$\mathfrak{aff}(1)$-INVARIANT n-ARY LINEAR DIFFERENTIAL OPERATOR AND COHOMOLOGY OF $\mathfrak{aff}(1)$ ACTING ON n-ARY LINEAR DIFFERENTIAL OPERATORS

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We consider the $\mathfrak{aff}(1)$-module structure on the spaces of n-ary linear differential operators acting on the spaces of weighted densities. We classify $\mathfrak{aff}(1)$-invariant n-ary linear differential operators acting on the spaces of weighted densities. We compute the first differential cohomology of the Lie algebra $\mathfrak{aff}(1)$ with coefficients in n-ary linear differential operators acting on the spaces of weighted densities.

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1. INTRODUCTION

Deformation theory plays a crucial role in all branches of mathematics and physics. In physics, the mathematical theory of deformations has proved to be a powerful tool in modeling physical reality. The concepts symmetry and deformations are considered to be two fundamental guiding principle for developing the physical theory further. Formal deformations of arbitrary rings and associative algebras, and the related cohomology questions, were first investigated by Gerstenhaber, in a series of articles [8–11]. The notion of deformation was applied to Lie algebras by Nijenhuis and Richardson [12,13]. Recently, deformations of Lie algebras with multi-parameters were intensively studied (see, *e.g.*, [5,6]).

Let $\mathrm{V}ect(\mathbb{R})$ be the Lie algebra of vector fields on $\mathbb{R}$. Denote by $\mathcal{F}_\lambda = \big\{ f dx^\lambda \mid f \in \mathcal{C}^\infty(\mathbb{R}) \big\}$ the space of weighted densities of weight $\lambda \in \mathbb{R}$, *i.e.*, the space of sections of the line bundle $(T^*\mathbb{R})^{\otimes\lambda}$, so its elements can be represented as $f(x)dx^\lambda$, where $f(x)$ is a function and dx^λ is a formal (for a time being) symbol [2]. This space coincides with the space of vector fields, functions and differential forms for $\lambda = -1$, 0 and 1, respectively.

The space $\mathcal{F}_\lambda$ is a V$ect(\mathbb{R})$-module for the action defined by

$$L^\lambda_{g\frac{d}{dx}}(fdx^\lambda) = (gf' + \lambda g'f)dx^\lambda.$$

Denote by $\mathcal{D}_{\underline{\lambda},\mu}$, where $\underline{\lambda} = (\lambda_1, \cdots, \lambda_n)$, the space of n-ary linear differential operators:

$$\underbrace{\mathcal{F}_{\lambda_1} \otimes \cdots \otimes \mathcal{F}_{\lambda_n}}_{n\otimes} \to \mathcal{F}_\mu,$$

for any $\lambda_1, \cdots, \lambda_n, \mu \in \mathbb{R}$. The Lie algebra V$ect(\mathbb{R})$ acts on the space

$$\mathcal{D}_{\underline{\lambda},\mu} := \mathrm{Hom}_{\mathrm{diff}}(\mathcal{F}_{\lambda_1} \otimes \cdots \otimes \mathcal{F}_{\lambda_n}, \mathcal{F}_\mu)$$

of these differential operators by:

$$X_h.A = L^\mu_{X_h} \circ A - A \circ L^{(\lambda_1,\cdots,\lambda_n)}_{X_h}, \tag{1.1}$$

where $L^{(\lambda_1,\cdots,\lambda_n)}_{X_h}$ is the Lie derivative on $\mathcal{F}_{\lambda_1} \otimes \cdots \otimes \mathcal{F}_{\lambda_n}$ defined by the Leibnitz rule:

$$L^{(\lambda_1,\cdots,\lambda_n)}_{X_h}(f_1dx^{\lambda_1} \otimes \cdots \otimes f_ndx^{\lambda_n}) = L^{\lambda_1}_{X_h}(f_1) \otimes \cdots \otimes f_ndx^{\lambda_n} + \cdots + f_1dx^{\lambda_1} \otimes \cdots \otimes L^{\lambda_n}_{X_h}(f_ndx^{\lambda_n}).$$

Thus the space of differential operators is a V$ect(\mathbb{R})$-module.

In [4], Bouarroudj computes the first differential cohomology of the Lie algebra $\mathfrak{sl}(2)$ with coefficients in bilinear differential operators acting on the spaces of weighted densities and in [3], a super analogue of [4]. Now, we restrict ourselves to the Lie algebra $\mathfrak{aff}(1)$ which is isomorphic to the Lie subalgebra of V$ect(\mathbb{R})$ spanned by

$$\{X_1, X_x\}.$$

In this paper, we are interested in the study of the first cohomology spaces $\mathrm{H}^1_{\mathrm{d}iff}(\mathfrak{aff}(1), \mathcal{D}_{\underline{\lambda},\mu})$ where $\mathcal{D}_{\underline{\lambda},\mu}$ is the space of n-ary linear differential operators from $\mathcal{F}_{\lambda_1} \otimes \cdots \otimes \mathcal{F}_{\lambda_n}$ to $\mathcal{F}_\mu$ and we classify $\mathfrak{aff}(1)$-invariant n-ary linear differential operators acting on the spaces of weighted densities.

2. DEFINITIONS AND NOTATIONS

2.1. COHOMOLOGY THEORY

Let us first recall some fundamental concepts from cohomology theory (see, *e.g.*, [7]). Let $\mathfrak{g}$ be a Lie algebra acting on a vector space V. The space of n-cochains of $\mathfrak{g}$ with values in V is the $\mathfrak{g}$-module

$$C^n(\mathfrak{g}, V) := \mathrm{Hom}(\Lambda^n\mathfrak{g}; V).$$

The *coboundary operator* $\delta_n : C^n(\mathfrak{g}, V) \longrightarrow C^{n+1}(\mathfrak{g}, V)$ is a $\mathfrak{g}$-map satisfying $\delta_n \circ \delta_{n-1} = 0$. The kernel of δ_n, denoted $Z^n(\mathfrak{g}, V)$, is the space of *n-cocycles*, among them, the elements in the range of δ_{n-1} are called *n-coboundaries*. We denote $B^n(\mathfrak{g}, V)$ the space of n-coboundaries.

By definition, the n^{th} cohomology space is the quotient space

$$\mathrm{H}^n(\mathfrak{g}, V) = Z^n(\mathfrak{g}, V)/B^n(\mathfrak{g}, V).$$

We will only need the formula of δ_n (which will be simply denoted δ) in degrees 0 and 1: for $\Xi \in C^0(\mathfrak{g}, V) = V$, $\delta\Xi(g) := g \cdot \Xi$, and for $\Lambda \in C^1(\mathfrak{g}, V)$,

$$\delta(\Lambda)(g, h) := g \cdot \Lambda(h) - h \cdot \Lambda(g) - \Lambda([g, h]) \quad \text{for any} \quad g, h \in \mathfrak{g}. \tag{2.2}$$

2.2. LIE ALGEBRA $\mathfrak{aff}(1)$

The Lie algebra $\mathfrak{aff}(1)$ is realized as subalgebra of the Lie algebra $\mathrm{V}ect(\mathbb{R})$ [1]:

$$\mathfrak{aff}(1) = Span(X_1 = \frac{d}{dx}, X_x = x\frac{d}{dx}).$$

The commutation relations are

$$[X_1, X_x] = X_1, \qquad [X_x, X_x] = 0, \qquad [X_1, X_1] = 0.$$

2.3. THE SPACE OF TENSOR DENSITIES ON $\mathbb{R}$

The Lie algebra, $\mathrm{V}ect(\mathbb{R})$, of vector fields on $\mathbb{R}$ naturally acts, by the Lie derivative, on the space

$$\mathcal{F}_\lambda = \left\{ f dx^\lambda \; : \; f \in C^\infty(S^1) \right\},$$

of weighted densities of degree λ. The Lie derivative L_X^λ of the space $\mathcal{F}_\lambda$ along the vector field $X\frac{d}{dx}$ is defined by

$$L_X^\lambda = X\partial_x + \lambda X', \tag{2.3}$$

where $X, f \in C^\infty(\mathbb{R})$ and $X' := \frac{dX}{dx}$. More precisely, for all $f dx^\lambda \in \mathcal{F}_\lambda$, we have

$$L_X^\lambda(f dx^\lambda) = (Xf' + \lambda f X')dx^\lambda.$$

In the paper, we restrict ourselves to the Lie algebra $\mathfrak{aff}(1)$ which is isomorphic to the Lie subalgebra of $\mathrm{V}ect(\mathbb{R})$ spanned by

$$\{X_1, X_x\}.$$

2.4. THE SPACE OF n-ARY LINEAR DIFFERENTIAL OPERATORS AS A $\mathfrak{aff}(1)$-module

The space of n-ary linear differential operators is a $\mathrm{V}ect(\mathbb{R})$-module, denoted

$$\mathcal{D}_{\underline{\lambda},\mu} := \mathrm{Hom}_{\mathrm{diff}}(\mathcal{F}_{\lambda_1} \otimes \cdots \otimes \mathcal{F}_{\lambda_n}, \mathcal{F}_{\mu}).$$

The $\mathrm{V}ect(\mathbb{R})$ action is:

$$L_X^{\underline{\lambda},\mu}(A) = L_X^{\mu} \circ A - A \circ L_X^{\underline{\lambda}}, \tag{2.4}$$

where $\underline{\lambda} = (\lambda_1, \cdots, \lambda_n)$ and $L_X^{(\lambda_1,\cdots,\lambda_n)}$ is the Lie derivative on $\mathcal{F}_{\lambda_1} \otimes \cdots \otimes \mathcal{F}_{\lambda_n}$ defined by the Leibnitz rule:

$$L_X^{(\lambda_1,\cdots,\lambda_n)}(f_1 dx^{\lambda_1} \otimes \cdots \otimes f_n dx^{\lambda_n}) = L_X^{\lambda_1}(f_1) \otimes \cdots \otimes f_n dx^{\lambda_n} + \cdots + f_1 dx^{\lambda_1} \otimes \cdots \otimes L_X^{\lambda_n}(f_n dx^{\lambda_n}).$$

3. $\mathfrak{aff}(1)$-INVARIANT n-ARY LINEAR DIFFERENTIAL OPERATORS

In this section, we will investigate differential operators on tensor densities that are $\mathfrak{aff}(1)$- invariant.

For $\underline{\lambda} = (\lambda_1, ..., \lambda_n) \in \mathbb{R}^n$ and $\mu \in \mathbb{R}$, we pose $\sigma = \mu - \lambda_1 - \lambda_2 - ... - \lambda_n$ and we are interested in the space $\mathcal{I}_{\underline{\lambda}}^{\mu}$ of the n-ary linear differential operators who are $\mathfrak{aff}(1)$-invariant. The elements of $\mathcal{I}_{\underline{\lambda}}^{\mu}$ are described as follows:

Theorem 3.1. *(a)If $\sigma \notin \mathbb{N}$, then $\mathcal{I}_{\underline{\lambda}}^{\mu} = \{0\}$.*
(b)There exist $\mathfrak{aff}(1)$-invariant n-ary linear differential operators

$$\mathcal{I}_{\underline{\lambda}}^{\mu} : \mathcal{F}_{\lambda_1} \otimes \cdots \otimes \mathcal{F}_{\lambda_n} \longrightarrow \mathcal{F}_{\lambda_1+\lambda_2+\cdots+\lambda_n+k}$$

given by

$$\begin{array}{rcl} \mathcal{I}_{\underline{\lambda}}^{\mu} : \mathcal{F}_{\lambda_1} \otimes \cdots \otimes \mathcal{F}_{\lambda_n} & \longrightarrow & \mathcal{F}_{\lambda_1+\lambda_2+\cdots+\lambda_n+k}, \\ \phi_1 \otimes \cdots \otimes \phi_n & \longmapsto & \displaystyle\sum_{i_1+i_2+\cdots+i_n=k} a_{i_1,i_2,\cdots,i_n} \ \phi_1^{(i_1)}\phi_2^{(i_2)}\cdots\phi_n^{(i_n)}, \end{array}$$

where $a_{i_1,\cdots,i_n}$ are constants.

Proof. Let A be an n-ary linear differential operator

$$\begin{array}{rcl} \mathrm{A} : \mathcal{F}_{\lambda_1} \otimes ... \otimes \mathcal{F}_{\lambda_n} & \longrightarrow & \mathcal{F}_{\mu}, \\ (\phi_1 \otimes \cdots \otimes \phi_n) & \longmapsto & \displaystyle\sum_{k=0}^{m} \sum_{i_1+...+i_n=k} a_{i_1,\cdots,i_n} \ \phi_1^{(i_1)}\cdots\phi_n^{(i_n)}, \end{array}$$

where $a_{i_1,\cdots,i_n}$ are, a priori, functions. A is $\mathfrak{aff}(1)$-invariant if and only if for all $X \in \mathfrak{aff}(1)$:

$$X.A = 0 \Longleftrightarrow L_X^{\lambda_1,\cdots,\lambda_n,\mu}(A)(\phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_n) = 0,$$

for all $(\phi_1, \cdots, \phi_n) \in \mathcal{F}_{\lambda_1} \times \mathcal{F}_{\lambda_2} \times \cdots \times \mathcal{F}_{\lambda_n}$.

• The invariance with respect to X_1 is reflected in:

$$\sum_{k=0}^{m} \sum_{i_1+i_2+\cdots+i_n=k} a'_{i_1,\cdots,i_n} \ \phi_1^{(i_1)} \cdots \phi_n^{(i_n)} = 0,$$

hence, $a_{i_1,\cdots,i_n}$ is a constant $\forall i_1, \cdots, i_n$.

Now $A_k(\phi_1 \otimes \cdots \otimes \phi_n) = \displaystyle\sum_{i_1+\cdots+i_n=k} a_{i_1,i_2,\cdots,i_n} \ \phi_1^{(i_1)} \cdots \phi_n^{(i_n)}$, denotes the homogeneous component of A of order k. We see that A is $\mathfrak{aff}(1)$-invariant if and only if each of its homogeneous components is $\mathfrak{aff}(1)$-invariant. So we can without loss of generality, assume that A is homogeneous of order k.

The $\mathfrak{aff}(1)$-invariance of A is reflected in:

$$L_X^{\lambda_1,\cdots,\lambda_n,\mu}(A)(\phi_1 \otimes \cdots \otimes \phi_n) = 0,$$

for all $X \in \mathfrak{aff}(1)$, $(\phi_1, \cdots, \phi_n) \in \mathcal{F}_{\lambda_1} \times \mathcal{F}_{\lambda_2} \times \cdots \times \mathcal{F}_{\lambda_n}$.

A direct computation proves that

$$L_X^{\lambda_1,\cdots,\lambda_n,\mu}(A)(\phi_1 \otimes \cdots \otimes \phi_n) = (\mu - \lambda_1 - \lambda_2 - \cdots - \lambda_n - k)\ X' \sum_{i_1+\cdots+i_n=k} a_{i_1,\cdots,i_n} \ \phi_1^{(i_1)} \cdots \phi_n^{(i_n)}.$$

So to examine the $\mathfrak{aff}(1)$-invariance of A we distinguish the following two cases:

• If $\sigma \neq k$, then $A = 0$.

• If $\sigma = k$, then

$$A = \sum_{i_1+\cdots+i_n=k} a_{i_1,\cdots,i_n} \ \phi_1^{(i_1)} \phi_2^{(i_2)} \cdots \phi_n^{(i_n)},$$

where $a_{i_1,i_2,\cdots,i_n}$ are constants.

Hence the result. $\square$

4. THE SPACE $\mathrm{H}^1_{diff}(\mathfrak{aff}(1), \mathcal{D}_{\underline{\lambda},\mu})$

In this section, we will compute the *differentiable* cohomology of the Lie algebra $\mathfrak{aff}(1)$ with coefficients in the space of n-ary linear differential operators $\mathcal{D}_{\underline{\lambda},\mu}$ where $\underline{\lambda} = (\lambda_1 \ \cdots, \lambda_n)$. Namely, we consider only cochains that are given by differentiable maps.

Any 1-cochain $c \in \mathrm{Z}^1_{\mathrm{d}iff}(\mathfrak{aff}(1), \mathcal{D}_{\underline{\lambda},\mu})$ has the following general form:

$$c(X,\phi_1,\cdots,\phi_n)=X\sum_{i_1,\cdots,i_n}\alpha_{i_1,\cdots,i_n}\,\phi_1^{(i_1)}\cdots\,\phi_n^{(i_n)}+X'\sum_{i_1,\cdots,i_n}\beta_{i_1,\cdots,i_n}\,\phi_1^{(i_1)}\cdots\,\phi_n^{(i_n)},$$

where $\alpha_{i_1,i_2,\cdots,i_n}$ and $\beta_{i_1,i_2,\cdots,i_n}$ are, a priori, functions.
So, for any integer $k \geq 0$, we define the k-homogeneous component of c by

$$\begin{aligned} c_k(X,\phi_1,\cdots,\phi_n) = X &\sum_{i_1+\cdots+i_n=k}\alpha_{i_1,\cdots,i_n}\,\phi_1^{(i_1)}\cdots\,\phi_n^{(i_n)} \\ &+X'\sum_{i_1+\cdots+i_n=k-1}\beta_{i_1,\cdots,i_n}\,\phi_1^{(i_1)}\cdots\,\phi_n^{(i_n)}. \end{aligned}$$

Of course, we suppose that $\beta_{i_1,i_2,\cdots,i_n}=0$ if $k=0$.

Lemma 4.1. *Any 1-cochain $c \in C^1_{\mathrm{diff}}(\mathfrak{aff}(1), \mathcal{D}_{\underline{\lambda},\mu})$ is a 1-cocycle if and only if each of its homogeneous components is a 1-cocycle.*

The following lemma gives the general form of any homogeneous 1-cocycle.

Lemma 4.2. *Up to a coboundary, any k-homogeneous 1-cocycle $c \in \mathrm{Z}^1_{\mathrm{d}iff}$ $(\mathfrak{aff}(1), \mathcal{D}_{\underline{\lambda},\mu})$ can be expressed as follows. For all $(\phi_1,\cdots,\phi_n) \in \mathcal{F}_{\lambda_1}\times\times\cdots\times\mathcal{F}_{\lambda_n}$ and for all $X \in \mathfrak{aff}(1)$:*

$$c(X,\phi_1,\cdots,\phi_n) = X'\sum_{i_1+i_2+\cdots+i_n=k-1}\beta_{i_1,i_2,\cdots,i_n}\,\phi_1^{(i_1)}\cdots\,\phi_n^{(i_n)},$$

where $\beta_{i_1,\cdots,i_n}$ are constants.

Proof. Any k-homogeneous 1-cocycle on $\mathfrak{aff}(1)$ has the following general form:

$$\begin{aligned} c(X,\phi_1,\cdots,\phi_n) = \; & X\sum_{i_1+\cdots+i_n=k}\alpha_{i_1,\cdots,i_n}\,\phi_1^{(i_1)}\cdots\,\phi_n^{(i_n)} \\ & +X'\sum_{i_1+\cdots+i_n=k-1}\beta_{i_1,\cdots,i_n}\,\phi_1^{(i_1)}\cdots\,\phi_n^{(i_n)}, \end{aligned}$$

where $\alpha_{i_1,\cdots,i_n}$ and $\beta_{i_1,\cdots,i_n}$ are, a priori, functions. First, we prove that the terms in X can be annihilated by adding a coboundary. Let $b : \mathcal{F}_{\lambda_1}\otimes\cdots\otimes\mathcal{F}_{\lambda_n}\to\mathcal{F}_\mu$ be a n-ary linear differential operator defined by

$$b(\phi_1,\phi_2,\cdots,\phi_n) = \sum_{i_1+\cdots+i_n=k} b_{i_1,\cdots,i_n}\,\phi_1^{(i_1)}\cdots\,\phi_n^{(i_n)},$$

where $b_{i_1,i_2,\cdots,i_n}$ are, a priori, functions and the coefficients $b_{i_1,\cdots,i_n}$ are functions satisfying

$$\frac{d}{dx}(b_{i_1,\cdots,i_n}) = \alpha_{i_1,\cdots,i_n}.$$

Then, for all $X \in \mathfrak{aff}(1)$, we have

$$\delta b(X,\phi_1,\cdots,\phi_n)=X\sum_{i_1+\cdots+i_n=k} b'_{i_1,\cdots,i_n}\,\phi_1^{(i_1)}\cdots\,\phi_n^{(i_n)}$$
$$+X'\sum_{i_1+\cdots+i_n=k} b_{i_1,\cdots,i_n}\,(\mu-\lambda_1-\lambda_2-\cdots-\lambda_n-k)\,\phi_1^{(i_1)}\cdots\,\phi_n^{(i_n)}.$$

We replace c by $\widetilde{c}=c-\delta b$ and then we see that the 1-cocycle $\widetilde{c}$ does not contain terms in X. So, up to a coboundary, any k-homogeneous 1-cocycle on $\mathfrak{aff}(1)$ can be expressed as follows:

$$c(X,\phi_1,\cdots,\phi_n)=X'\sum_{i_1+\cdots+i_n=k-1}\beta_{i_1,\cdots,i_n}\,\phi_1^{(i_1)}\cdots\,\phi_n^{(i_n)}.$$

Now, consider the 1-cocycle condition:

$$X\cdot c(Y,\phi_1,\cdots,\phi_n)-Y\cdot c(X,\phi_1,\cdots,\phi_n)-c([X,Y],\phi_1,\cdots,\phi_n)=0,$$

where $X,Y\in\mathfrak{aff}(1)$ and $(\phi_1,\cdots,\phi_n)\in\mathcal{F}_{\lambda_1}\times\cdots\times\mathcal{F}_{\lambda_n}$. A direct computation proves that we have

$$X\cdot c(Y,\phi_1,\cdots,\phi_n)=XY'\sum_{i_1+\cdots+i_n=k-1}\beta'_{i_1,\cdots,i_n}\,\phi_1^{(i_1)}\cdots\,\phi_n^{(i_n)}$$
$$+X'Y'\sum_{i_1+\cdots+i_n=k-1}\beta_{i_1,\cdots,i_n}\,(\mu-\lambda_1-\cdots-\lambda_n-k)\,\phi_1^{(i_1)}\cdots\,\phi_n^{(i_n)}.$$

$$Y\cdot c(X,\phi_1,\cdots,\phi_n)=X'Y\sum_{i_1+\cdots+i_n=k-1}\beta'_{i_1,\cdots,i_n}\,\phi_1^{(i_1)}\cdots\,\phi_n^{(i_n)}$$
$$+X'Y'\sum_{i_1+\cdots+i_n=k-1}\beta_{i_1,\cdots,i_n}\,(\mu-\lambda_1-\cdots-\lambda_n-k)\,\phi_1^{(i_1)}\cdots\,\phi_n^{(i_n)}.$$

$$c([X,Y],\phi_1,\cdots,\phi_n)=0.$$

$$X\cdot c(Y,\phi_1,\cdots,\phi_n)-Y\cdot c(X,\phi_1,\cdots,\phi_n)-c([X,Y],\phi_1,\cdots,\phi_n)=$$
$$(XY'-X'Y)\sum_{i_1+\cdots+i_n=k-1}\beta'_{i_1,\cdots,i_n}\,\phi_1^{(i_1)}\cdots\,\phi_n^{(i_n)}.$$

So, c is a 1-cocycle if and only if each of its $\beta_{i_1,\cdots,i_n}$ are constants. □

THEOREM 4.3.

$$\mathrm{H}^1_{\mathrm{d}iff}(\mathfrak{aff}(1),\mathcal{D}_{\underline{\lambda},\mu})\simeq\begin{cases}\mathbb{R}^{f_n(k)} & \textit{if}\quad \mu-\lambda_1-\lambda_2-\cdots-\lambda_n=k\in\mathbb{N},\\ 0 & \textit{otherwise.}\end{cases}$$

where

$$f_n(k)=\sum_{i=1}^{k+1}f_{n-1}(i-1),\quad f_1(k)=1.$$

Proof. By using Lemma 4.1 and Lemma 4.2 any 1-cocycle on $\mathfrak{aff}(1)$ has the following general form:

$$c(X,\phi_1,\cdots,\phi_n)=X'\sum_{i_1+\cdots+i_n=k}\beta_{i_1,\cdots,i_n}\ \phi_1^{(i_1)}\cdots\ \phi_n^{(i_n)}, \tag{4.5}$$

where $\beta_{i_1,\cdots,i_n}$ are constants.

Now we are going to deal with trivial 1-cocycles, and show how the general 1-cocycles (4.5) can be eventually trivial. Any trivial 1-cocycle should be of the form:

$$L_X^{\lambda_1,\lambda_2,\cdots,\lambda_n,\mu}(\mathcal{B}),$$

where

$$\mathcal{B}(\phi_1,\phi_2,\cdots,\phi_n)=\sum_{i_1+\cdots+i_n=k}\delta_{i_1,\cdots,i_n}\ \phi_1^{(i_1)}\cdots\ \phi_n^{(i_n)},$$

$\delta_{i_1,\cdots,i_n}$ are constants.
By a direct computation we have

$$\begin{aligned}&L_X^{\lambda_1,\cdots,\lambda_n,\mu}(\mathcal{B})(\phi_1,\cdots,\phi_n)\\&\qquad=X'\sum_{i_1+\cdots+i_n=k}\delta_{i_1,i_2,\cdots,i_n}\ (\mu-\lambda_1-\lambda_2-\cdots-\lambda_n-k)\ \phi_1^{(i_1)}\cdots\ \phi_n^{(i_n)}.\end{aligned}$$

Then to determine our space $\mathrm{H}^1_{\mathrm{d}iff}(\mathfrak{aff}(1),\mathcal{D}_{\underline{\lambda},\mu})$ we distinguish the following two cases:

- if $\mu-\lambda_1-\lambda_2-\cdots-\lambda_n\notin\mathbb{N}$, then c is trivial.
- if $\mu-\lambda_1-\lambda_2-\cdots-\lambda_n=k\in\mathbb{N}$, then

$$L_X^{\lambda_1,\lambda_2,\cdots,\lambda_n,\mu}(\mathcal{B})(\phi_1,\phi_2,\cdots,\phi_n)=0.$$

Thus the dimensions of $\mathrm{H}^1_{\mathrm{d}iff}(\mathfrak{aff}(1),\mathcal{D}_{\underline{\lambda},\mu})$ and $\mathrm{Z}^1_{\mathrm{d}iff}(\mathfrak{aff}(1),\mathcal{D}_{\underline{\lambda},\mu})$ are equal.
Now we are going to determine the dimension of $\mathrm{Z}^1_{\mathrm{d}iff}(\mathfrak{aff}(1),\mathcal{D}_{\underline{\lambda},\mu})$.
Let d be a 1-cocycle on $\mathfrak{aff}(1)$, then d has the following general form:

$$d(X,\phi_1,\cdots,\phi_n)=X'\sum_{i_1+\cdots+i_n=k}\kappa_{i_1,i_2,\cdots,i_n}\ \phi_1^{(i_1)}\cdots\ \phi_n^{(i_n)},$$

where $\kappa_{i_1,\cdots,i_n}$ are constants.
So $\mathrm{Z}^1_{\mathrm{d}iff}(\mathfrak{aff}(1),\mathcal{D}_{\underline{\lambda},\cdots,\lambda_n,\mu})$ is generated by the family

$$(X'\phi_1^{(i_1)}\phi_2^{(i_2)}\cdots\ \phi_n^{(i_n)})_{i_1+i_2+\cdots+i_n=k}.$$

And consequently the dimension of $\mathrm{Z}^1_{\mathrm{d}iff}(\mathfrak{aff}(1),\mathcal{D}_{\lambda_1,\cdots,\lambda_n,\mu})$ is the number of the constants $\kappa_{i_1,i_2,\cdots,i_n}$ such that $i_1+i_2+\cdots+i_n=k$.

The number of the constants $\kappa_{i_1,i_2,\cdots,i_n}$ such that $i_1 + i_2 + \cdots + i_n = k$ is determined by iteration and is equal to $f_n(k)$, where

$$f_n(k) = \sum_{i=1}^{k+1} f_{n-1}(i-1), \quad f_1(k) = 1. \quad \square$$

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