PSEUDO-PRIME SUBMODULES OF MODULES

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In this paper, we introduce the notion of pseudo-prime submodules of modules as a generalization of the prime ideal of commutative rings. We introduce a Zariski topology on the spectrum of pseudo-prime submodules of certain modules. We investigate this topology and clarify the interplay between the properties of this topological space and the algebraic properties of the module under consideration.

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1. INTRODUCTION

Inspired by the interplay between the Zariski topology defined on the prime spectrum of a commutative ring R and the ring theoretic properties of R in [3, 8, 18, 23, 25], we introduce in this paper a Zariski topology on the pseudo-prime spectrum X_M of pseudo-prime submodules of a certain module M over a commutative ring R and study the interplay between the properties of M and the topological space that we obtain.

We are going to show that the topological conditions on the pseudo-prime spectrum of modules such as connectedness, Noetherianness and irreducibility give more information about the algebraic structure of those modules. For example, we show that if the topology on the pseudo-prime spectrum of a Noetherian module is a T_1-space, then this module must be Artinian. Also, we study this topological space from the point of view of spectral spaces (a topological space which is homeomorphic to Spec(S) for some ring S).

Throughout the paper, all rings are commutative with identity and all modules are unital. For a submodule N of an R-module M, $(N :_R M)$ denotes the ideal $\{r \in R \mid rM \subseteq N\}$ and *annihilator* of M, denoted by $\mathrm{Ann}_R(M)$, is the ideal $(\mathbf{0} :_R M)$. If there is no ambiguity we will write $(N : M)$ (resp. $\mathrm{Ann}(M)$) instead of $(N :_R M)$ (resp. $\mathrm{Ann}_R(M)$).

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2. PSEUDO-PRIME SUBMODULES

Definition 2.1. Let M be an R-module.

(1) A proper submodule N of M is called *pseudo-prime* if $(N :_R M)$ is a prime ideal of R.

(2) We define the *pseudo-prime spectrum* of M to be the set of all pseudo-prime submodules of M and denote it by X_M^R. If there is no ambiguity we write only X_M instead of X_M^R. For any prime ideal $I \in X_R = \text{Spec}(R)$, the collection of all pseudo-prime submodules N of M with $(N : M) = I$ is designated by $X_{M,I}$.

(3) For a submodule N of M we define $V^M(N) = \{L \in X_M \mid L \supseteq N\}$. If there is no ambiguity we write $V(N)$ instead of $V^M(N)$.

(4) When $X_M \neq \emptyset$, the map $\psi : X_M \to \text{Spec}(R/\text{Ann}(M))$ defined by $\psi(L) = (L : M)/\text{Ann}(M)$ for every $L \in X_M$, will be called the *natural map of* X_M. An R-module M is called *pseudo-primeful* if either $M = (\mathbf{0})$ or $M \neq (\mathbf{0})$ and the natural map of X_M is surjective.

(5) M is called *pseudo-injective* if the natural map of X_M is injective.

By our definition, the prime ideals of the ring R and pseudo-prime submodules of the R-module R are the same. This shows that pseudo-prime submodule is a generalization of the notion of prime ideal to the modules.

We recall that a proper submodule N of an R-module M is said to be *prime* if M/N is a torsion-free $R/(N : M)$-module. The theory of prime submodules and Zariski topology on the prime spectrum of modules is studied by many algebraist (see [1, 6, 7, 9, 11, 14, 17, 19, 20]). Every prime submodule P of R-module M is pseudo-prime, because $(P : M) \in \text{Spec}(R)$. However, the converse is not true in general.

Example 2.2. Consider $M = \mathbb{Z} \oplus \mathbb{Z}$ as a $\mathbb{Z}$-module and $N = (2, 0)\mathbb{Z}$ is the submodule of M generated by $(2, 0) \in M$. Then $(N : M) = (0) \in \text{Spec}(\mathbb{Z})$, *i.e.*, $N \in X_M$ though N is not a prime submodule of M. Thus in general, a pseudo-prime submodule need not be a prime submodule, *i.e.*, $\text{Spec}(M) \subsetneqq X_M$, here $\text{Spec}(M)$ is the set of all prime submodules of M.

This example shows that the theory of pseudo-prime submodule and the theory of prime submodule are not the same. Indeed, we can find modules such as the $\mathbb{Z}$-module $\mathbb{Z}(p^\infty)$, where p is a prime integer, that has no prime submodules but every proper submodule of them is a pseudo-prime submodule. We show that the theory of pseudo-prime submodule of modules resembles to that theory of prime ideals of rings.

Example 2.3. Every free R-module F is pseudo-primeful (because for any prime ideal $\mathfrak{p}$ of R, $\mathfrak{p}F$ is a proper submodule of F such that $(\mathfrak{p}F : F) = \mathfrak{p}$).

However, the converse is not true in general. For example, consider $M = \mathbb{Z} \oplus \mathbb{Z}(p^\infty)$ as a $\mathbb{Z}$-module. Then M is pseudo-primeful which is not free.

We remark that pseudo-primeful modules and primeful modules which is introduced in [16] are not the same. More precisely, every primeful module is a pseudo-primeful module. However, the converse is not true in general.

Example 2.4. For example the $\mathbb{Z}$-module $\bigoplus_p(\mathbb{Z}/p\mathbb{Z})$, where p runs over the set of all prime integers is not primeful by [16, Result 2], but it is easy to see that this is pseudo-primeful.

We recall that an R-module M is called a *multiplication* module if every submodule N of M is of the form IM for some ideal I of R (see [4] and [10]).

Example 2.5. Every multiplication module is pseudo-injective. However, the converse is not true in general. For example, consider $L = (\mathbb{Z}/p\mathbb{Z}) \oplus \mathbb{Z}(p^\infty)$ as a $\mathbb{Z}$-module, where p is a prime integer. Let Q be a pseudo-prime submodule of L. Then $(Q : L)L \subseteq Q \neq L$. Since L is a torsion $\mathbb{Z}$-module, if $(Q : L) \neq p$, then $(Q : L)L = L$ which is a contradiction. Therefore, $(Q : L) = p$. Since $L/pL \cong \mathbb{Z}/p\mathbb{Z}$, $(Q : L)L$ is a maximal submodule of L, and so $Q = (Q : L)L$. This implies that L is pseudo-injective. It is easy to check that there does not exist an ideal I of $\mathbb{Z}$ such that $(\mathbb{Z}/p\mathbb{Z}) \oplus (0) = IM$, so that M is not a multiplication module.

We claim that every pseudo-injective finitely generated R-module is multiplication.

Lemma 2.6. *Let M be an R-module and consider the following statements.*

(1) *M is a multiplication module;*
(2) *M is a pseudo-injective module;*
(3) *$|X_{M,\mathfrak{m}}| \leq 1$ for every maximal ideal $\mathfrak{m}$ of R;*
(4) *$M/\mathfrak{m}M$ is cyclic for every maximal ideal $\mathfrak{m}$ of R.*

Then the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) *hold. Moreover, if M is finitely generated then* (4) *implies* (1).

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are clear. (3) $\Rightarrow$ (4) Let $\mathfrak{m}$ be a maximal ideal of R. If $\mathfrak{m}M = M$, then we are done. Suppose that $\mathfrak{m}M \neq M$ and $N/\mathfrak{m}M$ is a proper submodule of $M/\mathfrak{m}M$. Then, $\mathfrak{m} = (\mathfrak{m}M : M) = (N : M)$. Hence, N and $\mathfrak{m}M$ are belong to $X_{M,\mathfrak{m}}$. By (3), we have $N = \mathfrak{m}M$. This implies that $M/\mathfrak{m}M$ is a simple (and so cyclic) R-module. Now, let (4) hold and M is a finitely generated module. Then by [10, Corollary 1.5], M is multiplication. □

In the sequel, we use the notion of pseudo-prime submodules to define another new class of modules, namely *topological module.* We present some examples of topological modules (Theorem 2.10 and Theorem 2.11) and we investigate some algebraic properties of this new class. Afterward in the next section, we associate a topology to the set of all pseudo-prime submodules of topological modules, which is called *Zariski topology.* Let Y be a subset of X_M for an R-module M. We denote the intersection of all elements in Y by $\Im(Y)$.

Definition 2.7. Let M be an R-module.

(1) A submodule N of M is said to be *pseudo-semiprime* if it is an intersection of pseudo-prime submodules.
(2) A pseudo-prime submodule H of M is called *extraordinary* if $N \cap L \subseteq H$, where N and L are pseudo-semiprime submodules of M, then either $L \subseteq H$ or $N \subseteq H$.
(3) For a submodule N of M, the *pseudo-prime radical* of N, denoted by $\mathbb{P}\mathrm{rad}(N)$, is the intersection of all pseudo-prime submodules of M containing N, that is

$$\mathbb{P}\mathrm{rad}(N) = \Im(V(N)) = \bigcap_{P \in V(N)} P.$$

If $V(N) = \emptyset$, then we set $\mathbb{P}\mathrm{rad}(N) = M$.
(4) A submodule N of M is said to be a *pseudo-prime radical submodule* if $N = \mathbb{P}\mathrm{rad}(N)$.
(5) M is said to be *topological* if $X_M = \emptyset$ or every pseudo-prime submodule of M is extraordinary.

Remark 2.8.

(1) Every radical ideal of a ring R is a pseudo-semiprime submodule of the R-module R. For another example, every proper submodule of a co-semisimple module is a pseudo-semiprime submodule (see [2, p. 122]).
(2) Any prime ideal of the ring R is an extraordinary pseudo-prime submodule of the R-module R.
(3) It is not true that every pseudo-prime submodule is extraordinary. For example, consider the $\mathbb{Z}$-module $M = \mathbb{Q} \oplus (\mathbb{Z}/p\mathbb{Z})$, where p is a prime integer. It is easy to see that the submodules $(0) \oplus (\mathbb{Z}/p\mathbb{Z})$, $\mathbb{Q} \oplus (0)$ and $\mathbb{Z} \oplus (0)$ of M are pseudo-prime. We deduce from $((0) \oplus (\mathbb{Z}/p\mathbb{Z})) \cap (\mathbb{Q} \oplus (0)) \subseteq \mathbb{Z} \oplus (0)$ that $\mathbb{Z} \oplus (0)$ is not extraordinary. Therefore, M is *not* a topological module.
(4) The notion of top modules is introduced in [19] and by definition, every topological R-module is a top module. However, the converse is not true

in general. For example, the above mentioned $\mathbb{Z}$-module M is a top module by [19, Example 2.6].

We recall that an R-module is called *uniserial* if its submodules are linearly ordered by inclusion (see [24]). Obviously, any uniserial module is a topological module. Hence, the $\mathbb{Z}$-module $\mathbb{Z}(p^\infty)$, where p is a prime integer is a topological $\mathbb{Z}$-module.

THEOREM 2.9. *Let M be a topological R-module.*

(1) *Any R-homomorphic image of M is a topological R-module.*

(2) *$M_\mathfrak{p}$ is a topological $R_\mathfrak{p}$-module for each prime ideal $\mathfrak{p}$ of R.*

Proof. (1) It suffices to show that $M' := M/N$ is a topological R-module for each submodule N of M. We may assume that $X_{M'}$ is not an empty set. Let H/N be a pseudo-prime submodule of M'. Since $(H/N : M') = (H : M)$, H is a pseudo-prime submodule of M. Let L_1/N and L_2/N be two pseudo-semiprime submodule of M' such that $(L_1/N) \cap (L_2/N) \subseteq (H/N)$. By above discussion, L_1 and L_2 are pseudo-semiprime submodules of M and $L_1 \cap L_2 \subseteq H$. Since M is a topological module, we have $L_1 \subseteq H$ or $L_2 \subseteq H$. This implies that $L_1/N \subseteq H/N$ or $L_2/N \subseteq H/N$. Therefore M' is a topological R-module.

(2) Let H be any pseudo-prime submodule of the $R_\mathfrak{p}$-module $M_\mathfrak{p}$. We claim that $H \cap M$ (the contraction of H with respect to the canonical map $M \to M_\mathfrak{p}$) is a pseudo-prime submodule of M. Let I and J be two ideals of R such that $IJ \subseteq (H \cap M :_R M)$. Hence

$$I_\mathfrak{p} J_\mathfrak{p} M_\mathfrak{p} \subseteq H = (H \cap M)_\mathfrak{p}.$$

Since H is a pseudo-prime submodule of the $R_\mathfrak{p}$-module $M_\mathfrak{p}$, $I_\mathfrak{p} \subseteq (H : M_\mathfrak{p})$ or $J_\mathfrak{p} \subseteq (H : M_\mathfrak{p})$. This shows that

$$IM \subseteq (IM)_\mathfrak{p} \cap M \subseteq H \cap M$$

or

$$JM \subseteq H \cap M.$$

Therefore $H \cap M$ is a pseudo-prime submodule of M. Now let L_1 and L_2, be pseudo-semiprime submodules of $M_\mathfrak{p}$ with $L_1 \cap L_2 \subseteq H$. Then $L_1 \cap M$ and $L_2 \cap M$ are pseudo-semiprime submodules of M with

$$(L_1 \cap M) \cap (L_2 \cap M) = (L_1 \cap L_2) \cap M \subseteq H \cap M.$$

So, $L_1 \cap M \subseteq H \cap M$ or $L_2 \cap M \subseteq H \cap M$. It follows that

$$L_1 = (L_1 \cap M)_\mathfrak{p} \subseteq (H \cap M)_\mathfrak{p} = H$$

or $L_2 \subseteq H$. Thus, H is extraordinary and $M_\mathfrak{p}$ is a topological $R_\mathfrak{p}$-module. □

It is clear that if R is any ring then the pseudo-prime submodules of R (as an R-module) are the pseudo-prime ideals, and hence the R-module R is a topological module. Theorem 2.9 shows that every cyclic R-module is a topological module. In Theorem 2.10, we generalized this fact to multiplication modules (recall that every cyclic module is multiplication, see [4]).

THEOREM 2.10. *Consider the following statements for an R-module M.*

(1) *M is a multiplication module;*

(2) *For every submodule N of M there exists an ideal I of R such that $V(N) = V(IM)$;*

(3) *M is a topological module;*

Then the implications (1) ⇒ (2) ⇒ (3) hold. Moreover, if M is finitely generated then (3) implies (1).

Proof. (1) ⇒ (2) This is clear by definition of multiplication modules.

(2) ⇒ (3) Let K be a pseudo-prime submodule of M and let N and L be pseudo-semiprime submodules of M such that $N \cap L \subseteq K$. By assumption, there are ideals I and J of R such that $V(N) = V(IM)$ and $V(L) = V(JM)$. Suppose that $N = \bigcap_{\lambda\in\Lambda} P_\lambda$, for some collection of pseudo-prime submodules $\{P_\lambda\}_{\lambda\in\Lambda}$. For each $\lambda \in \Lambda$,

$$P_\lambda \in V(N) \subseteq V(N) \cup V(L) = V(IM) \cup V(JM) = V((I \cap J)M),$$

so that $(I \cap J)M \subseteq P_\lambda$. Thus

$$(I \cap J)M \subseteq \bigcap_{\lambda\in\Lambda} P_\lambda = N.$$

Similarly $(I \cap J)M \subseteq L$. Therefore

$$(I \cap J)M \subseteq N \cap L \subseteq K.$$

Now we have $I \cap J \subseteq (K : M)$. It follows that $K \in V(IM) = V(N)$ or $K \in V(JM) = V(L)$, *i.e.*, $N \subseteq K$ or $L \subseteq K$.

(3) ⇒ (1) Since every topological module is a top module, according to [19, Theorem 3.5], $M/\mathfrak{m}M$ is cyclic for every maximal ideal $\mathfrak{m}$ of R. Hence, by Lemma 2.6, M is multiplication. □

In the next theorem, we present some examples of topological modules. For any element x of an R-module M, we define

$$c(x) := \bigcap\{A | A \text{ is an ideal of } R \text{ and } x \in AM\}.$$

We recall that an R-module M is called a *content* R-module if, for every $x \in M$, $x \in c(x)M$. Every free module, or more generally, every projective module, is a content R-module [22, p. 51]. M is a content R-module if and only if for every family $\{A_i | i \in J\}$ of ideals of R,

$$(\bigcap_{i \in J} A_i)M = \bigcap_{i \in J}(A_i M).$$

Also, every faithful multiplication module is a content module [10, Theorem 1.6].

Theorem 2.11. *The R-module M is topological in each of the following cases:*

(1) *M is a content and pseudo-injective module.*
(2) *$\mathbb{P}\mathrm{rad}(N) = \sqrt{(N : M)}M$ for each submodule N of M.*

Proof. (1) Let N be a submodule of M. If $\mathbb{P}\mathrm{rad}(N) = M$, then $V(N) = V(RM)$. If $\mathbb{P}\mathrm{rad}(N) \neq M$, then $\mathbb{P}\mathrm{rad}(N)$ is a pseudo-semiprime submodule of M. Let $\mathbb{P}\mathrm{rad}(N) = \bigcap_{\lambda \in \Lambda} P_\lambda$, where P_λ is pseudo-prime submodule of M for each $\lambda \in \Lambda$ with $(P_\lambda : M) = \mathfrak{p}_\lambda \in \mathrm{Spec}(R)$. Since

$$\mathfrak{p}_\lambda M = (P_\lambda : M)M = ((P_\lambda : M)M : M)$$

and M is pseudo-injective, for each $\lambda \in \Lambda$, $P_\lambda = \mathfrak{p}_\lambda M$. Since M is a content module, we have

$$\begin{aligned}
\mathbb{P}\mathrm{rad}(N) &= \bigcap_{\lambda \in \Lambda} P_\lambda = \bigcap_{\lambda \in \Lambda}(\mathfrak{p}_\lambda M) = (\bigcap_{\lambda \in \Lambda} \mathfrak{p}_\lambda)M \\
&= (\bigcap_{\lambda \in \Lambda}(P_\lambda : M))M = (\bigcap_{\lambda \in \Lambda} P_\lambda : M)M = (\mathbb{P}\mathrm{rad}(N) : M)M.
\end{aligned}$$

Hence

$$V(N) = V(\mathbb{P}\mathrm{rad}(N)) = V((\mathbb{P}\mathrm{rad}(N) : M)M).$$

By Theorem 2.10, M is a topological module.

(2) Let N be a submodule of M. Then

$$V(N) = V(\mathbb{P}\mathrm{rad}(N)) = V(\sqrt{(N : M)}M).$$

By Theorem 2.10, M is a topological module. □

3. ON THE PSEUDO-PRIME SPECTRUM OF TOPOLOGICAL MODULES

For the remainder of this paper, we assume that M is always a topological R-module. Then $\emptyset = V(M)$, $X_M = V(\mathbf{0})$ and for any family of submodules $\{N_i\}_{i \in I}$ of M,

$$\bigcap_{i \in I} V(N_i) = V(\sum_{i \in I} N_i).$$

Also for any submodules N and L of M there exists a submodule K of M such that $V(N) \cup V(L) = V(K)$. Thus if $\zeta(M)$ denotes the collection of all subsets $V(N)$ of X_M then $\zeta(M)$ satisfies the axioms of a topological space for the closed subsets. This topology is called the *Zariski topology.*

In the sequel, we investigate the topological properties of this topology and we find more results about the relationship between algebraic properties of topological modules and topological properties of the Zariski topology on the pseudo-prime spectrum of them. Modules whose Zariski topology is irreducible or Noetherian are studied, and several characterizations of such modules are given. Also, we investigate this topological space from the point of view of spectral spaces.

We remark that for any R-module M, the $\mathrm{Spec}(M)$ when equipped the topology Zariski which is introduced in [19] is a subspace of X_M. There are modules M' such that $\mathrm{Spec}(M')$ is empty but $X_{M'}$ is not an empty set. Hence, we can talk about the relationship between M' and the topological property of $X_{M'}$.

Our first application of Zariski topology is to prove that the connectedness property of X_M has some results about the certain elements of the ring R. For the remainder of this paper, for every ideal $I \in V^R(\mathrm{Ann}(M))$, $\overline{R}$ and $\overline{I}$ will denote respectively $R/\mathrm{Ann}(M)$ and $I/\mathrm{Ann}(M)$.

Theorem 3.1. *Let M be a pseudo-primeful R-module such that X_M is connected. Then $X_{\bar{R}}$ is connected and the ring $\bar{R}$ contains no idempotent other than $\bar{0}$ and $\bar{1}$.*

Proof. Since the natural map $\psi : X_M \to \mathrm{Spec}(R/\mathrm{Ann}(M))$ is surjective, it suffices to show that ψ is a continuous map with respect to the Zariski topology. To do this, let I be an ideal of R containing $\mathrm{Ann}(M)$ and let $L \in \psi^{-1}(V^{\overline{R}}(\bar{I}))$. Then there exists some $\bar{J} \in V^{\overline{R}}(\bar{I})$ such that $\psi(L) = \bar{J}$. Hence $J = (L : M) \supseteq I$, and so $IM \subseteq L$. Therefore $L \in V^M(IM)$.

Now, let $K \in V^M(IM)$. Then

$$(K : M) \supseteq (IM : M) \supseteq I,$$

and so $K \in \psi^{-1}(V^{\overline{R}}(\bar{I}))$. Consequently $\psi^{-1}(V^{\overline{R}}(\bar{I})) = V^M(IM)$, *i.e.*, ψ is continuous. □

In the next proposition, we show that if the topological space X_M is a T_1-space, then we can obtain some properties of the pseudo-prime submodules of M.

Proposition 3.2. *Let M be an R-module and $Y \subseteq X_M$ and let $L \in X_{M,I}$, for some $I \in \mathrm{Spec}(R)$.*

(1) $Cl(Y) = V(\Im(Y))$. *Hence, Y is closed if and only if $Y = V(\Im(Y))$. In particular, $Cl(\{L\}) = V(L)$;*
(2) *If $(\mathbf{0}) \in Y$, then Y is dense in X_M;*
(3) *X_M is a T_0-space;*
(4) *X_M is a T_1-space if and only if each pseudo-prime submodule of M is a maximal element in the set of all pseudo-prime submodules of M.*
(5) *If* $\mathrm{Spec}(R)$ *is a T_1-space, then X_M is a T_1-space.*

Proof. (1) Clearly, $Y \subseteq V(\Im(Y))$. Let $V(N)$ be any closed subset of X_M containing Y. Because of $\Im(Y) \supseteq \Im(V(N))$, we have

$$V(\Im(Y)) \subseteq V(\Im(V(N))) = V(\mathbb{P}\mathrm{rad}(N)) = V(N).$$

This proves that $V(\Im(Y))$ is the smallest closed subset of X_M containing Y. Hence, $Cl(Y) = V(\Im(Y))$.

(2) This is clear by (1).

(3) We recall that a topological space is a T_0-space if and only if the closures of distinct points are distinct. Let N and L be two distinct points of X_M. Then by (1),

$$Cl(\{N\}) = V(N) \neq V(L) = Cl(\{L\}).$$

We deduce that, X_M is a T_0-space.

(4) We recall that a topological space is a T_1-space if and only if every singleton subset is closed. Suppose P is a maximal element in the set of all pseudo-prime submodules of M, then by (1), we have that

$$Cl(\{P\}) = V(P) = \{P\}$$

so that $\{P\}$ is closed. Thus X_M is a T_1-space.

Conversely, since X_M is a T_1-space, $\{P\}$ is closed, hence

$$\{P\} = Cl(\{P\}) = V(\Im(\{P\})) = V(P),$$

thus P is a maximal element in the set of all pseudo-prime submodules of M.

(5) Suppose H is a pseudo-prime submodule of M. Then $Cl(\{H\}) = V(H)$ by (1). Let $L \in V(H)$. Then by assumption we have

$$(H : M) = (L : M) \in \mathrm{Max}(R).$$

Hence, H and L are prime submodule of M. Now, by Theorem 2.10, $L = H$. Therefore $Cl(\{H\}) = \{H\}$ and this implies that X_M is a T_1-space. □

Corollary 3.3. *If R is an absolutely flat ring and M is an R-module, then X_M is a T_1-space.*

Proof. We know that in an absolutely flat ring, maximal ideals and prime ideals are the same (see [3, p. 55, Exercise 3 and p. 35, Exercise 27]). Hence, by Proposition 3.2, X_M is a T_1-space. □

Noetherian and Artinian modules are two well-known classes of modules and there are many papers devoted to the relationships between these classes. Here, we present a topological condition which implies that a Noetherian module is Artinian.

THEOREM 3.4. *Let M be an R-module and X_M be a T_1-space. If M is a Noetherian R-module, then M is a Artinian cyclic module.*

Proof. By Theorem 2.10, M is a multiplication R-module. By Proposition 3.2, every pseudo-prime submodule of M is a maximal element of X_M. Since M is finitely generated, every pseudo-prime submodule of M is maximal. Thus by [5, Theorem 4.9], M is Artinian. Now the result follows from [10, Corollary 2.9]. □

A topological space T is *irreducible* if and only if for any decomposition $T = A_1 \cup A_2$ with closed subsets A_i of T with $i = 1, 2$, we have $A_1 = T$ or $A_2 = T$. By an *irreducible component* of a topological space T we mean a maximal irreducible subset of T. Since every singleton subset of X_M is irreducible, its closure is also irreducible. Now, applying (1) of Proposition 3.2, we obtain that

COROLLARY 3.5. *$V(L)$ is an irreducible closed subset of X_M for every pseudo-prime submodule L of an R-module M.*

The next theorem shows that the irreducible subsets of the topological space X_M have a close relationship to the pseudo-prime submodules of the R-module M. It is well-known that in a ring R, a subset T of $\mathrm{Spec}(R)$ is irreducible if and only if $\Im(T)$ is a prime ideal of R. The next theorem is a generalization of this fact for topological modules.

THEOREM 3.6. *Let M be an R-module and Y be a subset of X_M. Then $\Im(Y)$ is a pseudo-prime submodule of M if and only if Y is an irreducible space.*

Proof. Let Y be irreducible, I and J be ideals of R such that $IJ \subseteq (\Im(Y) : M)$. Then $(IJ)M \subseteq \Im(Y)$. Now, we have

$$Y \subseteq V(\Im(Y)) \subseteq V((IJ)M) = V(IM) \cup V(JM).$$

Since Y is irreducible, so either $Y \subseteq V(IM)$ or $Y \subseteq V(JM)$. Hence, either

$$\Im(Y) \supseteq \Im(V(IM)) = \mathbb{P}\mathrm{rad}(IM) \supseteq IM$$

or $\Im(Y) \supseteq JM$. This implies that either $I \subseteq (\Im(Y) : M)$ or $J \subseteq (\Im(Y) : M)$. Therefore, $\Im(Y)$ is a pseudo-prime submodule of M. Suppose that $\Im(Y)$ is a pseudo-prime submodule of M and that $Y \subseteq Y_1 \cup Y_2$ where Y_1 and Y_2 are two closed subsets of X_M. Then there are submodules N and L of M such that $Y_1 = V(N)$ and $Y_2 = V(L)$. Hence

$$\Im(Y) \supseteq \Im(V(N) \cup V(L)) = \Im(V(N)) \cap \Im(V(L)) = \mathbb{P}\text{rad}(N) \cap \mathbb{P}\text{rad}(L).$$

Since M is a topological module, $\Im(Y)$ is an extraordinary submodule. Hence, we may have $\mathbb{P}\text{rad}(N) \subseteq \Im(Y)$ or $\mathbb{P}\text{rad}(L) \subseteq \Im(Y)$. Thus

$$Y \subseteq V(\Im(Y)) \subseteq V(\mathbb{P}\text{rad}(N)) = V(N) = Y_1$$

or $Y \subseteq Y_2$. This implies that Y is irreducible. $\square$

Corollary 3.7. *Let M be an R-module and N be a submodule of M.*

1. *$V(N)$ is an irreducible space if and only if $\mathbb{P}\text{rad}(N)$ is a pseudo-prime submodule of M.*
2. *X_M is a irreducible space if and only if $\mathbb{P}\text{rad}(\boldsymbol{0})$ is a pseudo-prime submodule of M.*
3. *Let $X_{M,I} \neq \emptyset$ for some $I \in \text{Spec}(R)$. Then $X_{M,I}$ is an irreducible space.*
4. *Let R be a quasi-local ring. Then $\text{Max}(M)$ is an irreducible space (here the $\text{Max}(M)$ is the set of all maximal submodules of M).*

Proof. (1) Since $\mathbb{P}\text{rad}(N) = \Im(V(N))$, the result follows immediately from Theorem 3.6.

(2) Take $N = (\mathbf{0})$ in (1).

(3) We have

$$(\Im(X_{M,I}) : M) = \bigcap_{Q \in X_{M,I}} (Q : M) = I \in \text{Spec}(R)$$

and so the result follows from Theorem 3.6.

(4) Use Theorem 3.6 and the fact that $(\Im(\text{Max}(M)) : M) \in \text{Max}(R)$. $\square$

Corollary 3.8. *Let M be an R-module such that $(\boldsymbol{0}) \in X_M$. Since*

$$\Im(X_M) = (0) \in X_M,$$

by Theorem 3.6, X_M is an irreducible space. In particular, if R is an integral domain and M is a torsion-free R-module, then by [15, Lemma 4.5],

$$(0 : M) = (0) \in \text{Spec}(R).$$

Hence, X_M is an irreducible space, by Theorem 3.6 again.

Example 3.9. As we have seen in the paragraph after Remark 2.8, the $\mathbb{Z}$-module $M = \mathbb{Z}(p^\infty)$, where p is a prime integer, is a topological module. By [13, p. 3745], every proper submodule N of M is pseudo-prime with $(N : M) = (0)$. Therefore, $X_M = X_{M,(0)}$ is an irreducible space.

LEMMA 3.10. *Let M be a nonzero pseudo-primeful R-module and let I be a radical ideal of R. Then $(IM : M) = I$ if and only if* $\mathrm{Ann}(M) \subseteq I$*. In particular, $\mathfrak{q}M$ is a pseudo-prime submodule of M for every $\mathfrak{q} \in V^R(\mathrm{Ann}(M))$.*

Proof. The necessity is clear. For sufficiency, since I is a radical ideal, one has $\mathrm{Ann}(M) \subseteq I = \cap_i \mathfrak{q}_i$, where $\mathfrak{q}_i$ runs through $V^R(I)$. Since M is a pseudo-primeful R-module and $\mathfrak{q}_i \in V^R(\mathrm{Ann}(M))$, there exists a pseudo-prime submodule L_i of M such that $(L_i : M) = \mathfrak{q}_i$. Now we have

$$I \subseteq (IM : M) = ((\cap_i \mathfrak{q}_i)M : M) \subseteq \cap_i (\mathfrak{q}_i M : M) \subseteq \cap_i (L_i : M) = \cap_i \mathfrak{q}_i = I.$$

Thus $(IM : M) = I$. □

One of the cornerstone theorem in algebra is Nakayama's Lemma. Since every finitely generated module is pseudo-primeful, the next proposition is a generalization of Nakayama's Lemma to the class of pseudo-primful modules.

PROPOSITION 3.11. *Let M be a pseudo-primeful R-module. If I is an ideal of R contained in the Jacobson radical* $\mathrm{Rad}(R)$ *such that $IM = M$, then $M = (\mathbf{0})$.*

Proof. Suppose that $M \neq (\mathbf{0})$. Then $\mathrm{Ann}(M) \neq R$. If $\mathfrak{m}$ is any maximal ideal containing $\mathrm{Ann}(M)$, then

$$I \subseteq \mathrm{Rad}(R) \subseteq \mathfrak{m}$$

and

$$IM = M = \mathfrak{m}M$$

whence

$$(\mathfrak{m}M : M) = R \neq \mathfrak{m},$$

a contradiction to Lemma 3.10. □

Let Y be a closed subset of a topological space. An element $y \in Y$ is called a *generic point of* Y if $Y = Cl(\{y\})$. In Proposition 3.2(1), we have seen that every element L of X_M is a generic point of the irreducible closed subset $V(L)$. Note that a generic point of a closed subset Y of a topological space is unique if the topological space is a T_0-space (see the Proposition 3.2). The next theorem is a good application of the Zariski topology on modules. Indeed, the next theorem shows that there is a correspondence between irreducible closed subsets of X_M and the pseudo-prime submodules of the R-module M.

THEOREM 3.12. *Suppose that M is an R-module and let $Y \subseteq X_M$.*

(1) *Then Y is an irreducible closed subset of X_M if and only if $Y = V(L)$ for some $L \in X_M$. Thus every irreducible closed subset of X_M has a generic point.*

(2) *The correspondence $V(L) \mapsto L$ is a bijection of the set of all irreducible components of X_M onto the set of all minimal elements of X_M.*

(3) *Let M be a pseudo-primeful R-module. Then the set of all irreducible components of X_M is of the form*

$$T = \{V^M(IM) \mid I \text{ is a minimal element of } V(\mathrm{Ann}(M))\}.$$

(4) *Let R be a Laskerian ring (i.e., every proper ideal of R has a primary decomposition) and let M be a nonzero pseudo-primeful R-module. Then X_M has only finitely many irreducible components.*

Proof. (1) By Corollary 3.5, $Y = V(L)$ is an irreducible closed subset of X_M for any $L \in X_M$. Conversely, if Y is an irreducible closed subset of X_M, then $Y = V(N)$ for some $N \leq M$ and

$$\Im(Y) = \Im(V(N)) = \mathbb{P}\mathrm{rad}(N) \in X_M$$

by Theorem 3.6. Hence $Y = V(N) = V(\mathbb{P}\mathrm{rad}(N))$, as desired.

(2) Let Y be an irreducible component of X_M. Each irreducible component of X_M is a maximal element of the set $\{V(Q) \mid Q \in X_M\}$ by (1), so we have $Y = V(P)$ for some $P \in X_M$. Obviously, P is a minimal element of X_M, for if T is a pseudo-prime submodule of M with $T \subseteq P$, then $V(P) \subseteq V(T)$ so that $P = T$. Now, let P be a minimal element of X_M with $V(P) \subseteq V(Q)$ for some $Q \in X_M$. Then

$$Q = \mathbb{P}\mathrm{rad}(Q) = \Im(V(Q)) \subseteq \Im(V(P)) = \mathbb{P}\mathrm{rad}(P) = P,$$

hence $P = Q$. This implies that $V(P)$ is an irreducible component of X_M.

(3) Let Y be an irreducible component of X_M. By part (1), $Y = V(L)$ for some $L \in X_M$. It is evident that $(L : M)M$ is a pseudo-prime submodule of M. Since $(L : M)M \subseteq L$, we have

$$Y = V(L) \subseteq V((L : M)M).$$

Since Y is an irreducible component, $V(L) = V((L : M)M)$, and so $L = (L : M)M$. We must show that $l := (L : M)$ is a minimal element of $V^R(\mathrm{Ann}(M))$. To see this let $J \in V^R(\mathrm{Ann}(M))$ and $J \subseteq l$. Then $J/\mathrm{Ann}(M) \in \mathrm{Spec}(R/\mathrm{Ann}(M))$, and there exists an element $Q \in X_M$ such that $(Q : M) = J$ because of M is a pseudo-primeful R-module. Thus $Y = V^M(L) \subseteq V^M(Q)$. Hence $Y = V^M(L) = V^M(Q)$ due to the maximality of $V^M(L)$. Thus we have that $l = J$. Conversely, assume that $Y \in T$. There exists a minimal element

I in $V^R(\mathrm{Ann}(M))$ such that $Y = V^M(IM)$. Since M is pseudo-primeful, IM is a pseudo-prime submodule of M by Lemma 3.10. Thus Y is an irreducible space by part (1). Suppose $Y = V^M(IM) \subseteq V^M(Q)$, where Q is an element of X_M. Since $IM \in V^M(Q)$ and I is minimal, we have $I = (IM : M) = (Q : M)$. Now,

$$Y = V^M(IM) = V^M((Q : M)M) \supseteq V^M(Q).$$

Therefore, $Y = V^M(IM) = V^M(Q)$.

(4) By assumption, every proper ideal of R has a primary decomposition. So, if I is a minimal element of $V(\mathrm{Ann}(M))$ and if $\mathrm{Ann}(M) = \cap_{i=1}^n Q_i$ is a minimal primary decomposition of $\mathrm{Ann}(M)$, then for some $1 \le i \le n$, we must have $Q_i \subseteq I$. By minimality of I, we get $I = \sqrt{Q_i}$. Hence, the irreducible components of X_M are $V^M(Q_iM)$'s, by part (3). □

In the next proposition, we show that the irreducibility of X_M (a topological property) implies that $\sqrt{\mathrm{Ann}(M)}$ is a prime ideal of R (an algebraic property).

Proposition 3.13. *The following statements are equivalent for a nonzero pseudo-primeful R-module M:*

(1) X_M *is an irreducible space.*
(2) $\mathrm{Spec}(R/\mathrm{Ann}(M))$ *is an irreducible space.*
(3) $V(\mathrm{Ann}(M))$ *is an irreducible space.*
(4) $\sqrt{\mathrm{Ann}(M)}$ *is a prime ideal of R.*
(5) $X_M = V^M(IM)$ *for some $I \in V(\mathrm{Ann}(M))$.*

Proof. (1) ⇒ (2) As we have seen in the proof of Theorem 3.1, the natural map ψ is continuous and by assumption ψ is surjective. Hence $Im(\psi) = \mathrm{Spec}(R/\mathrm{Ann}(M))$ is also irreducible.

(2) ⇒ (3) It is well-known that the mapping

$$\begin{array}{rcl} \varphi : \mathrm{Spec}(R/\mathrm{Ann}(M)) & \to & \mathrm{Spec}(R) \\ J/\mathrm{Ann}(M) & \mapsto & J \end{array}$$

is a homeomorphism. This implies that $V(\mathrm{Ann}(M))$ is an irreducible space.

(3) ⇒ (4) By Theorem 3.6, $\Im(V(\mathrm{Ann}(M))) = \sqrt{\mathrm{Ann}(M)}$ is a prime ideal of R.

(4) ⇒ (5) By Lemma 3.10, $\sqrt{\mathrm{Ann}(M)}M$ is a pseudo-prime submodule of M. Now, let $L \in X_M$. Then $\sqrt{\mathrm{Ann}(M)} \subseteq (L : M)$, and so $\sqrt{\mathrm{Ann}(M)}M \subseteq L$. Therefore $X_M = V(\sqrt{\mathrm{Ann}(M)}M)$, where $\sqrt{\mathrm{Ann}(M)} \in V(\mathrm{Ann}(M))$.

(5) ⇒ (1) By Lemma 3.10, IM is a pseudo-prime submodule of M. By Corollary 3.5, $V^M(IM) = X_M$ is irreducible. □

Example 3.14. Consider the faithful $\mathbb{Z}$-module $M = (\bigoplus_p \mathbb{Z}/p\mathbb{Z}) \oplus \mathbb{Q}$, where p runs through the set if all are prime integers. Let q be a prime integer. Then

$$M/qM \cong \frac{(\bigoplus_p \mathbb{Z}/p\mathbb{Z}) \oplus \mathbb{Q}}{(\bigoplus_p q(\mathbb{Z}/p\mathbb{Z})) \oplus q\mathbb{Q}} \cong \mathbb{Z}/q\mathbb{Z}.$$

This implies that M is a pseudo-primeful module. By Proposition 3.13, X_M is an irreducible space.

We recall that a topological space X is said to be a *Noetherian* space if the open subsets of X satisfy the ascending chain condition. The next result, as another application of the topology on modules, shows that the Noetherianness of the pseudo-prime spectrum of modules impose a chain condition on the modules. Recall that a ring has Noetherian spectrum if and only if the ascending chain condition ACC for radical ideals holds [21, p. 631]. The next theorem is a generalization of this fact to modules.

THEOREM 3.15. *An R-module M has a Noetherian pseudo-prime spectrum if and only if the ACC holds for pseudo-prime radical submodules of M.*

Proof. Suppose the ACC holds for pseudo-prime radical submodules of M. Let

$$V(N_1) \supseteq V(N_2) \supseteq \cdots$$

be a descending chain of close subsets of X_M, where $N_i \leq M$. Then

$$\Im(V(N_1)) \subseteq \Im(V(N_2)) \subseteq \cdots$$

is an ascending chain of pseudo-prime radical submodules $\Im(V(N_i)) = \mathbb{P}\mathrm{rad}(N_i)$ of M. So, by assumption there exists $k \in \mathbb{N}$ such that for all $i \in \mathbb{N}$,

$$\Im(V(N_k)) = \Im(V(N_{k+i})).$$

Now, by Proposition 3.2,

$$V(N_k) = V(\Im(V(N_k))) = V(\Im(V(N_{k+i}))) = V(N_{k+i}).$$

Hence the first chain is stationary, *i.e.*, X_M is a Noetherian space. Conversely, suppose that M has a Noetherian pseudo-prime spectrum. Let $N_1 \subseteq N_2 \subseteq \cdots$ be an ascending chain of pseudo-prime radical submodules of M. Thus

$$N_i = \Im(V(N_i)) = \mathbb{P}\mathrm{rad}(N_i).$$

Hence

$$V(N_1) \supseteq V(N_2) \supseteq \cdots$$

is a descending chain of close subsets of X_M. By assumption there is $k \in \mathbb{N}$ such that for all $i \in \mathbb{N}$, $V(N_k) = V(N_{k+i})$. Therefore,

$$N_k = \mathbb{P}\mathrm{rad}(N_k) = \Im(V(N_k)) = \Im(V(N_{k+i})) = \mathbb{P}\mathrm{rad}(N_{k+i}) = N_{k+i}. \quad \square$$

Following M. Hochster [12], we say that a topological space Y is a *spectral* space in case Y is homeomorphic to $\mathrm{Spec}(S)$, with the Zariski topology, for some ring S. In the sequel, we present algebraic conditions that under which the pseudo-prime spectrum of modules is a spectral space. Spectral spaces have been characterized by Hochster [12, p. 52, Proposition 4] as the topological space Y which satisfies the following conditions: (1) Y is a T_0-space; (2) Y is quasi-compact; (3) the quasi-compact open subsets of Y are closed under finite intersection and these form an open base; (4) each irreducible closed subset of Y has a generic point. Note that a Noetherian space is spectral if and only if it is T_0 and every non-empty irreducible closed subspace has a generic point [12, pp. 57–58]. We recall that if M is a topological R-module, then X_M is T_0-space (see Proposition 3.2) and every non-empty irreducible closed subset of X_M has a generic point (see Theorem 3.12).

THEOREM 3.16. *Let M be an R-module. Then X_M is a spectral space in each of the following cases:*

(1) *Let R be a Noetherian ring and let for every submodule N of M there exists an ideal I of R such that $V(N) = V(IM)$.*
(2) *Let $\mathrm{Spec}(R)$ be a Noetherian topological space and let M be a content pseudo-injective R-module.*

Proof. (1) We will show that every open subset of X_M is quasi-compact. Let H be an open subset of X_M and let $\{E_\lambda\}_{\lambda\in\Lambda}$ be an open cover of H. Then there are submodules N and N_λ such that $H = X_M \setminus V(N)$, $E_\lambda = X_M \setminus V(N_\lambda)$ for each $\lambda \in \Lambda$ and

$$H \subseteq \bigcup_{\lambda\in\Lambda} E_\lambda = X_M \setminus \bigcap_{\lambda\in\Lambda} V(N_\lambda).$$

By hypothesis, for each $\lambda \in \Lambda$ there exists an ideal J_λ in R such that $V(N_\lambda) = V(J_\lambda M)$. Then

$$H \subseteq X_M \setminus V(\sum_{\lambda\in\Lambda} J_\lambda M) = X_M \setminus V((\sum_{\lambda\in\Lambda} J_\lambda)M).$$

Since R is a Noetherian ring, there exists a finite subset Λ' of Λ such that

$$H \subseteq \bigcup_{\lambda\in\Lambda'} E_\lambda.$$

Therefore X_M is a Noetherian space and whence a spectral space.

(2) We will show that X_M is Noetherian. Let

$$V(N_1) \supseteq V(N_2) \supseteq \cdots$$

be a descending chain of closed subsets of X_M. Then we have

$$\mathbb{P}\mathrm{rad}(N_1) \subseteq \mathbb{P}\mathrm{rad}(N_2) \subseteq \cdots .$$

Since $\operatorname{Spec}(R)$ is Noetherian, the ascending chain

$$(\mathbb{P}\mathrm{rad}(N_1) : M) \subseteq (\mathbb{P}\mathrm{rad}(N_2) : M) \subseteq \cdots$$

of radical ideals must be stationary by Theorem 3.15. Thus there is an integer k such that for each $i = 1, 2, \ldots,$

$$(\mathbb{P}\mathrm{rad}(N_k) : M) = (\mathbb{P}\mathrm{rad}(N_{k+i}) : M) = \cdots .$$

As we have seen in the proof of Theorem 2.11, for each $\lambda \in \mathbb{N}$,

$$\mathbb{P}\mathrm{rad}(N_\lambda) = (\mathbb{P}\mathrm{rad}(N_\lambda) : M)M.$$

Therefore, for each $i = 1, 2, \ldots$ we have $\mathbb{P}\mathrm{rad}(N_k) = \mathbb{P}\mathrm{rad}(N_{k+i}) = \cdots$. This implies that

$$V(N_k) = V(\mathbb{P}\mathrm{rad}(N_k)) = V(\mathbb{P}\mathrm{rad}(N_{k+i})) = V(N_{k+i}) = \cdots .$$

Thus X_M is Noetherian and the proof is completes. $\square$

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