}x)(t) + \int_0^t \tfrac{\partial S(t-s)}{\partial s}(\widehat{G}x)(s)\mathrm{d}s + \int_0^t S(t-s)(\widehat{F}_1 x)(s)\mathrm{d}s \\ &\quad + \int_0^t S(t-s)(\widehat{F}_2 x)(s)\mathrm{d}w(s) + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-)),\end{aligned}$$

where, for $i = 1,2$

$$(\widehat{F}_i x)(t) = F_i\left(t, x(t), f_{i,1}(\eta x(t)), f_{i,2}(\delta x(t)), f_{i,3}(\xi x(t))\right),$$
$$(\widehat{G}x)(t) = G\left(t, x(t), g(\eta x(t))\right),$$

and $S(t)$ is the resolvent matrix [5] which satisfies

$$(4)\quad \begin{cases} \frac{\partial S(T-s)}{\partial s} + S(T-s)A(s) + \int_s^T S(T-r)H(r,s)\mathrm{d}r = 0, \\ S(0) = I, \quad 0 \le s \le t \le T. \end{cases}$$

The solution of Eq (2) can be represented in the following integral form [18]:

$$\begin{aligned} x(t) &= S(t)\left[x_0 - G(0,x_0,0)\right] + (\widehat{G}x)(t) + \int_0^t \frac{\partial S(t-s)}{\partial s}(\widehat{G}x)(s)\mathrm{d}s \\ &\quad + \int_0^t S(t-s)\left(B(s)u(s) + (\widehat{F}_1 x)(s)\right)\mathrm{d}s + \int_0^t S(t-s)(\widehat{F}_2 x)\mathrm{d}w(s) \end{aligned}$$

$$+\sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-)),$$

then, we can written

$$x(t) = S(t)\left[x_0 - G(0,x_0,0)\right] + (\Psi_1 x)(t) + \int_0^t S(t-s)B(s)u(s)\mathrm{d}s. \tag{5}$$

Let $\mathcal{L}(X,Y)$ the space of all linear bounded operators from a Banach space X to a Banach space Y. Let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbf{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ generated by $\{w(s): 0\leq s\leq t\}$ and $\mathcal{F}=\mathcal{F}_T$. Let $L_2(\Omega,\mathcal{F}_T,R^n)$ be the Hilbert space of all $\mathcal{F}_T$-measurable square integrable variables with values in R^n. Let $U_{ad} := L_2^{\mathcal{F}}([0,T],R^m)$ be the Hilbert space of all square integrable and $\mathcal{F}_t$-measurable processes with values in R^m. Let $PC([0;T];R^n)$ be the space of function from $[0;T]$ into R^n such that $x(t)$ is continuous at $t\neq t_k$ and left continuous at $t=t_k$ and the right limit $X(t_k^+)$ exists for $k=1,2,...r$. Further, let $\mathbb{H}_2 := PC^b_{\mathcal{F}_t}([0,T],L_2(\Omega,\mathcal{F}_t,R^n))$ be the Banach space of all bounded $\mathcal{F}_t$-measurable, $PC([0;T];R^n)$ valued random variables φ satisfying

$$\|\varphi\|^2 = \sup_{t\in[0,T]} E\|\varphi(t)\|^2.$$

Now let us introduce the following operators and sets.

- The linear bounded operator $L_0^T \in \mathcal{L}\left(L_2^{\mathcal{F}}([0,T],R^m),L_2(\Omega,\mathcal{F}_T,R^n)\right)$ is defined by

$$L_0^T = \int_0^T S(T-s)B(s)u(s)\mathrm{d}s.$$

- The controllability operator Π_0^T associated with (1) is

$$\Pi_0^T(.) = \int_0^T S(T-t)B(t)B^*(t)S^*(T-t)E(.\mid\mathcal{F}_t)\mathrm{d}t, \tag{6}$$

  which belongs to $\mathcal{L}\left(L_2(\Omega,\mathcal{F}_T,R^n),L_2(\Omega,\mathcal{F}_T,R^n)\right)$, $B^*(t)$ is the adjoint operator of $B(t)$, and the controllability matrix $\Gamma_s^T\in\mathcal{L}(R^n,R^n)$

$$\Gamma_s^T = \int_s^T S(T-t)B(t)B^*(t)S^*(T-t)\mathrm{d}t, \quad 0\leq s\leq t.$$

- The set of all states attainable from x_0 in time $T>0$ is

$$\mathcal{R}_t(x_0) = \{x(t,x_0,u): u\in U_{ad}\},$$

  where $x(t,x_0,u)$ is the solution of (2) corresponding to $x_0\in R^n$ and $u(.)\in U_{ad}$.

Definition 1. The system (2) is completely controllable on $[0,T]$ if

$$\mathcal{R}_T(x_0) = L_2(\Omega,\mathcal{F}_T,R^n),$$

that is, if all the points in $L_2(\Omega,\mathcal{F}_T,R^n)$ can be reached from the point x_0 at time T.

The following lemma gives a formula for a control steering the state x_0 to an arbitrary final point x_T.

LEMMA 1. *Assume the operator Π_0^T is invertible, then for arbitrary $x_T \in L_2(\Omega, \mathcal{F}_T, R^n)$ the control*

$$u(t) = B^*(t)S^*(T-t)E\left\{(\Pi_0^T)^{-1}(x_T - S(T)[x_0 - G(0,x_0,0)] - (\Psi_1 x)(T)) \mid \mathcal{F}_t\right\} \tag{7}$$

transfers the system (5) from $x_0 \in R^n$ to $x_T \in R^n$ at time T.

Proof. By substituting (7) in (5), we obtain

$$\begin{aligned} x(t) &= S(t)\left[x_0 - G(0,x_0,0)\right] + (\Psi_1 x)(t) \\ &+ \int_0^t S(t-s)B(s)B^*(s)S^*(T-s) \\ &\times E\left\{(\Pi_0^T)^{-1}\left(x_T - S(T)[x_0 - G(0,x_0,0)] - (\Psi_1 x)(T)\right) \mid \mathcal{F}_s\right\} \mathrm{d}s, \end{aligned} \tag{8}$$

we have

$$S^*(T-s) = S^*\left[(T-t) + (t-s)\right] = S^*(t-s)S^*(T-t),$$

then

$$\begin{aligned} x(t) &= S(t)\left[x_0 - G(0,x_0,0)\right] + (\Psi_1 x)(t) \\ &+ \int_0^t S(t-s)B(s)B^*(s)S^*(t-s)S^*(T-t) \\ &\times E\left\{(\Pi_0^T)^{-1}\left(x_T - S(T)[x_0 - G(0,x_0,0)] - (\Psi_1 x)(T)\right) \mid \mathcal{F}_s\right\} \mathrm{d}s, \end{aligned} \tag{9}$$

from (6), we have

$$\Pi_0^t(.) = \int_0^t S(t-s)B(s)B^*(s)S^*(t-s)E(. \mid \mathcal{F}_s)\mathrm{d}s,$$

then

$$\begin{aligned} x(t) &= S(t)\left[x_0 - G(0,x_0,0)\right] + (\Psi_1 x)(t) + \Pi_0^t(S^*(T-t)(\Pi_0^T)^{-1} \\ &\times \left(x_T - S(T)[x_0 - G(0,x_0,0)] - (\Psi_1 x)(T)\right)). \end{aligned} \tag{10}$$

Writing $t = T$ in (10), we see that the control $u(.)$ transfers the system (5) from x_0 to x_T. □

3. CONTROLLABILITY RESULTS FOR STOCHASTIC IMPULSIVE SYSTEMS

In this section, we derive controllability conditions for the nonlinear stochastic integrodifferential impulsive system (2) using the contraction mapping principle. For the study of this problem we hence introduce the following hypotheses.

(A1) The functions $F_i,\ f_{i,j},\ G,\ g,\quad i=1,2,\ j=1,3$ satisfies the Lipschitz condition:
there are constants $L_1,\ N_1,\ K_1,\ C_1,\ q_k>0$ for $x_h,\ y_h,\ v_h,\ z_h\in R^n$, $h=1,2$ and $0\le s\le t\le T$

$$\begin{aligned}
&\|F_i(t,x_1,y_1,v_1,z_1)-F_i(t,x_2,y_2,v_2,z_2)\|^2\\
&\le L_1\left(\|x_1-x_2\|^2+\|y_1-y_2\|^2+\|v_1-v_2\|^2+\|z_1-z_2\|^2\right),\\
&\|G(t,x_1,y_1)-G(t,x_2,y_2)\|^2\le N_1\left(\|x_1-x_2\|^2+\|y_1-y_2\|^2\right),\\
&\|f_{i,j}(t,s,x_1(s))-f_{i,j}(t,s,x_2(s))\|^2\le K_1\|x_1-x_2\|^2,\\
&\|g(t,s,x_1(s))-g(t,s,x_2(s))\|^2\le C_1\|x_1-x_2\|^2,\\
&\|I_k(x)-I_k(y)\|^2\le q_k\|x-y\|^2,\quad k\in\{1,...,r\}.
\end{aligned}$$

(A2) The functions $F_i,\ f_{i,j},\ G,\ g,\quad i=1,2,\ j=1,3$ are continuous and there exist constants $L_2,\ N_2,\ K_2,\ C_2,\ d_k>0>0$ for $x,\ y,\ v,\ z\in R^n$ and $0\le t\le T$

$$\begin{aligned}
&\|F_i(t,x,y,v,z)\|^2\le L_2\left(1+\|x\|^2+\|y\|^2+\|v\|^2+\|z\|^2\right),\\
&\|G(t,x,y)\|^2\le N_2\left(1+\|x\|^2+\|y\|^2\right),\\
&\|f_{i,j}(t,s,x(s))\|^2\le K_2\left(1+\|x\|^2\right),\\
&\|g(t,s,x)\|^2\le C_2\left(1+\|x\|^2\right),\\
&\|I_k(x)\|^2\le d_k\left(1+\|x\|^2\right),\quad k\in\{1,...,r\}.
\end{aligned}$$

(A3) The linear system (1) is completely controllable.

Now for convenience, let us introduce the following notations:

$$\begin{aligned}
&l_1=\max\left\{\|S(t)\|^2,\ t\in[0,T]\right\},\ l_2=\max\left\{\left\|\tfrac{\partial S}{\partial t}\right\|^2,\ t\in[0,T]\right\}\\
&M=\max\left\{\left\|\Gamma_s^T\right\|^2,\ s\in[0,T]\right\}.
\end{aligned}$$

LEMMA 2 ([15]). *For every $z\in L_2(\Omega,\mathcal{F}_T,R^n)$, we have*

1. $E\left\|\Pi_0^t z\right\|^2\le ME\|z\|^2$.
2. *If the linear system (1) is completely controllable, then there exist $l_3>0$ such that*
$$E\left\|(\Pi_0^T)^{-1}\right\|^2\le l_3$$

LEMMA 3. *Under the condition (A1) and (A2), there exist a real constant $M_1,M_2>0$ such that for $x,\ y\in\mathbb{H}_2$, we have*

$$E\|(\Psi_1x)(t)-(\Psi_1y)(t)\|^2\le M_1\left(\sup_{s\in[0,T]}E\|x(s)-y(s)\|^2\right),\tag{11}$$

$$E\left\|(\Psi_1 x)(t)\right\|^2 \leq M_2\left(1+T\sup_{s\in[0,T]} E\left\|x(s)\right\|^2\right). \tag{12}$$

Proof. First, we will provide the proof of inequality (11), since (12) can be established in a similar way. For $i=1,2$, let $x,\ y\in\mathbb{H}_2$.
It follows from condition (A1), Holder inequality and Ito isometry that

$$\begin{aligned}\left\|(\widehat{F}_i x)(t)-(\widehat{F}_i y)(t)\right\|^2 &\leq L_1\Big(\|x(t))-y(t))\|^2+\|f_{i,1}(\eta x(t))-f_{i,1}(\eta y(t))\|^2\\ &+\|f_{i,2}(\delta x(t))-f_{i,2}(\delta y(t))\|^2+\|f_{i,3}(\xi x(t))-f_{i,3}(\xi y(t))\|_Q^2\Big),\\ &\leq L_1(1+2T^2K_1+TK_1)\sup_{s\in[0,T]}\|x(s)-y(s)\|^2,\end{aligned}$$

from which it follows that

$$E\left(\int_0^t\left\|(\widehat{F}_i x)(s)-(\widehat{F}_i y)(s)\right\|^2 \mathrm{d}s\right)\leq L_1T(1+2T^2K_1+TK_1)\sup_{s\in[0,T]}E\|x(s)-y(s)\|^2. \tag{13}$$

We have

$$\begin{aligned}\left\|(\widehat{G}x)(t)-(\widehat{G}y)(t)\right\|^2 &\leq N_1\left(\|x(t)-y(t)\|^2+\|g(\eta x(t))-g(\eta y(t))\|^2\right),\\ &\leq N_1(1+T^2C_1)\left(\sup_{s\in[0,T]}\|x(s)-y(s)\|^2\right),\end{aligned}$$

Then we obtain that

$$E\left(\int_0^t\left\|(\widehat{G}x)(s)-(\widehat{G}y)(s)\right\|^2\mathrm{d}s\right)\leq N_1T(1+T^2C_1)\left(\sup_{s\in[0,T]}E\left\|x(s)-y(s)\right\|^2\right). \tag{14}$$

It follows from the above inequality, Holder inequality and Ito isometry that

$$\begin{aligned}E\left\|(\Psi_1 x)(t)-(\Psi_1 y)(t)\right\|^2 &\leq 5E\left\|\int_0^t\frac{\partial S(t-s)}{\partial s}\left[(\widehat{G}x)(s)-(\widehat{G}y)(s)\right]\mathrm{d}s\right\|^2\\ &+5E\left\|\int_0^t S(t-s)\left[(\widehat{F}_1x)(s)-(\widehat{F}_1y)(s)\right]\mathrm{d}s\right\|^2\\ &+5E\left\|\int_0^t S(t-s)\left[(\widehat{F}_2x)(s)-(\widehat{F}_2y)(s)\right]\mathrm{d}w(s)\right\|^2\\ &+5E\left\|\sum_{0<t_k<t}S(t-t_k)\left[I_k(x(t_k^-))-I_k(y(t_k^-))\right]\right\|^2\\ &+5E\left\|(\widehat{G}x)(t)-(\widehat{G}y)(t)\right\|^2,\end{aligned}$$

then, we have

$$
\begin{aligned}
E\left\|(\Psi_1x)(t)-(\Psi_1y)(t)\right\|^2 \leq{}& 5Tl_1l_2E\int_0^t\left\|(\widehat{G}x)(s)-(\widehat{G}y)(s)\right\|^2\mathrm{d}s\\
&+5Tl_1E\int_0^t\left\|(\hat{F}_1x)(s)-(\hat{F}_1y)(s)\right\|^2\mathrm{d}s\\
&+5l_1E\int_0^t\left\|(\hat{F}_2x)(s)-(\hat{F}_2y)(s)\right\|^2\mathrm{d}s\\
&+5l_1r\sum_{k=1}^{r}E\left\|I_k(x(t_k^-))-I_k(y(t_k^-))\right\|^2\\
&+5E\left\|(\widehat{G}x)(t)-(\widehat{G}y)(t)\right\|^2.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&E\left\|(\Psi_1x)(t)-(\Psi_1y)(t)\right\|^2\\
&\leq\Big(10T^2l_1l_2N_1(1+T^2C_1)+15l_1(T+1)L_1T(1+2T^2K_1+TK_1)\\
&\quad+5l_1r\left(\sum_{k=1}^{r}q_k\right)+10N_1(1+T^2C_1)\Big)\sup_{s\in[0,T]}E\left\|x(s)-y(s)\right\|^2\\
&=M_1\sup_{s\in[0,T]}E\left\|x(s)-y(s)\right\|^2,
\end{aligned}
$$

where

$$
\begin{aligned}
M_1 &= 5T^2l_1l_2N_1(1+T^2C_1)+5l_1r\left(\sum_{k=1}^{r}q_k\right)\\
&\quad+5N_1(1+T^2C_1)+5l_1(T+1)L_1T(1+2T^2K_1+TK_1). \qquad \square
\end{aligned}
$$

THEOREM 1. *Assume that the conditions (A1), (A2) and (A3) hold. If the inequality*

$$
2M_1(1+Ml_1l_3)<1 \tag{15}
$$

hold, then the stochastic control system (2) is completely controllable on $[0,T]$.

Proof. Define a nonlinear operator $\Psi_2:\mathbb{H}_2\to\mathbb{H}_2$ by

$$
(\Psi_2x)(t)=S(t)\left[x_0-G(0,x_0,0)\right]+(\Psi_1x)(t)+\int_0^t S(t-s)B(s)u(s)\mathrm{d}s
$$

where

(16)

$$
u(t)=B^*(t)S^*(T-t)E\left\{\left(\Pi_0^T\right)^{-1}\left(x_T-S(T)[x_0-G(0,x_0,0)]-(\Psi_1x)(T)\right)\mid\mathcal{F}_t\right\}
$$

From Lemma (1), the control (16) transfers the system (5) from the initial state x_0 to the final state x_T provided that the operator Ψ_2 has a fixed point. So, if the operator Ψ_2 has a fixed point then the system (2) is completely controllable. As mentioned above, to prove the complete controllability it is enough to show that Ψ_2 has a fixed point in $\mathbb{H}_2$. To do this, we use the contraction mapping principle. To apply the contraction principle, first we show that Ψ_2 maps $\mathbb{H}_2$ into itself.

In order to prove the complete controllability of the stochastic system (2) it is enough to show that Ψ_2 has a fixed point in $\mathbb{H}_2$. To apply the contraction principle, first we show that Ψ_2 maps $\mathbb{H}_2$ into itself.

Let $x \in \mathbb{H}_2$. Now by Lemma (1), we have for $t \in [0, T]$

$$
\begin{aligned}
E\left\|(\Psi_2 x)(t)\right\|^2 &= E\big\|S(t)\left[x_0 - G(0, x_0, 0)\right] + (\Psi_1 x)(t) \\
&+\Pi_0^t S^*(T-t)(\Pi_0^T)^{-1}\left(x_T - S(T)[x_0 - G(0, x_0, 0)] - (\Psi_1 x)(T)\right)\big\|^2, \\
&\leq 3E\left\|S(t)\left[x_0 - G(0, x_0, 0)\right]\right\|^2 + 3E\left\|(\Psi_1 x)(t)\right\|^2 \\
&+3E\left\|\Pi_0^t S^*(T-t)(\Pi_0^T)^{-1}\left(x_T - S(T)[x_0 - G(0, x_0, 0)] - (\Psi_1 x)(T)\right)\right\|^2.
\end{aligned}
$$

It follows from Lemma (2) that

$$
\begin{aligned}
E\left\|(\Psi_2 x)(t)\right\|^2 &\leq 6l_1\left(\|x_0\|^2 + \|G(0, x_0, 0)\|^2\right) + 3E\left\|(\Psi_1 x)(t)\right\|^2 \\
&+9Ml_1l_3\left(E\|x_T\|^2 + 2l_1\left[\|x_0\|^2 + \|G(0, x_0, 0)\|^2\right] + E\left\|(\Psi_1 x)(T)\right\|^2\right), \\
&\leq 6l_1\left(\|x_0\|^2 + \|G(0, x_0, 0)\|^2\right) \\
&+9Ml_1l_3\left(E\|x_T\|^2 + 2l_1\left[\|x_0\|^2 + \|G(0, x_0, 0)\|^2\right]\right) \\
&+3(1 + 3Ml_1l_3)M_2\left(1 + T\sup_{s\in[0,T]} E\|x(s))\|^2\right),
\end{aligned}
$$

therefore, we obtain that $\|(\Psi_2 x)(t)\|_{\mathbb{H}_2}^2 < \infty$. Since Ψ_2 maps $\mathbb{H}_2$ into itself.

Secondly, we show that Ψ_2 is a contraction mapping. To see this let $x,\ y \in \mathbb{H}_2$, so from Lemma (2) and inequality (11), we have

$$
\begin{aligned}
&E\left\|(\Psi_2 x)(t) - (\Psi_2 y)(t)\right\|^2 \\
&= E\left\|(\Psi_1 x)(t) - (\Psi_1 y)(t) + \Pi_0^t S^*(T-t)(\Pi_0^T)^{-1}\left((\Psi_1 x)(T) - (\Psi_1 y)(T)\right)\right\|^2, \\
&\leq 2E\left\|(\Psi_1 x)(t) - (\Psi_1 y)(t)\right\|^2 + 2Ml_1l_3E\left\|(\Psi_1 x)(T) - (\Psi_1 y)(T)\right\|^2, \\
&\leq 2(1 + Ml_1l_3)\sup_{\mathbf{s}\in[\mathbf{0},\mathbf{T}]} E\left\|\Psi_1(x(s)) - \Psi_1(y(s))\right\|^2, \\
&\leq 2(1 + Ml_1l_3)M_1\left(\sup_{s\in[0,T]} E\|x(s) - y(s)\|^2\right).
\end{aligned}
$$

It result that

$$\sup_{s\in[0,T]} E\left\|(\Psi_2 x)(s)-(\Psi_2 y)(s)\right\|^2 \le 2M_1(1+Ml_1l_3)\left(\sup_{s\in[0,T]} E\left\|x(s)-y(s)\right\|^2\right).$$

Therefore Ψ_2 is contraction mapping if the inequality (15) holds. Then the mapping Ψ_2 has a unique fixed point $x(.)$ in $\mathbb{H}_2$ which is the solution of the equation (5). Thus the system (5) is completely controllable. □

4. EXAMPLE

Consider a scalar nonlinear stochastic integrodifferential control system

$$(17)\quad \begin{cases} \mathrm{d}\left[x(t)-(\widehat{G}x)(t)\right] = \left(e^{-7(T-t)}-7\right)\left[x(t)+\int_0^t 7e^{-13(t-s)}x(s)\mathrm{d}s\right]\mathrm{d}t \\ +\left\{e^{-2t}u(t)+(\widehat{F}_1x)(t)\right\}\mathrm{d}t+(\widehat{F}_2x)(t)\mathrm{d}w(t), \quad t\in[0,T], \quad t\neq t_k, \\ \Delta x(t_k)=0,24e^{0,03}(x(t_k^-)), \quad t=t_k, \\ x(0)=x_0\in R^n, \end{cases}$$

where $t_k = t_{k-1}+0,5$ for $k=1,2,...,r$. Here we have $A(t)=e^{-7(T-t)}-7$, $H(t,s)=7e^{-13(t-s)}$, $B(t)=e^{-2t}$

$$\begin{aligned} (\widehat{F}_1x)(t) &= 2x(t)+(2t^2+1)e^{-t}+\int_0^t 2se^{-s}x(s)\mathrm{d}s \\ &\quad +\int_0^T \frac{1}{\sqrt{1+|x(s)|}}\mathrm{d}s+\int_0^t \cos(x(s))\mathrm{d}w(s), \end{aligned}$$

$$\begin{aligned} (\widehat{F}_2x)(t) &= e^{-t}(x(t)+1)+\int_0^t (2s^2+3)x(s)\mathrm{d}s \\ &\quad +\int_0^T \sin(x(s))\mathrm{d}s+\int_0^t \log(1+|x(s)|)\mathrm{d}w(s), \end{aligned}$$

$$(\widehat{G}x)(t)=\log\left[(3t+e^{2t})\left|\int_0^t e^{-s}(x(s)+3)\mathrm{d}s\right|+1\right].$$

Obviously $S(t,s)=e^{-6(t-s)}$ satisfies

$$\begin{cases} \frac{\partial S(T,s)}{\partial s}+S(T,s)A(s)+\int_s^T S(T,r)H(r,s)\mathrm{d}r=0, \\ S(t,t)=I, \quad 0\le s\le t\le T. \end{cases}$$

So that

$$\begin{aligned} \Gamma_0^T &= \int_0^T S(T-s)B(s)B^*(s)S^*(T-s)\mathrm{d}s \\ &= \int_0^T e^{-4T}\mathrm{d}s = Te^{-4T}>0, \text{ for some } T>0. \end{aligned}$$

It can be easily seen that F_1, F_2 and G satisfy the hypotheses (A1)–(A2) of Theorem 1. Hence, the stochastic system (17) is completely controllable on $[0, T]$.

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Received 13 October 2014

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