SUM OF THE FIBONOMIAL COEFFICIENTS AT MOST ONE AWAY FROM FIBONACCI NUMBERS

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In this note, we show that the equation

$$\sum_{k=0}^{m} \begin{bmatrix} 2m+1 \\ k \end{bmatrix}_F \pm 1 = F_n$$

has no solution except that $(m, n) = (1, 3)$, $(3, 14)$ where $\begin{bmatrix} m \\ k \end{bmatrix}_F$ is Fibonomial coefficient.

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1. INTRODUCTION

The Fibonacci sequence $\{F_n\}_{n\geq 2}$ is given by the following recurrence relation

$$F_{n+2} = F_{n+1} + F_n$$

with initial conditions $F_0 = 0$, $F_1 = 1$. For $1 \leq k \leq m$, the Fibonomial coefficient $\begin{bmatrix} m \\ k \end{bmatrix}_F$ is defined by

$$\begin{bmatrix} m \\ k \end{bmatrix}_F = \frac{F_m F_{m-1} F_{m-2} \dots F_{m-k+1}}{F_1 F_2 \dots F_k}. \tag{1.1}$$

It is surprising that the equation (1.1) always takes integer values.

There are some diophantine equations involving the Fibonomial coefficients and Fibonacci numbers. For example, Marques [5] investigated the solutions of the Fibonacci version of the Brocard-Ramanujan Diophantine equation and showed that the diophantine equation

$$F_n F_{n+1} \dots F_{n+k-1} + 1 = F_m^2 \tag{1.2}$$

has no solution in positive integer m and n. Although the idea of the proof is clear and correct, the solutions $F_4 + 1 = F_3^2$ and $F_6 + 1 = F_4^2$ were not observed because of some inaccuracy in the evaluation which is noted in [7]. Then, Marques [4] generalized the equation (1.2) one step more and showed that the equation

$$F_nF_{n-1}\dots F_1+1=F_m^t \tag{1.3}$$

has at most finitely many solutions in positive integers n, m, where t is previously fixed. Moreover, it is proven that there is no solution of the equation (1.3) in the same paper. Afterwards, Szalay [7] generalized the diophantine equation (1.2) as

$$G_{n_1}G_{n_2}\dots G_{n_k}+1=G_m^2$$

where the binary recurrence $\{G_n\}$ is the Fibonacci sequence, the Lucas sequence, the sequence of Balancing numbers, respectively. In [3], Marques focused on the following diophantine equation

$$\begin{bmatrix} m \\ k \end{bmatrix}_F \pm 1 = F_n \tag{1.4}$$

and proved that there is no solution of the equation (1.4) without $(m, k, n) = (3, 2, 4)$ and $(m, k, n) = (3, 2, 1)$, $(3, 2, 2)$ according to sign $+$ and $-$, respectively.

In this paper, we prove the following theorem.

Theorem 1. *Let m and n are positive integers. Then the solutions of the Diophantine equation*

$$\sum_{k=0}^{m} \begin{bmatrix} 2m+1 \\ k \end{bmatrix}_F \pm 1 = F_n \tag{1.5}$$

are $(m, n) = (1, 3)$, $(3, 14)$ according to the sign $-$. If the sign is $+$, then there is no solution.

2. THE PROOF OF THE THEOREM

2.1. AUXILIARY RESULTS

Before proceeding further, some results will be needed in order to prove Theorem 1. The sequence of the Lucas numbers is given by the following recurrence

$$L_{n+2}=L_{n+1}+L_n$$

with the initial conditions $L_0 = 2$ and $L_1 = 1$ for $n \geq 2$.

The another important item for this paper is about the half sum of Fibonomial coefficients. Kılıç, Akkuş and Ohtsuka [6] showed that

$$\sum_{k=0}^{m} \begin{bmatrix} 2m+1 \\ k \end{bmatrix}_F = \prod_{k=1}^{m} L_{2k} \tag{2.1}$$

for positive integer m.

For all $n \geq 1$, we have

(1)

$$F_{2n} = L_n F_n \tag{2.2}$$

(2) (**Binet formulae**)

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

and $L_n = \alpha^n + \beta^n$, where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

(3) $\alpha^{n-2} \leq F_n \leq \alpha^{n-1}$ where $\alpha = \frac{1+\sqrt{5}}{2}$.

(4) (**Primitive Divisor Theorem**) A primitive divisor p of F_n is a prime factor of F_n which does not divide $\prod_{j=1}^{n-1} F_j$. It is known that a primitive divisor p of F_n exists whenever $n \geq 13$ (for more details, see [2]).

(5) The factorization of the $F_n \pm 1$ depends on the class of n modul 4, namely, the identities for the case sign $+$

$$\begin{aligned} F_{4l} + 1 &= F_{2l-1} L_{2l+1} \\ F_{4l+1} + 1 &= F_{2l+1} L_{2l} \\ F_{4l+2} + 1 &= F_{2l+2} L_{2l} \\ F_{4l+3} + 1 &= F_{2l+1} L_{2l+2} \end{aligned}$$

hold. Similarly, the identities

$$\begin{aligned} F_{4l} - 1 &= F_{2l+1} L_{2l-1} \\ F_{4l+1} - 1 &= F_{2l} L_{2l+1} \\ F_{4l+2} - 1 &= F_{2l} L_{2l+2} \\ F_{4l+3} - 1 &= F_{2l+2} L_{2l+1} \end{aligned}$$

hold for the case $-$. The above identities can be proven by using Binet formulas for Fibonacci and Lucas numbers.

Now, we are ready to prove Theorem 1.

2.2. PROOF

We focus on the diophantine equation (2.3) with the case $-$. Let $m \geq 4$. Using identity 5, we have only four possibilities for this diophantine equation.

$$\sum_{k=0}^{m} \begin{bmatrix} 2m+1 \\ k \end{bmatrix}_F = F_{2l-1} L_{2l+1}$$

$$\begin{aligned}
\sum_{k=0}^{m} \begin{bmatrix} 2m+1 \\ k \end{bmatrix}_F &= F_{2l+1}L_{2l} \\
\sum_{k=0}^{m} \begin{bmatrix} 2m+1 \\ k \end{bmatrix}_F &= F_{2l+2}L_{2l} \\
\sum_{k=0}^{m} \begin{bmatrix} 2m+1 \\ k \end{bmatrix}_F &= F_{2l+1}L_{2l+2}
\end{aligned}$$

We shall work only with the first equation. The equation (2.1) yields that

$$L_2L_4 \dots L_{2m} = F_{2l-1}L_{2l+1}.$$

together with the equation (2.1). When we multiply both sides of the equation above with $F_2F_4 \dots F_{2m}$ and F_{2l+1}, we obtain

$$F_4F_8 \dots F_{4m}F_{2l+1} = F_{2l-1}F_{4l+2}F_2F_4 \dots F_{2m} \tag{2.3}$$

by identity (2.2).

If m is even integer, we get the followings after simplifying the common terms.

$$F_{2l-1}F_{4l+2}F_2F_6F_{10} \dots F_{2m-2} = F_{2m+4}F_{2m+8} \dots F_{4m}F_{2l+1} \tag{2.4}$$

Since $4m \geq 16$, the Primitive Divisor Theorem yields $4m = 4l+2$, which gives the equations $2l-1 = 2m-2$ and $2l+1 = 2m$. So, the equation (2.4) turns to

$$F_2F_6F_{10} \dots F_{2m-6}F_{2m-2}^2 = F_{2m}F_{2m+4}F_{2m+8} \dots F_{4m-4} \tag{2.5}$$

which is impossible since the left hand side of the equation (2.5) is greater than the right hand side for $m \geq 4$.

If m is odd integer, we have

$$F_{2l-1}F_{4l+2}F_2F_6F_{10} \dots F_{2m-2} = F_{2m+2}F_{2m+6} \dots F_{4m}F_{2l+1}$$

after simplifying the equation (2.3). The Primitive Divisor Theorem yields that $4m = 4l-2$. So, we have

$$F_2F_6F_{10} \dots F_{2m-2}^2 = F_{2m}F_{2m+2}F_{2m+6} \dots F_{4m-4}. \tag{2.6}$$

It is obvious that the equation (2.6) does not hold for $m \geq 4$.

Therefore, we only need to consider the case $1 \leq m \leq 3$. When we put $m = 1, 2$ and 3 in equation (2.3) for the sign $-$, we obtain that

$$\begin{aligned}
\sum_{k=0}^{1} \begin{bmatrix} 3 \\ k \end{bmatrix}_F - 1 &= 2 = F_3 \\
\sum_{k=0}^{2} \begin{bmatrix} 5 \\ k \end{bmatrix}_F - 1 &= 20
\end{aligned}$$

$$\sum_{k=0}^{3} \begin{bmatrix} 7 \\ k \end{bmatrix}_F - 1 \quad = \quad 377 = F_{14}$$

which means that the solutions of the equation are $(m, n) = (1, 3)$ and $(3, 14)$. In order to prove the remaining cases, we can follow the above similar ways . To cut the unnecessary repetitions, we do not give them.

Hence, we prove Theorem 1.

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