A VARIATIONAL APPROACH TO PERTURBED ELASTIC BEAM PROBLEMS WITH NONLINEAR BOUNDARY CONDITIONS

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The existence of infinitely many generalized solutions for perturbed fourth-order nonlinear boundary value problems with nonlinear boundary conditions is investigated. The approach is based on variational methods and critical point theory.

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1. INTRODUCTION

The aim of this paper is to ensure the existence of infinitely many generalized solutions for the following perturbed problem

$$\begin{cases} u^{(4)}(t) = \lambda f(t, u(t)) + \mu g(t, u(t)) + p(u(t)), & 0 < t < 1, \\ u(0) = u'(0) = 0, \\ u''(1) = 0, u'''(1) = h(u(1)) \end{cases} \tag{1.1}$$

where $\lambda > 0, \mu \geq 0$ are two parameters, f, g are two L^2-Carathéodory functions, and $p, h : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous functions with the Lipschitz constants $L_1 > 0$ and $L_2 > 0$, respectively, *i.e.*

$$|p(\xi_1) - p(\xi_2)| \leq L_1|\xi_1 - \xi_2| \quad \text{and} \quad |h(\xi_1) - h(\xi_2)| \leq L_2|\xi_1 - \xi_2|$$

for every $\xi_1, \xi_2 \in \mathbb{R}$ such that $p(0) = h(0) = 0$.

This kind of problems arises in the study of deflections of elastic beams on nonlinear elastic foundations. The problem (1.1) has the following physical descriptions: a thin flexible elastic beam of length 1 is clamped at its left end $t = 0$ and resting on an elastic device at its right end $t = 1$, which is given by h. Then the problem models the static equilibrium of the beam under a load, along its length, characterized by f, g, p and h. The derivation of the model can be found in [3, 20].

In recent years, fourth-order boundary value problems modeling bending equilibria of elastic beams have been extensively studied by many researchers. We refer the reader to [1–8, 11, 13, 14, 17–20, 22–24]. For example, in [4] the authors using a fixed-point theorem and degree theory discussed the existence of one or two positive solutions for nonlinear fourth-order beam equations. Moreover, under the assumption that the nonlinear term is monotone increasing respect to the second variable, they got the uniqueness result and the result of the existence of infinitely many positive solutions for the problem. In [17] Li based on the fixed point index theory in cones discussed the existence of positive solutions for fourth-order periodic boundary value problems. In [20], Ma using variational methods and a maximum principle for fourth-order equations discussed the existence of positive solutions for the problem (1.1), in the case $\lambda = 1$, $\mu = 0$ and $p \equiv 0$. In [18], by using monotone operator theory and critical point theory, Li *et al.* established some sufficient conditions for the nonlinear term to guarantee that a class of fourth-order boundary value problems has an unique solution, at least one nonzero solution, or infinitely many solutions. In a later paper [13], employing the critical point theory and the subsolution and supersolution method studied some fourth-order boundary value problems, and obtained several new existence theorems on multiple positive, negative and sign-changing solutions for the problems. In [14] Han and Xu employing the Morse theory obtained some existence theorems on three solutions and infinitely many solutions for a fourth-order beam equation. Bonanno and Di Bella in [7] using an infinitely many critical points theorem, without symmetric condition on the nonlinear term established existence results of infinitely many solutions for a fourth-order nonlinear boundary value problem. In [8] the authors employing a local minimum theorem for differentiable functionals investigated the existence of at least one non-trivial solution to a boundary value problem for fourth order elastic beam equations, under a non-standard growth condition of the nonlinear term. In [2, 6] based on variational methods and critical point theory the existence of at least three solutions for fourth-order elastic beam equations was discussed. Yang *et al.* in [23] by using variational methods and a three-critical-point theorem, established sufficient conditions under which the problem (1.1), in the case $\mu = 0$ and $p \equiv 0$ possesses two solutions generated from the boundary condition h. Recently, Gao in [11] based on variational methods and critical point theory established some existence results of three solutions for the problem (1.1), in the case $p \equiv 0$. Song in [22] by using the smooth version of [10, Theorem 2.1] established infinitely many solutions for the problem (1.1), in the case $p \equiv 0$. Very recently, in [5] the authors based on variational methods studied the existence of nonzero solutions for a fourth-order differential equation with nonlinear boundary conditions. In [1] based on recent variational methods for smooth functionals defined on reflexive Banach spaces, the existence of three distinct generalized solutions for the problem (1.1) under suitable assumptions on the nonlinear terms was established.

Motivated by the above works, in the present paper, by employing a smooth version of [10, Theorem 2.1], which is a more precise version of Ricceri's Variational Principle [21, Theorem 2.5] under some hypotheses on the behavior of the nonlinear terms at infinity, under conditions on f and g we prove the existence of a definite interval about λ and μ in which the problem (1.1) admits a sequence of solutions which is unbounded in the space E which will be introduced later (Theorem 3.1). Furthermore, some consequences of Theorem 3.1 are listed. Replacing the conditions at infinity of the nonlinear terms, by a similar one at zero, we obtain a sequence of solutions strongly converging to zero; see Theorem 3.4. At the end, two examples of applications are pointed out (see Examples 3.1 and 3.2).

2. PRELIMINARIES

Our main tool to ensure the existence of infinitely many solutions for the problem (1.1) is a smooth version of Theorem 2.1 of [10] which is a more precise version of Ricceri's Variational Principle [21] that we now recall here.

THEOREM 2.1. *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous, and coercive and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let us put*

$$\varphi(r) := \inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\sup_{v\in\Phi^{-1}(]-\infty,r[)} \Psi(v) - \Psi(u)}{r - \Phi(u)}$$

and

$$\gamma := \liminf_{r\to+\infty} \varphi(r), \quad \delta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

Then, one has

(a) for every $r > \inf_X \Phi$ and every $\lambda \in]0, \frac{1}{\varphi(r)}[$, the restriction of the functional $I_\lambda = \Phi - \lambda\Psi$ to $\Phi^{-1}(]-\infty, r[)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.

(b) If $\gamma < +\infty$ then, for each $\lambda \in]0, \frac{1}{\gamma}[$, the following alternative holds: either

(b_1) I_λ possesses a global minimum,

or

(b_2) *there is a sequence* $\{u_n\}$ *of critical points (local minima) of* I_λ *such that*

$$\lim_{n\to+\infty} \Phi(u_n) = +\infty.$$

(c) *If* $\delta < +\infty$ *then, for each* $\lambda \in]0, \frac{1}{\delta}[$, *the following alternative holds: either*

(c_1) *there is a global minimum of* Φ *which is a local minimum of* I_λ, *or*

(c_2) *there is a sequence of pairwise distinct critical points (local minima) of* I_λ *which weakly converges to a global minimum of* Φ.

Denote

$$\mathrm{E} = \left\{u \in \mathrm{H}^2(0,1);\;\; u(0) = u'(0) = 0\right\},$$

where $\mathrm{H}^2(0,1)$ is the Sobolev space of all functions $u : [0,1] \to \mathbb{R}$ such that u and its distributional derivative u' are absolutely continuous and u'' belongs to $\mathrm{L}^2(0,1)$. Then E is a Hilbert space equipped with the following inner product

$$\langle u, v\rangle = \int_0^1 u''(t)v''(t)\mathrm{d}t, \quad \text{for all}\, u, v \in \mathrm{E},$$

and its corresponding norm is defined by

$$\|u\| = \|u''\|_2 = \left(\int_0^1 (|u''(t)|^2 \mathrm{d}t\right)^{\frac{1}{2}} \quad \text{for all}\, u \in \mathrm{E}.$$

Obviously, E is a separable and uniformly convex Banach space. In addition, $(\mathrm{E}, \|\cdot\|)$ is compactly embedded in the space $\mathrm{C}([0,1])$, therefore, there exists a constant S such that

$$\|u\|_\infty \le S\|u\|, \tag{2.1}$$

where $\|u\|_\infty = \max_{t\in[0,1]} |u(t)|$ for all $u \in \mathrm{E}$. Now, we put

$$\begin{aligned} C_1 &:= \tfrac{1}{2}(1 - L_1S^2 - L_2S^2),\\ C_2 &:= \tfrac{1}{2}(1 + L_1S^2 + L_2S^2). \end{aligned} \tag{2.2}$$

To state our results concisely we introduce the following assumption:

$(A1)$ there exist two constants $\bar{t}$ and $\tilde{t}$ with $0 < \bar{t} < \tilde{t} < 1$ such that there exist two functions $d \in \mathrm{C}^2([0,\bar{t}])$ and $e \in \mathrm{C}^2([\tilde{t}, 1])$ satisfying

$$d(0) = d'(0) = 0, \qquad d(\bar{t}) = e(\tilde{t}) = 1, \qquad d'(\bar{t}) = e'(\tilde{t}) = 0 \tag{2.3}$$

and $D := \sqrt{\int_0^{\bar{t}} |d''(t)|^2 \mathrm{d}t + \int_{\tilde{t}}^1 |e''(t)|^2 \mathrm{d}t} \neq 0$.

Corresponding to the functions f and p we introduce the functions

$$F(t,x)=\int_0^x f(t,\xi)\mathrm{d}\xi \quad \text{and} \quad P(x)=\int_0^x p(\xi)\mathrm{d}\xi$$

for all $t\in[0,1]$ and $x\in\mathbb{R}$.

Let us recall that a weak solution of the problem (1.1) is a function $u\in \mathrm{E}$ if

$$\int_0^1 (u''(t)v''(t)-p(u(t))v(t))\mathrm{d}t+h(u(1))v(1)-\lambda\int_0^1 f(t,u(t))v(t)\mathrm{d}t \\ -\mu\int_0^1 g(t,u(t))v(t)\mathrm{d}t=0 \tag{2.4}$$

for every $v\in \mathrm{E}$.

A function $u:[0,1]\to\mathbb{R}$ is a generalized solution to the problem (1.1) if $u\in C^3([0,1])$, $u'''\in \mathrm{AC}([0,1])$, $u(0)=u'(0)=u''(1)=0$, $u'''(1)=h(u(1))$, and $u^{(4)}(t)=\lambda f(t,u(t))+\mu g(t,u(t))+p(u(t))$ for almost every $t\in(0,1)$. If f and g are continuous in $[0,1]\times\mathbb{R}$, therefore each generalized solution u is a classical solution.

Standard methods (see [6, Proposition 2.2]) show that a weak solution to (1.1) is a generalized one when f,g are L^2-Carathéodory functions.

We suppose that $S^2(L_1+L_2)<1$.

A special case of our main result is the following theorem.

Theorem 2.2. *Suppose that (A1) holds. Let $f:\mathbb{R}\longrightarrow\mathbb{R}$ is a nonnegative function and let $F(x)=\int_0^x f(\xi)d\xi$ for all $x\in\mathbb{R}$. Assume that*

$$\liminf_{\xi\longrightarrow+\infty}\frac{F(\xi)}{\xi^2}=0 \quad \textit{and} \quad \limsup_{\xi\longrightarrow+\infty}\frac{F(\xi)}{\xi^2}=+\infty.$$

Then, for every continuous function $g:\mathbb{R}\longrightarrow\mathbb{R}$ whose $G(x)=\int_0^x g(\xi)d\xi$ for every $x\in\mathbb{R}$, is a nonnegative function satisfying the condition

$$g_\star:=\lim_{\xi\longrightarrow\infty}\frac{\sup_{|x|\le\xi}G(x)}{\frac{C_1\xi^2}{S^2}}<+\infty \tag{2.5}$$

and for every $\mu\in[0,\mu_{\star,\lambda}[$ where $\mu_{\star,\lambda}:=\frac{1}{g_\star}\left(1-\lambda S^2\liminf_{\xi\to+\infty}\frac{F(\xi)}{C_1\xi^2}\right)$, the problem

$$\begin{cases} u^{(4)}(t)=f(u(t))+\mu g(u(t))+p(u(t)), & 0<t<1,\\ u(0)=u'(0)=0, & \\ u''(1)=0, u'''(1)=h(u(1)) & \end{cases}$$

has an unbounded sequence of generalized solutions.

3. MAIN RESULTS

We present our main result as follows.

THEOREM 3.1. *Suppose that* (A1) *holds. Assume that*

(A2) $F(t,x) \geq 0$ *for all* $t \in [0,\bar{t}] \cup [\bar{t},\tilde{t}]$ *and* $x \in \mathbb{R}$;

(A3) $\liminf_{\xi \longrightarrow +\infty} \frac{\int_0^1 \sup_{|x|\leq\xi} F(t,x)\mathrm{d}t}{\xi^2} < \frac{C_1}{D^2S^2C_2} \limsup_{\xi\longrightarrow+\infty} \frac{\int_{\bar{t}}^{\tilde{t}} F(t,\xi)\mathrm{d}t}{\xi^2}$.

Then, for each $\lambda \in]\lambda_1,\lambda_2[$ *where*

$$\lambda_1 := \frac{1}{\limsup_{\xi\longrightarrow+\infty} \frac{\int_{\bar{t}}^{\tilde{t}} F(t,\xi)\mathrm{d}t}{D^2C_2\xi^2}} \quad \text{and} \quad \lambda_2 := \frac{1}{\liminf_{\xi\longrightarrow+\infty} \frac{\int_0^1 \sup_{|x|\leq\xi} F(t,x)\mathrm{d}t}{\frac{C_1\xi^2}{S^2}}},$$

for every L^2*-Carathéodory function* $g : [0,1]\times\mathbb{R} \longrightarrow \mathbb{R}$ *whose* $G(t,x) = \int_0^x g(t,\xi)d\xi$ *for every* $(t,x) \in [0,1]\times\mathbb{R}$, *is a nonnegative function satisfying the condition*

$$g_\infty := \lim_{\xi\longrightarrow\infty} \frac{\int_0^1 \sup_{|x|\leq\xi} G(t,x)\mathrm{d}t}{\frac{C_1\xi^2}{S^2}} < \infty \tag{3.1}$$

and for every $\mu \in [0,\mu_{g,\lambda}[$ *where*

$$\mu_{g,\lambda} := \frac{1}{g_\infty}\Big(1 - \lambda S^2 \liminf_{\xi\to+\infty} \frac{\int_0^1 \sup_{|x|\leq\xi} F(t,x)\mathrm{d}t}{C_1\xi^2}\Big),$$

the problem (1.1) *has an unbounded sequence of generalized solutions.*

Proof. Fix $\overline{\lambda} \in]\lambda_1,\lambda_2[$ and let g be a function satisfying the condition (3.1). Since, $\overline{\lambda} < \lambda_2$, one has $\mu_{g,\overline{\lambda}} > 0$. Fix $\overline{\mu} \in [0,\mu_{g,\overline{\lambda}}[$ and put $\nu_1 := \lambda_1$ and $\nu_2 := \frac{\lambda_2}{1+\frac{\overline{\mu}}{\overline{\lambda}}\lambda_2 g_\infty}$. If $g_\infty = 0$, clearly, $\nu_1 = \lambda_1$, $\nu_2 = \lambda_2$ and $\overline{\lambda} \in]\nu_1,\nu_2[$. If $g_\infty \neq 0$, since $\overline{\mu} < \mu_{g,\overline{\lambda}}$, we obtain $\frac{\overline{\lambda}}{\lambda_2} + \overline{\mu}g_\infty < 1$, and so $\frac{\lambda_2}{1+\frac{\overline{\mu}}{\overline{\lambda}}\lambda_2 g_\infty} > \overline{\lambda}$, namely, $\overline{\lambda} < \nu_2$. Hence, since $\overline{\lambda} > \lambda_1 = \nu_1$, one has $\overline{\lambda} \in]\nu_1,\nu_2[$. Now, set $Q(t,x) = F(t,x) + \frac{\overline{\mu}}{\overline{\lambda}}G(t,x)$ for all $(t,x) \in [0,1]\times\mathbb{R}$. Take $X = \mathrm{E}$ and define in X two functionals Φ and Ψ by setting, for each $u \in X$, as follows

$$\Phi(u) = \frac{1}{2}\|u\|^2 + \int_0^{u(1)} h(s)\mathrm{d}s - \int_0^1 P(u(t))\mathrm{d}t = \phi_1(u) + \phi_2(u) - \phi_3(u)$$

and

$$\Psi(u) = \int_0^1 F(t,u(t))\mathrm{d}t + \frac{\overline{\mu}}{\overline{\lambda}}\int_0^1 G(t,u(t))\mathrm{d}t$$

where

$$\phi_1(u) = \frac{1}{2}\|u\|^2, \qquad \phi_2(u) = \int_0^{u(1)} h(s)\mathrm{d}s \quad \text{and} \quad \phi_3(u) = \int_0^1 P(u(t))\mathrm{d}t.$$

It is well known that Ψ is a Gâteaux differentiable functional and sequentially weakly upper semi-continuous whose Gâteaux derivative at the point $u \in X$ is the functional $\Psi'(u) \in X^*$, given by

$$\Psi'(u)v = \int_0^1 f(t, u(t))v(t)\mathrm{d}t + \frac{\overline{\mu}}{\overline{\lambda}} \int_0^1 g(t, u(t))v(t)\mathrm{d}t$$

for every $v \in X$, and $\Psi' : X \to X^*$ is a compact operator. Moreover, Φ is a Gâteaux differentiable functional which Gâteaux derivative at the point $u \in X$ is the functional $\Phi'(u) \in X^*$, given by

$$\Phi'(u)v = \int_0^1 u''(t)v''(t)\mathrm{d}t - \int_0^1 p(u(t))v(t)\mathrm{d}t + h(u(1))v(1)$$

for every $v \in X$. Furthermore, Φ is sequentially weakly lower semi-continuous. Indeed, obviously ϕ_1 is weakly lower semi-continuous in X. Therefore, by continuity of P it suffices to show that ϕ_2 is sequentially weakly continuous in X. In fact, if $\{u_n\} \subset X$ and $u_n \rightharpoonup u$ in X, $\{u_n\}$ converges uniformly to u on $[0, 1]$. Then, there exists $M > 0$ such that

$$\|u_n\|_\infty \le M, \quad \text{for all}\, n \in \mathbb{N}.$$

Therefore, we have

$$\begin{aligned} |\phi_2(u_n) - \phi_2(u)| &= \Big| \int_0^{u_n(1)} h(t)\mathrm{d}t - \int_0^{u(1)} h(t)\mathrm{d}t \Big| \\ &= \Big| \int_{u(1)}^{u_n(1)} h(t)\mathrm{d}t \Big| \\ &\le \max\{M, u(1)\} L_2 \|u_n - u\|_\infty \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Thus ϕ_2 is sequentially weakly continuous. So, Φ is sequentially weakly lower semi-continuous in X. Now from the facts $-L_1|\xi| \le p(\xi) \le L_1|\xi|$ and $-L_2|\xi| \le h(\xi) \le L_2|\xi|$ for every $\xi \in \mathbb{R}$, and taking (2.1) into account, for every $u \in X$ we have

$$C_1\|u\|^2 \le \Phi(u) \le C_2\|u\|^2. \tag{3.2}$$

Put $I_{\overline{\lambda}} := \Phi - \overline{\lambda}\Psi$. Similar to the proof of Lemma 1 of [11], we observe that the weak solutions of the problem (1.1) are exactly the solutions of the equation $I'_{\overline{\lambda}}(u) = 0$ and they are also generalized solutions. So, our goal is to apply Theorem 2.1 to Φ and Ψ. Now, we wish to prove that $\gamma < +\infty$, where γ is defined in Theorem 2.1. Let $\{\xi_n\}$ be a real sequence such that $\xi_n > 0$ for all $n \in \mathbb{N}$ and $\xi_n \to +\infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{\int_0^1 \sup_{|x|\le\xi_n} Q(t, x)\mathrm{d}t}{\xi_n^2} = \liminf_{\xi\to+\infty} \frac{\int_0^1 \sup_{|x|\le\xi} Q(t, x)\mathrm{d}t}{\xi^2}.$$

Put $r_n = \frac{C_1\xi_n^2}{S^2}$ for all $n \in \mathbb{N}$. Since $\{\xi_n\}$ is a positive sequence, $r_n > 0$ for all $n \in \mathbb{N}$. Now let $u \in \Phi^{-1}(]-\infty, r_n])$, owing to (3.2), we have that

$$C_1\|u\|^2 \leq \Phi(u) \leq r_n. \tag{3.3}$$

Combining (3.3) with (2.1) yields $\|u\|_\infty \leq \xi_n$. Thus

$$\Phi^{-1}(]-\infty, r_n[) \subseteq \{u : \|u\|_\infty \leq \xi_n\}. \tag{3.4}$$

Hence, taking into account that $\Phi(0) = \Psi(0) = 0$, for every n large enough, one has

$$\begin{aligned}
\varphi(r_n) &= \inf_{u\in\Phi^{-1}(]-\infty,r_n[)} \frac{(\sup_{v\in\Phi^{-1}(]-\infty,r_n[)} \Psi(v)) - \Psi(u)}{r_n - \Phi(u)} \\
&\leq \frac{\sup_{v\in\Phi^{-1}(]-\infty,r_n[)} \Psi(v)}{r_n} \\
&\leq \frac{\int_0^1 \sup_{|x|\leq\xi_n} Q(t,x)\mathrm{d}t}{\frac{C_1\xi_n^2}{S^2}} = \frac{\int_0^1 \sup_{|x|\leq\xi_n} F(t,x) + \frac{\overline{\mu}}{\overline{\lambda}}G(t,x)\mathrm{d}t}{\frac{C_1\xi_n^2}{S^2}} \\
&\leq \frac{\int_0^1 \sup_{|x|\leq\xi_n} F(t,x)\mathrm{d}t}{\frac{C_1\xi_n^2}{S^2}} + \frac{\overline{\mu}}{\overline{\lambda}} \frac{\int_0^1 \sup_{|x|\leq\xi_n} G(t,x)\mathrm{d}t}{\frac{C_1\xi_n^2}{S^2}}.
\end{aligned}$$

Moreover, it follows from Assumption $(A3)$ that

$$\liminf_{\xi\longrightarrow+\infty} \frac{\int_0^1 \sup_{|x|\leq\xi} F(t,x)\mathrm{d}t}{\frac{C_1\xi^2}{S^2}} < +\infty,$$

which concludes

$$\lim_{n\to\infty} \frac{\int_0^1 \sup_{|x|\leq\xi_n} F(t,x)\mathrm{d}t}{\frac{C_1\xi_n^2}{S^2}} < +\infty. \tag{3.5}$$

Then, in view of (3.1) and (3.5), we have

$$\lim_{n\to\infty} \frac{\int_0^1 \sup_{|x|\leq\xi_n} F(t,x)\mathrm{d}t}{\frac{C_1\xi_n^2}{S^2}} + \lim_{n\to\infty} \frac{\overline{\mu}}{\overline{\lambda}} \frac{\int_0^1 \sup_{|x|\leq\xi_n} G(t,x)\mathrm{d}t}{\frac{C_1\xi_n^2}{S^2}} < +\infty,$$

which follows

$$\lim_{n\to\infty} \frac{\int_0^1 \sup_{|x|\leq\xi_n} F(t,x) + \frac{\overline{\mu}}{\overline{\lambda}}G(t,x)\mathrm{d}t}{\frac{C_1\xi_n^2}{S^2}} < +\infty.$$

Therefore,

$$\gamma \leq \liminf_{n\to+\infty} \varphi(r_n) \leq \lim_{n\to\infty} \frac{\int_0^1 \sup_{|x|\leq\xi_n} F(t,x) + \frac{\overline{\mu}}{\overline{\lambda}}G(t,x)\mathrm{d}t}{\frac{C_1\xi_n^2}{S^2}} < +\infty. \tag{3.6}$$

Since

$$\frac{\int_0^1 \sup_{|x|\le\xi_n} Q(t,x)\mathrm{d}t}{\frac{C_1\xi_n^2}{S^2}} \le \frac{\int_0^1 \sup_{|x|\le\xi_n} F(t,x)\mathrm{d}t}{\frac{C_1\xi_n^2}{S^2}} + \frac{\overline{\mu}}{\overline{\lambda}}\frac{\int_0^1 \sup_{|x|\le\xi_n} G(t,x)\mathrm{d}t}{\frac{C_1\xi_n^2}{S^2}},$$

taking (3.1) into account, one has

$$\liminf_{\xi\to+\infty} \frac{\int_0^1 \sup_{|x|\le\xi} Q(t,x)\mathrm{d}t}{\frac{C_1\xi^2}{S^2}} \le \liminf_{\xi\to+\infty} \frac{\int_0^1 \sup_{|x|\le\xi} F(t,x)\mathrm{d}t}{\frac{C_1\xi^2}{S^2}} + \frac{\overline{\mu}}{\overline{\lambda}}g_\infty. \tag{3.7}$$

Moreover, since G is nonnegative, we have

$$\limsup_{|\xi|\to+\infty} \frac{\int_{\bar{t}}^{\tilde{t}} Q(t,\xi)\mathrm{d}t}{D^2C_2\xi^2} \ge \limsup_{|\xi|\to+\infty} \frac{\int_{\bar{t}}^{\tilde{t}} F(t,\xi)\mathrm{d}t}{D^2C_2\xi^2}. \tag{3.8}$$

Therefore, from (3.7) and (3.8), and from Assumption $(A3)$ and (3.6) one has

(3.9)

$$\overline{\lambda} \in]\nu_1,\nu_2[\subseteq \left]\frac{1}{\limsup_{|\xi|\longrightarrow+\infty} \frac{\int_{\bar{t}}^{\tilde{t}} Q(t,\xi)dt}{D^2C_2\xi^2}}, \frac{1}{S^2 \liminf_{\xi\longrightarrow+\infty} \frac{\int_0^1 \sup_{|x|\le\xi} Q(t,x)\mathrm{d}t}{C_1\xi^2}}\right[\subseteq \left]0,\frac{1}{\gamma}\right[.$$

For the fixed $\overline{\lambda}$, the inequality (3.6) assures that the condition (b) of Theorem 2.1 can be used and either $I_{\overline{\lambda}}$ has a global minimum or there exists a sequence $\{u_n\}$ of solutions of the problem (1.1) such that $\lim_{n\to\infty}\|u\| = +\infty$.
The other step is to verify that the functional $I_{\overline{\lambda}}$ has no global minimum. Since

$$\frac{1}{\overline{\lambda}} < \limsup_{|\xi|\to+\infty} \frac{\int_{\bar{t}}^{\tilde{t}} F(t,\xi)\mathrm{d}t}{D^2C_2\xi^2},$$

we can consider a real sequence $\{\gamma_n\}$ and a positive constant τ such that $\gamma_n \to +\infty$ as $n \to \infty$ and

$$\frac{1}{\overline{\lambda}} < \tau < \frac{\int_{\bar{t}}^{\tilde{t}} F(t,\gamma_n)\mathrm{d}t}{D^2C_2\gamma_n^2} \tag{3.10}$$

for each $n \in \mathbb{N}$ large enough. Thus, if we consider a sequence $\{w_n\}$ in X defined by setting

$$w_n(t) = \begin{cases} d(t)\gamma_n & \text{if } t\in[0,\bar{t}), \\ \gamma_n & \text{if } t\in[\bar{t},\tilde{t}], \\ e(t)\gamma_n & \text{if } t\in(\tilde{t},1], \end{cases} \tag{3.11}$$

one has

$$||w_n||^2 = \gamma_n^2 \left[\int_0^{\bar{t}} |d''(t)|^2 \mathrm{d}t + \int_{\tilde{t}}^1 |e''(t)|^2 \mathrm{d}t \right] = D^2\gamma_n^2,$$

and in particular, taking (3.2) into account, it follows

$$\Phi(w_n) \le D^2 C_2 \gamma_n^2. \tag{3.12}$$

On the other hand, since G is nonnegative, we observe

$$\Psi(w_n) \ge \int_{\bar{t}}^{\tilde{t}} F(t, \gamma_n)\mathrm{d}t. \tag{3.13}$$

So, from (3.10), (3.12) and (3.13) we conclude

$$I_{\overline{\lambda}}(w_n) = \Phi(w_n) - \overline{\lambda}\Psi(w_n) \le D^2 C_2 \gamma_n^2 - \overline{\lambda}(\int_{\bar{t}}^{\tilde{t}} F(t, \gamma_n)\mathrm{d}t) < (1 - \overline{\lambda}\tau) D^2 C_2 \gamma_n^2,$$

for every $n \in \mathbb{N}$ large enough. Hence, the functional $I_{\overline{\lambda}}$ is unbounded from below, and it follows that $I_{\overline{\lambda}}$ has no global minimum. Therefore, Theorem 2.1 assures that there is a sequence $\{u_n\} \subset X$ of critical points of $I_{\overline{\lambda}}$ such that $\lim_{n\to\infty} \Phi(u_n) = +\infty$, which from (3.2) it follows that $\lim_{n\to\infty} \|u_n\| = +\infty$. Hence, we have the conclusion. $\square$

Remark 3.1. Under the conditions

$$\liminf_{\xi \longrightarrow +\infty} \frac{\int_0^1 \sup_{|x|\le\xi} F(t,x)\mathrm{d}t}{\xi^2} = 0 \quad \text{and} \quad \limsup_{\xi \longrightarrow +\infty} \frac{\int_{\bar{t}}^{\tilde{t}} F(t,\xi)\mathrm{d}t}{\xi^2} = \infty,$$

Theorem 3.1 assures that for every $\lambda > 0$ and for each $\mu \in [0, \frac{1}{g_\infty}[$ the problem (1.1) admits infinitely many generalized solutions. Moreover, if $g_\infty = 0$, the result holds for every $\lambda > 0$ and $\mu \ge 0$.

Now, we give an application of Theorem 3.1.

Example 3.1. Let $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by

$$f(t,x) = \begin{cases} f^*(t)xe^x(2+x-\cos(\ln(|x|))-(2+x)\sin(\ln(|x|))) & \text{if } (t,x)\in[0,1]\times(\mathbb{R}\setminus\{0\}), \\ 0 & \text{if } (t,x)\in[0,1]\times\{0\}, \end{cases}$$

where $f^* : [0,1] \longrightarrow \mathbb{R}$ is a non-negative continuous function, and let $p(x) = \frac{1}{2S^2}\arctan x$, $h(x) = \frac{1}{3S^2}\sin x$ for each $x \in \mathbb{R}$. A direct calculation shows

$$F(t,x) = \begin{cases} f^*(t)x^2e^x(1-\sin(\ln(|x|))) & \text{if } (t,x) \in [0,1] \times (\mathbb{R}\setminus\{0\}), \\ 0 & \text{if } (t,x) \in [0,1] \times \{0\}. \end{cases}$$

Now by $\bar{t} = \frac{1}{4}$, $\tilde{t} = \frac{3}{4}$, $d(t) = 48t^2 - 128t^3$ and $e(t) = \frac{8}{3}t - \frac{16}{9}t^2$ we have $D = \frac{2^5\sqrt{61}}{9}$, and since $S, C_1, C_2, < \infty$, we have

$$\liminf_{\xi \longrightarrow +\infty} \frac{\int_0^1 \sup_{|x|\le\xi} F(t,x)\mathrm{d}t}{\xi^2} = 0 \quad \text{and} \quad \limsup_{\xi \longrightarrow +\infty} \frac{\int_{\frac{1}{4}}^{\frac{3}{4}} F(t,\xi)\mathrm{d}t}{\xi^2} = \infty.$$

Hence, note that $L_1 = \frac{1}{2S^2}$ and $L_2 = \frac{1}{3S^2}$ using Theorem 3.1, the problem (1.1) in this case with $g(t,x) = e^{t-x^+}(x^+)^\omega$ for all $(t,x) \in [0,1] \times \mathbb{R}$ where $x^+ = \max\{x, 0\}$ and ω is a positive real number, for every $(\lambda, \mu) \in]0, +\infty[\times[0, +\infty[$ has an unbounded sequence of generalized solutions.

Remark 3.2. Assumption $(A3)$ in Theorem 3.1 could be replaced by the following more general condition

$(A'3)$ there exist two sequence $\{\theta_n\}$ and $\{\eta_n\}$ with $\eta_n > 0$ for every $n \in \mathbb{N}$ and $D^2C_2\theta_n^2 < \frac{C_1}{S^2}\eta_n^2$ for all $n \in \mathbb{N}$ and $\lim_{n\to+\infty} \eta_n = +\infty$ such that

$$\lim_{n\to+\infty} \frac{\int_0^1 \sup_{|x|\le\eta_n} F(t,x)\mathrm{d}t - \int_{\bar{t}}^{\tilde{t}} F(t,\theta_n)\mathrm{d}t}{\frac{C_1}{S^2}\eta_n^2 - D^2C_2\theta_n^2} < \limsup_{\xi \longrightarrow +\infty} \frac{\int_{\bar{t}}^{\tilde{t}} F(t,\xi)\mathrm{d}t}{D^2C_2\xi^2}.$$

Indeed, clearly, by choosing $\theta_n = 0$ for all $n \in \mathbb{N}$ from $(A'3)$ we obtain $(A3)$. Moreover, if we assume $(A'3)$ instead of $(A3)$ and we choose $r_n = \frac{C_1\eta_n^2}{S^2}$ for all $n \in \mathbb{N}$, by the same arguments as in Theorem 3.1, we obtain

$$\begin{aligned} \varphi(r_n) &\le \frac{\sup_{v\in\Phi^{-1}(]-\infty,r_n])} \Psi(v) - \int_0^1 F(t, w_n(t))\mathrm{d}t}{r_n - \int_0^1 \frac{1}{2}|w_n'(t)|^2\mathrm{d}t - \int_0^{w_n(1)} h(t)\mathrm{d}t + \int_0^1 P(w_n(t))\mathrm{d}t} \\ &\le \frac{\int_0^1 \sup_{|x|\le\eta_n} F(t,x)\mathrm{d}t - \int_{\bar{t}}^{\tilde{t}} F(t,\theta_n)\mathrm{d}t}{\frac{C_1}{S^2}\eta_n^2 - D^2C_2\theta_n^2} \end{aligned}$$

where $w_n(t)$ is the same as (3.11) but γ_n replaced by θ_n. We have the same conclusion as in Theorem 3.1 with the interval $]\lambda_1, \lambda_2[$ replaced by the interval

$$\Lambda' = \left] \frac{1}{\limsup_{\xi\longrightarrow+\infty} \frac{\int_{\bar{t}}^{\tilde{t}} F(t,\xi)\mathrm{d}t}{D^2C_2\xi^2}}, \frac{1}{S^2 \lim_{n\to+\infty} \frac{\int_0^1 \sup_{|x|\le\eta_n} F(t,x)\mathrm{d}t - \int_{\bar{t}}^{\tilde{t}} F(t,\theta_n)\mathrm{d}t}{C_1\eta_n^2 - D^2S^2C_2\theta_n^2}} \right[.$$

Here, we point out a simple consequence of Theorem 3.1.

Corollary 3.2. *Assume that Assumptions* $(A1)$ *and* $(A2)$ *hold. Furthermore, suppose that*

$(B1)$ $\liminf_{\xi\longrightarrow+\infty} \frac{\int_0^1 \sup_{|x|\le\xi} F(t,x)\mathrm{d}t}{\xi^2} < \frac{C_1}{S^2}$

$(B2)$ $\limsup_{\xi\longrightarrow+\infty} \frac{\int_{\bar{t}}^{\tilde{t}} F(t,\xi)\mathrm{d}t}{\xi^2} > D^2C_2$.

Then, for every arbitrary L^2-Carathéodory function $g : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ whose $G(t,x) = \int_0^x g(t,\xi)\mathrm{d}\xi$ for every $(t,x) \in [0,1] \times \mathbb{R}$ is a nonnegative function satisfying the condition (3.1) *and for every $\mu \in [0, \mu_{g,1}[$ where $\mu_{g,1} := \frac{1}{g_\infty}\Big(1 - S^2 \liminf_{\xi\to+\infty} \frac{\int_0^1 \sup_{|x|\leq\xi} F(t,x)\mathrm{d}t}{C_1\xi^2}\Big)$, the problem*

$$\begin{cases} u^{(4)}(t) = f(t,u(t)) + \mu g(t,u(t)) + p(u(t)), & 0 < t < 1, \\ u(0) = u'(0) = 0, \\ u''(1) = 0, u'''(1) = h(u(1)), \end{cases}$$

has an unbounded sequence of generalized solutions.

Remark 3.3. Theorem 2.2 is an immediate consequence of Corollary 3.2 when $\mu = 0$.

We here give the following consequence of the main result.

COROLLARY 3.3. *Let $f_1 : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ be an* L^2*-Carathéodory function and let $F_1(t,x) = \int_0^x f_1(t,\xi)\mathrm{d}\xi$ for all $x \in \mathbb{R}$. Assume that*

(D1) $\liminf_{\xi\longrightarrow+\infty} \frac{\int_0^1 \sup_{|x|\leq\xi} F_1(t,x)\mathrm{d}t}{\xi^2} < +\infty$;

(D2) $\limsup_{\xi\longrightarrow+\infty} \frac{\int_{\bar{t}}^{\tilde{t}} F_1(t,\xi)\mathrm{d}t}{\xi^2} = +\infty$.

Then, for every L^2*-Carathéodory function $f_i : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$, denoting $F_i(t,x) = \int_0^x f_i(t,\xi)\mathrm{d}\xi$ for all $x \in \mathbb{R}$ for $2 \leq i \leq n$, satisfying*

$$\min\left\{ \inf_{(t,\xi)\in([0,\bar{t}]\cup[\tilde{t},1])\times\mathbb{R}} F_i(t,\xi);\ 2 \leq i \leq n \right\} \geq 0$$

and

$$\min\left\{ \liminf_{\xi\to+\infty} \frac{F_i(t,\xi)}{\xi^2};\ 2 \leq i \leq n \right\} < +\infty,$$

for each

$$\lambda \in \left]0,\ \frac{1}{S^2 \liminf_{\xi\to+\infty} \frac{F_1(t,\xi)}{C_1\xi^2}}\right[,$$

for every arbitrary L^2*-Carathéodory function $g : [0,1]\times\mathbb{R} \longrightarrow \mathbb{R}$ whose $G(t,x) = \int_0^x g(t,\xi)\mathrm{d}\xi$ for every $(t,x) \in [0,1] \times \mathbb{R}$, is a non-negative function satisfying the condition* (3.1) *and for every $\mu \in [0, \mu_{g,\lambda}[$ where$\mu_{g,\lambda} := \frac{1}{g_\infty}\Big(1 - \lambda S^2 \liminf_{\xi\to+\infty} \frac{\int_0^1 \sup_{|x|\leq\xi} F_1(t,x)\mathrm{d}t}{C_1\xi^2}\Big)$, the problem*

$$\begin{cases} u^{(4)}(t) = \lambda \sum_{i=1}^n f_i(t,u(t)) + \mu g(t,u(t)) + p(u(t)), & 0 < t < 1, \\ u(0) = u'(0) = 0, \\ u''(1) = 0, u'''(1) = h(u(1)), \end{cases}$$

has an unbounded sequence of generalized solutions.

Proof. Set $F(t,\xi) = \sum_{i=1}^{n} F_i(t,\xi)$ for all $\xi \in \mathbb{R}$. From the assumption $(D2)$ and the condition

$$\min\left\{ \inf_{(t,\xi)\in([0,\bar{t}]\cup[\tilde{t},1])\times\mathbb{R}} F_i(t,\xi);\ 2 \le i \le n \right\} \ge 0$$

we conclude

$$\limsup_{\xi\longrightarrow+\infty} \frac{\int_{\bar{t}}^{\tilde{t}} F(t,\xi)\mathrm{d}t}{D^2 C_2 \xi^2} = \limsup_{\xi\to+\infty} \frac{\sum_{i=1}^{n} \int_{\bar{t}}^{\tilde{t}} F_i(t,\xi)\mathrm{d}t}{D^2 C_2 \xi^2} = +\infty.$$

Moreover, from the assumption $(D1)$ and the condition

$$\min\left\{ \liminf_{\xi\to+\infty} \frac{F_i(t,\xi)}{\xi^2};\ 2 \le i \le n \right\} < +\infty,$$

we obtain

$$\liminf_{\xi\longrightarrow+\infty} \frac{\int_0^1 \sup_{|x|\le\xi} F(t,x)\mathrm{d}t}{\xi^2} \le \liminf_{\xi\longrightarrow+\infty} \frac{\int_0^1 \sup_{|x|\le\xi} F_1(t,x)\mathrm{d}t}{\xi^2} < +\infty.$$

Hence, the conclusion follows from Theorem 3.1. □

Arguing as in the proof of Theorem 3.1, but using conclusion (c) of Theorem 2.1 instead of (b), one establishes the following result.

THEOREM 3.4. *Assume that Assumptions* $(A1)$ *and* $(A2)$ *hold. Furthermore, suppose that*

$(E1)$ $\liminf_{\xi\longrightarrow0^+} \frac{\int_0^1 \sup_{|x|\le\xi} F(t,x)\mathrm{d}t}{\xi^2} < \frac{C_1}{D^2 S^2 C_2} \limsup_{\xi\longrightarrow0^+} \frac{\int_{\bar{t}}^{\tilde{t}} F(t,\xi)\mathrm{d}t}{\xi^2}$.

Then, for each $\lambda \in]\lambda_3, \lambda_4[$ *where*

$$\lambda_3 := \frac{1}{\limsup_{\xi\longrightarrow0^+} \frac{\int_{\bar{t}}^{\tilde{t}} F(t,\xi)\mathrm{d}t}{D^2 C_2 \xi^2}} \quad \text{and} \quad \lambda_4 := \frac{1}{\liminf_{\xi\longrightarrow0^+} \frac{\int_0^1 \sup_{|x|\le\xi} F(t,x)\mathrm{d}t}{\frac{C_1 \xi^2}{S^2}}},$$

for every arbitrary L^2*-Carathéodory function* $g : [0,1]\times\mathbb{R} \longrightarrow \mathbb{R}$ *whose* $G(t,x) = \int_0^x g(t,\xi)\mathrm{d}\xi$ *for every* $(t,x) \in [0,1]\times\mathbb{R}$ *is a nonnegative function satisfying the condition*

$$(3.14)\qquad g_0 := \lim_{\xi\longrightarrow0^+} \frac{\int_0^1 \sup_{|x|\le\xi} G(t,x)\mathrm{d}t}{\frac{C_1 \xi^2}{S}^2} < +\infty$$

and for every $\mu \in [0, \mu_{g,\lambda}[$ *where* $\mu_{g,\lambda} := \frac{1}{g_0}\left(1 - \lambda S^2 \liminf_{\xi\to0^+} \frac{\int_0^1 \sup_{|x|\le\xi} F(t,x)\mathrm{d}t}{C_1 \xi^2}\right)$, *the problem* (1.1) *has a sequence of pairwise distinct generalized solutions, which strongly converges to* 0 *in* E.

Proof. Fix $\overline{\lambda} \in]\lambda_3, \lambda_4[$ and let g is the function satisfying the condition (3.14). Since, $\overline{\lambda} < \lambda_2$, one has $\mu_{g,\overline{\lambda}} > 0$. Fix $\overline{\mu} \in]0, \mu_{g,\overline{\lambda}}[$ and set $\nu_3 := \lambda_3$ and $\nu_4 := \frac{\lambda_4}{1+\frac{\overline{\mu}}{\overline{\lambda}}\lambda_4 g_0}$. If $g_0 = 0$, clearly, $\nu_3 = \lambda_3$, $\nu_4 = \lambda_4$ and $\lambda \in]\nu_3, \nu_4[$. If $g_0 \neq 0$, since $\overline{\mu} < \mu_{g,\overline{\lambda}}$, one has

$$\frac{\overline{\lambda}}{\lambda_4} + \overline{\mu} g_0 < 1,$$

and so

$$\frac{\lambda_4}{1 + \frac{\overline{\mu}}{\overline{\lambda}}\lambda_4 g_0} > \overline{\lambda},$$

namely, $\overline{\lambda} < \nu_4$. Hence, recalling that $\overline{\lambda} > \lambda_3 = \nu_3$, one has $\overline{\lambda} \in]\nu_3, \nu_4[$.
Now, put $Q(t,u) = F(t,u) + \frac{\overline{\mu}}{\overline{\lambda}} G(t,u)$ for all $u \in \mathbb{R}$ and $t \in [0,1]$. Since

$$\frac{\int_0^1 \sup_{|x|\leq\xi} Q(t,x)\mathrm{d}t}{\frac{C_1\xi^2}{S^2}} \leq \frac{\int_0^1 \sup_{|x|\leq\xi} F(t,x)\mathrm{d}t}{\frac{C_1\xi^2}{S^2}} + \frac{\overline{\mu}}{\overline{\lambda}} \frac{\int_0^1 \sup_{|x|\leq\xi} G(t,x)\mathrm{d}t}{\frac{C_1\xi^2}{S^2}},$$

taking (3.14) into account, one has

$$\liminf_{\xi\to 0^+} \frac{\int_0^1 \sup_{|x|\leq\xi} Q(t,x)\mathrm{d}t}{\frac{C_1\xi^2}{S^2}} \leq \liminf_{\xi\to 0^+} \frac{\int_0^1 \sup_{|x|\leq\xi} F(t,x)\mathrm{d}t}{\frac{C_1\xi^2}{S^2}} + \frac{\overline{\mu}}{\overline{\lambda}} g_0. \tag{3.15}$$

Moreover, since G is nonnegative, from Assumption $(E1)$ we have

$$\limsup_{\xi\longrightarrow 0^+} \frac{\int_{\bar{t}}^{\tilde{t}} Q(t,\xi)\mathrm{d}t}{D^2 C_2 \xi^2} \geq \limsup_{\xi\longrightarrow 0^+} \frac{\int_{\bar{t}}^{\tilde{t}} F(t,\xi)\mathrm{d}t}{D^2 C_2 \xi^2}. \tag{3.16}$$

Therefore, from (3.15) and (3.16), we obtain

$$\begin{aligned} &\overline{\lambda} \in]\nu_3, \nu_4[\\ &\subseteq \left] \frac{1}{\limsup_{|\xi|\longrightarrow 0^+} \frac{\int_{\bar{t}}^{\tilde{t}} Q(t,\xi)\mathrm{d}t}{D^2 C_2 \xi^2}}, \frac{1}{S^2 \liminf_{\xi\longrightarrow 0^+} \frac{\int_0^1 \sup_{|x|\leq\xi} Q(t,x)\mathrm{d}t}{C_1\xi^2}} \right[\subseteq]\lambda_3, \lambda_4[. \end{aligned}$$

We take X, Φ, Ψ and $I_{\overline{\lambda}}$ as in the proof of Theorem 3.1. We prove that $\delta < +\infty$. For this, let $\{\xi_n\}$ be a sequence of positive numbers such that $\xi_n \to 0^+$ as $n \to +\infty$ and

$$\lim_{n\to\infty} \frac{\int_0^1 \sup_{|x|\leq\xi_n} F(t,x)\mathrm{d}t}{\xi_n^2} < +\infty.$$

Put $r_n = \frac{C_1\xi_n^2}{S^2}$ for all $n \in \mathbb{N}$. Let us show that the functional $I_{\overline{\lambda}}$ has not a local minimum at zero. For this, let $\{\gamma_n\}$ be a sequence of positive numbers and

$\tau > 0$ such that $\gamma_n \to 0^+$ as $n \to \infty$ and

$$\frac{1}{\overline{\lambda}} < \tau < \frac{\int_{\bar{t}}^{\tilde{t}} F(t,\gamma_n)\mathrm{d}t}{D^2 C_2 \gamma_n^2} \tag{3.17}$$

for each $n \in \mathbb{N}$ large enough. Let $\{w_n\}$ be a sequence in X defined by (3.11). So, owing to (3.12), (3.13) and (3.17) we obtain

$$I_{\overline{\lambda}}(w_n) = \Phi(w_n) - \overline{\lambda}\Psi(w_n) \le D^2 C_2 \gamma_n^2 \overline{\lambda} \int_{\bar{t}}^{\tilde{t}} F(t,\gamma_n)\mathrm{d}t < (1-\overline{\lambda}\tau) D^2 C_2 \gamma_n^2 < 0$$

for every $n \in \mathbb{N}$ large enough. Since $I_{\overline{\lambda}}(0) = 0$, that means that 0 is not a local minimum of the functional $I_{\overline{\lambda}}$. Hence, the part (c) of Theorem 2.1 ensures that there exists a sequence $\{u_n\}$ in X of critical points of $I_{\overline{\lambda}}$ such that $\|u_n\| \to 0$ as $n \to \infty$, and the proof is complete. $\square$

Remark 3.4. Applying Theorem 3.4, results similar to Remark 3.2, Corollaries 3.2 and 3.3 can be obtained.

We end this paper by giving the following example as an application of Theorem 3.4.

Example 3.2. Let $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by

$$f(t,x) = \begin{cases} f^*(t)\big(1 - \cos(\ln(|x|)) - \sin(\ln(|x|))\big) & \text{if } (t,x) \in [0,1] \times (\mathbb{R} \setminus \{0\}), \\ 0 & \text{if } (t,x) \in [0,1] \times \{0\}, \end{cases}$$

where $f^* : [0,1] \longrightarrow \mathbb{R}$ is a non-negative continuous function, and let $p(x) = \ln(1+x^2)^{1/4S^2}$, $h(x) = \frac{1}{5S^2}|x|$ for each $x \in \mathbb{R}$. A direct calculation shows

$$F(t,x) = \begin{cases} f^*(t)x\big(1 - \sin(\ln(|x|))\big) & \text{if } (t,x) \in [0,1] \times (\mathbb{R} \setminus \{0\}), \\ 0 & \text{if } (t,x) \in [0,1] \times \{0\}. \end{cases}$$

Now by $\bar{t} = \frac{8}{10}$, $\tilde{t} = \frac{9}{10}$, $d(t) = \frac{75}{16}t^2 - \frac{375}{96}t^3$ and $e(t) = \frac{29}{9}t - \frac{100}{81}t^2$ we have $D = \frac{25\sqrt{4039}}{324}$, and since $S, C_1, C_2 < \infty$, we have

$$\liminf_{\xi \longrightarrow 0^+} \frac{\int_0^1 \sup_{|x| \le \xi} F(t,x)\mathrm{d}t}{\xi^2} = 0 \quad \text{and} \quad \limsup_{\xi \longrightarrow 0^+} \frac{\int_{\frac{8}{10}}^{\frac{9}{10}} F(t,\xi)\mathrm{d}t}{\xi^2} = +\infty.$$

Hence, note that $L_1 = \frac{1}{2S^2}$ and $L_2 = \frac{2}{5S^2}$ using Theorem 3.4 the problem (1.1) in this case, with $g(t,x) = tx$ for all $(t,x) \in [0,1] \times \mathbb{R}$, for every $(\lambda,\mu) \in]0,+\infty[\times [0, \frac{4C_1}{S^2}[$ has an unbounded sequence of generalized solutions.

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