COMMUTATIVE RINGS WHOSE MAXIMAL IDEALS ARE DIRECT SUM OF COMPLETELY CYCLIC MODULES*

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Communicated by Vasile Brînzănescu

In this paper, we study commutative rings R whose maximal ideals are direct sums of completely cyclic modules (an R-module M is called *completely cyclic* if each submodule of M is cyclic). It is shown that if every maximal ideal of R is a direct sum of completely cyclic R-modules, then $\dim(R) \leq 1$ and either R is a local ring such that every prime ideal of R is a direct sum of uniserial Noetherian R-modules, or R is a Noetherian ring and there exists a positive integer n such that every prime ideal of R is a direct sum of at most n completely cyclic modules. In particular, if R is a commutative Artinian ring, then every maximal ideal of R is a direct sum of completely cyclic R-modules if and only if every maximal ideal of R is cyclic, if and only if, R is a Köthe-ring (*i.e.*, every R-module is a direct sum of cyclic R-modules).

AMS 2010 Subject Classification: Primary 13C05, 13E05, 13F10; Secondary 13E10, 13H99.

Key words: cyclic modules, completely cyclic modules, principal ideal rings, direct sum decomposition.

1. INTRODUCTION

Throughout this paper, all rings are commutative with identity and all modules are unital. For a ring R we denote by $\mathrm{Spec}(R)$ and $\mathrm{Max}(R)$ the set of prime ideals and the set of maximal ideals of R, respectively. A ring R is called *local* (resp. *semilocal*) if R has a unique maximal ideal (resp. R has a finite number of maximal ideals). In this paper, $(R, \mathcal{M})$ will be a local ring with maximal ideal $\mathcal{M}$. Let R be a ring, and let M be an R-module. Choose a nonempty subset X of M. The *annihilator of* X, denoted $\mathrm{Ann}_R(X)$, is the ideal $\mathrm{Ann}_R(X) = \{a \in R \mid aX = 0\}$. The annihilator of a single element x is usually written $\mathrm{Ann}_R(x)$ instead of $\mathrm{Ann}_R(\{x\})$. An ideal I of R is called an *annihilator ideal* if $I = \mathrm{Ann}_R(S)$ for some $S \subseteq R$.

*The research of the first author was in part supported by a grant from IPM (No. 93130413). This research is partially carried out in the IPM-Isfahan Branch.

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We recall that a commutative ring in which every ideal is principal is called a *principal ideal ring*. A *principal ideal domain* (PID) is an integral domain that is principal ideal ring. An R-module M is called *completely cyclic* if each submodule of M is cyclic. Completely cyclic modules are obvious generalizations of principal ideal rings. Also, an R-module M is called *uniserial* if its submodules are linearly ordered by inclusion. If R is an Artinian ring, then every uniserial module is completely cyclic (see [10, Lemma 13.9]).

It was shown by Köthe [12] and Cohen-Kaplansky [6] that "a commutative ring R has the property that every module is a direct sum of cyclic modules if and only if R is an Artinian principal ideal ring." In fact, over an Artinian principal ideal ring every module is a direct sum of completely cyclic modules. The study of commutative rings R that the ideals of R are direct sums of (completely) cyclic modules was initiated by Behboodi et al., in [2–4].

In this paper, we study commutative rings R whose maximal ideals are direct sums of completely cyclic modules. In Section 2, it is shown that if every maximal ideal of R is a direct sum of completely cyclic R-modules, then $\dim(R) \leq 1$ (see Proposition 2.4). Moreover, in the local case every prime ideal is a direct sum of uniserial Noetherian R-modules (see Theorem 2.10), and in the non-local case R is a Noetherian ring and there exists a positive integer n such that every prime ideal of R is a direct sum of at most n completely cyclic modules (see Theorem 2.12). In particular, it is shown that for any commutative Artinian ring R every maximal ideal of R is a direct sum of completely cyclic R-modules if and only if R is a Köthe-ring (*i.e.*, every R-module is a direct sum of cyclic R-modules (see Proposition 2.13). Finally, some relevant examples and counterexamples are indicated in Section 3.

2. MAIN RESULTS

We begin with the following two famous results from commutative algebra.

LEMMA 2.1 (Cohen [5, Theorem 2]). *Let R be a commutative ring. Then R is a Noetherian ring if and only if every prime ideal of R is finitely generated.*

LEMMA 2.2 (Kaplansky, [11, Theorem 12.3]). *A commutative Noetherian ring R is a principal ideal ring if and only if every maximal ideal of R is principal.*

The following theorem is an analogue of Kaplanskys (Cohen) theorem.

PROPOSITION 2.3. *Let R be a ring such that every maximal ideal of R is a direct sum of completely cyclic R-modules. Then*

(a) *R is a principal ideal ring if and only if every maximal ideal of R is principal.*
(b) *R is Noetherian if and only if every maximal ideal of R is finitely generated.*

Proof. We only prove (a), since the proof of (b) is similar.
(a) ($\Rightarrow$) is clear.
(a) ($\Leftarrow$). Assume $\mathcal{M}$ is a maximal ideal of R. Then $\mathcal{M} = \bigoplus_{j\in J} Rx_j$ where each Rx_i is completely cyclic R-module. By the assumption $\mathcal{M} = Ry$, for some $y \in R$. Thus $y = r_1x_{j_1} + ... + r_nx_{j_n}$ for some $x_{j_1}, ..., x_{j_n} \in \mathcal{M}$ and $r_1, ..., r_n \in R$. It follows that $\mathcal{M} = Rx_{j_1} \oplus ... \oplus Rx_{j_n}$, and so $\mathcal{M}$ is a finite direct sum of completely cyclic R-module and so $\mathcal{M}$ is a Noetherian R-module. Since $R/\mathcal{M}$ is also Noetherian, R is a Noetherian ring, and so by Lemma 2.2, R is a principal ideal ring. □

Proposition 2.4. *Let R be a commutative ring. If every maximal ideal of R is a direct sum of completely cyclic R-modules, then* $\dim(R) \leq 1$.

Proof. Let P be a minimal prime ideal of R. Since each maximal ideal of R is a direct sum of completely cyclic R-modules, so each maximal ideal of the ring R/P is completely cyclic. Now by Proposition 2.3, R/P is a PID. Thus $\dim(R/P) \leq 1$ for each minimal prime ideal P of R. It follows that $\dim(R) \leq 1$. □

Next we need the following two lemmas.

Lemma 2.5 (See [13, Lemma 1.1]). *Let R be a ring (not necessarily commutative) and let M be an R-module. If $\{e_i \mid i \in I\}$ is a minimal generating set of M where the cardinality I is infinite, then M cannot be generated by fewer than $|I|$ elements.*

Lemma 2.6 (See [2, Proposition 2.15]). *Let R be a ring (not necessarily commutative). Then the following statements are equivalent:*
(1) *R is a local ring.*
(2) *If $\bigoplus_{i=1}^{n} Rx_i \cong \bigoplus_{j=1}^{m} Ry_j$ where $n, m \in \mathbb{N}$ and Rx_i, Ry_j are nonzero cyclic R-modules, then $n = m$.*
(3) *If $\bigoplus_{i\in I} Rx_i \cong \bigoplus_{j\in J} Ry_j$ where I, J are index sets and Rx_i, Ry_j are nonzero cyclic R-modules, then $|I| = |J|$.*

Let $R = R_1 \times \cdots \times R_k$ where $k \in \mathbb{N}$ and each R_i is a nonzero ring. One can easily see that, each prime ideal P of R is of the form $P = R_1 \times \cdots \times R_{i-1} \times P_i \times R_{i+1} \times \cdots \times R_k$ where each P_i is a prime ideal of R_i. Also, if P_i is a direct sum of Λ completely cyclic (respectively, cyclic) R_i-modules, then it is easy to see that P is also a direct sum of Λ completely cyclic (respectively, cyclic) R-module. Thus the ring R has the property that whose prime ideals are direct sum of completely cyclic (respectively, cyclic) R-modules if and only if for each i the ring R_i has this property.

PROPOSITION 2.7. *Let R be a commutative semilocal ring and let $\mathcal{M} \in \mathrm{Max}(R)$. If $\mathcal{M} = \bigoplus_{i\in I} Rx_i = \bigoplus_{j\in J} Ry_j$ where I, J are index sets and Rx_i, Ry_j are nonzero cyclic R-modules, then $|I| = |J|$.*

Proof. Suppose that $\mathcal{M} = \bigoplus_{i\in I} Rx_i = \bigoplus_{j\in J} Ry_j$ where I or J is infinite. Then by Lemma 2.5, $|I| = |J|$. Thus we can assume that $I = \{1, ..., n\}$ and $J = \{1, ..., m\}$, where $n, m \in \mathbb{N}$. Since R is semilocal, we can write $R = R_1 \times ... \times R_t$, where each R_i is an indecomposable ring (*i.e.*, R_i has no nontrivial idempotent elements). By above argument, without loss of generality, we can assume that R is indecomosable. Set $\bar{R} = S^{-1}R, \bar{\mathcal{M}} = S^{-1}\mathcal{M}, \bar{x_i} = x_i/1$ and $\bar{y_i} = y_i/1$, where $S = R \setminus \mathcal{M}$. Then $\bar{R}$ is a local ring with the maximal ideal $\bar{\mathcal{M}} = \bigoplus_{i=1}^{n} \bar{R}\bar{x_i} = \bigoplus_{i=1}^{m} \bar{R}\bar{y_i}$. Now we claim that for each i, j the R-modules $\bar{R}\bar{x_i}, \bar{R}\bar{y_j}$ are nonzero. In fact, for instance if $\bar{R}\bar{x_i} = 0$ for some i, then there exists a $s \in S$ such that $sx_i = 0$. Since $s \notin \mathcal{M}$ and $\mathcal{M} + Rs = R$, so there exist $m \in \mathcal{M}$ and $r \in R$, such that $m + rs = 1$. But $m = r_1x_1 + ... + r_nx_n$ for some $r_1, \ldots, r_n \in R$ and so $x_i = mx_i + rsx_i$, *i.e.*, $x_i = mx_i = r_ix_i^2$. It follows that $(1 - r_ix_i)x_i = 0$ and so $R = Rx_i \oplus R(1 - r_ix_i)$, a contradiction (since R is indecomposable). Thus $\bar{R}\bar{x_i}, \bar{R}\bar{y_j}$ are nonzero cyclic modules and so by Lemma 2.6, $n = m$. $\square$

COROLLARY 2.8. *Let R be a commutative semilocal ring. If $\mathcal{M} = \bigoplus_{i\in I} Rx_i$ where $\mathcal{M} \in \mathrm{Max}(R)$ and for each $i \in I$, Rx_i is a nonzero completely cyclic R-module, then for each $i \in I$, Rx_i is an indecomposable R-module.*

A local Artinian principal ideal ring is called a special principal ring and has an extremely simple ideal structure: there are only finitely many ideals, each of which is a power of the maximal ideal. A principal ideal ring R can be written as a direct product $\prod_{i=1}^{n} R_i$, where each R_i is either a principal ideal domain or a *special principal ring* (see [14, p. 245, Theorem 33] and [9, Theorem 1]).

LEMMA 2.9. *Let R be a commutative local ring and M be an R-module. If $M = \bigoplus_{i\in I} Rx_i$, where for each $i \in I$, Rx_i is a nonzero completely cyclic R-module, then for each $i \in I$, Rx_i is a uniserial R-module.*

Proof. By Lemma 2.6, Rx_i is an indecomposable R-module for each $i \in I$. On the other hand, for each $i \in I$, $Rx_i \cong R/\mathrm{Ann}(x_i)$ is a principal ideal ring so it can be written as a direct product $\prod_{j=1}^{s} R_j$, where each R_i is either a principal ideal domain or a special principal ring (see above argument). Now since for each $i \in I$, Rx_i is indecomosable so every $Rx_i \cong R/\mathrm{Ann}(x_i)$ is either a PID or a special principal ring that both are uniserial rings (note that a local PID is a chain ring) and so M is a direct sum of uniserial R-module. $\square$

THEOREM 2.10. *Let $(R, \mathcal{M})$ be a commutative local ring. Then the following statements are equivalent:*

(1) *$\mathcal{M}$ is a direct sum of completely cyclic R-modules.*
(2) *$\mathcal{M} = \bigoplus_{\lambda\in\Lambda} Rw_\lambda$, where Λ is an index set and each Rw_λ is a completely cyclic R- module for every $\lambda \in \Lambda$.*
(3) *$\mathcal{M} = \bigoplus_{\lambda\in\Lambda} Rw_\lambda$, where Λ is an index set and each Rw_λ is a uniserial Noetherian R-module for every $\lambda \in \Lambda$.*
(4) *Every prime ideal of R is a direct sum of at most $|\Lambda|$ uniserial Noetherian R-module, where Λ is an index set.*
(5) *Every prime ideal of R is a direct sum of at most $|\Lambda|$ completely cyclic R-module, where Λ is an index set.*

Proof. (1) $\Rightarrow$ (2) is clear.
(2) $\Rightarrow$ (3) is by Lemma 2.9.
(3) $\Rightarrow$ (4). Let P be a non-maximal ideal of R. Then $P \subsetneqq \mathcal{M} = \bigoplus_{\lambda\in\Lambda} Rw_\lambda$, where Λ is an index set, and for each $\lambda \in \Lambda$, Rw_λ is a uniserial Noetherian R-module. There exists a $\lambda_0 \in \Lambda$, such that $w_{\lambda_0} \notin P$ and since $w_\lambda w_{\lambda_0} = 0 \in P$, so $\oplus_{\lambda\in\Lambda\setminus\{\lambda_0\}} Rw_\lambda \subseteq P$. Now by modular property, we have

$$P = P \cap \mathcal{M} = (P \cap Rw_{\lambda_0}) \oplus (\bigoplus_{\lambda\in\Lambda\setminus\{\lambda_0\}} Rw_\lambda).$$

Since $P \cap Rw_{\lambda_0} \subseteq Rw_{\lambda_0}$ and Rw_{λ_0} is a uniserial Noetherian R-module, so P is a direct sum of at most $|\Lambda|$ uniserial Noetherian R-modules.
(4) $\Rightarrow$ (5) $\Rightarrow$ (1) is clear. $\square$

The following theorem shows that if R is a commutative ring such that every maximal ideal is a direct sum of completely cyclic modules, then either R is a local ring or R is a Noetherian ring.

THEOREM 2.11. *Let R be a non-local commutative ring. If every maximal ideal is a direct sum of completely cyclic R-modules, then R is a Noetherian ring.*

Proof. First we will show that every maximal ideal of R is a finite direct sum of completely cyclic R-modules, for this let $\mathrm{Max}(R) = \{\mathcal{M}_i \mid i \in \{1, 2\} \cup I\}$ where I is an index set, that can be empty. Now let $\mathcal{M}_1 = \bigoplus_{\lambda\in\Lambda} Rx_\lambda, \mathcal{M}_2 = \bigoplus_{\gamma\in\Gamma} Ry_\gamma$ where Λ, Γ are index sets, and for each $\lambda \in \Lambda, \gamma \in \Gamma$, Rx_λ, Ry_γ are completely cyclic R-modules be two distinct maximal ideals of R. Now there exists $y_\gamma \in \mathcal{M}_2$ such that $y_\gamma \notin \mathcal{M}_1$, and so $\mathcal{M}_1 + Ry_\gamma = R$, from this we infer that there exist $x_{\lambda_1}, x_{\lambda_2}, ..., x_{\lambda_n} \in \mathcal{M}_1$ such that

$$Rx_{\lambda_1} + Rx_{\lambda_2} \ldots + Rx_{\lambda_n} + Ry_\gamma = R.$$

Let $\Lambda' = \Lambda \setminus \{\lambda_1, \lambda_2, ..., \lambda_n\}$. Then it is easy to check that $\bigoplus_{\Lambda'} Rx_\lambda \subsetneq Ry_\gamma$. Since Ry_γ is a completely cyclic R-module, so $\bigoplus_{\Lambda'} Rx_\lambda = Rz$, for some $z \in R$ and it follows that

$$\mathcal{M}_1 = Rx_{\lambda_1} \oplus Rx_{\lambda_2} \oplus \ldots \oplus Rx_{\lambda_n} \oplus Rz.$$

From this we infer that every maximal ideal of R is a finite direct sum of completely cyclic R-modules and so it is a finitely generated ideal. Now part (b) of Proposition 2.3, follows that R is a Noetherian ring. $\square$

A *uniform module* is a nonzero module M such that the intersection of any two nonzero submodules of M is nonzero. Let M be an R-module and n a nonnegative integer. We recall that M has *finite rank* n (denoted by $\mathrm{u.dim}(M_R) = n$) if M contains a direct sum of n nonzero submodules but no direct sum of $n+1$ nonzero submodules (see for instance [8, Proposition 5.50]). We note that every Noetherian module has finite rank (see for instance [8, Section 5]).

THEOREM 2.12. *Let R be a non-local commutative ring. Then the following statements are equivalent:*

(1) *Every prime ideal of R is a direct sum of completely cyclic R-modules.*
(2) *Every maximal ideal of R is a direct sum of completely cyclic R-modules.*
(3) *There exists a positive integer n such that every maximal ideal of R is a direct sum of at most n completely cyclic modules.*
(4) *There exists a positive integer n such that every prime ideal of R is a direct sum of at most n completely cyclic modules.*
(5) *R is a Noetherian ring and every maximal ideal of R is a direct sum of completely cyclic modules.*
(6) *R is a Noetherian ring and every prime ideal of R is a direct sum of completely cyclic modules.*

Proof. (1) $\Rightarrow$ (2) is clear.
(2) $\Rightarrow$ (3). By Theorem 2.11, R is a Noetherian ring and hence R has a finite rank. Thus $\mathrm{u.dim}(R_R) = n$ for some $n \in \mathbb{N}$. It follows that every maximal ideal is a direct sum of at most n completely cyclic modules.
(3) $\Rightarrow$ (4). Let P be a non-maximal ideal of R. Then $P \subsetneqq \mathcal{M} = \bigoplus_{i=1}^n Rw_i$, where $\mathcal{M}$ is a maximal ideal of R and for each i, Rw_i is a completely cyclic R-module. Without loss of generality we can assume that $Rx_1 \nsubseteq P$, and so $\bigoplus_{i=2}^n Rw_1 \subseteq P$. Now by modular property, we have $P = P \cap \mathcal{M} = (P \cap Rw_1) \oplus (\bigoplus_{i=2}^n Rw_i)$. Since $P \cap Rw_1 \subseteq Rw_1$ and Rw_1 is a completely cyclic R-module,

so P is a direct sum of at most n completely cyclic R-modules.
(4) $\Rightarrow$ (5) is by Proposition 2.3.
(5) $\Rightarrow$ (6) and (6) $\Rightarrow$ (1) are clear. $\square$

We conclude this section with the following result for commutative Artinian rings.

PROPOSITION 2.13. *Let R be a commutative Artinian ring. Then the following statements are equivalent:*

(1) *Every maximal ideal of R is a direct sum of completely cyclic R-modules.*
(2) *Every maximal ideal of R is a direct sum of cyclic R-modules.*
(3) *Every maximal ideal of R is cyclic.*
(4) *R is a principal ideal ring.*
(5) *R is a Köthe-ring (i.e., every R-module is a direct sum of cyclic R-modules).*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is clear.
(3) $\Rightarrow$ (4) is by Lemma 2.2.
(4) $\Rightarrow$ (5) is by Cohen-Kaplansky [6] .
(5) $\Rightarrow$ (1) is clear. $\square$

3. SOME RELEVANT EXAMPLES AND COUNTEREXAMPLES

The following example shows that there exists a local ring $(R, \mathcal{M})$ with $\dim(R) = 1$ such that every maximal (prime) ideal of R is an infinite direct sum of uniserial completely cyclic modules.

Example 3.1. Let F be a field. Then the ring

$$R = F[[\{X_i \mid i \in \mathbb{N}\}]]/\langle\{X_iX_j \mid i \neq j\} \cup \{X_i^2 \mid i \geq 2\}\rangle,$$

is a local commutative non-domain, non-Noetherian ring with $\dim(R) = 1$ and $\mathcal{M} = \bigoplus_{i=1}^{\infty} Rx_i$, (where $x_i = X_i + \{\langle X_iX_j \mid i \neq j\} \cup \{X_i^2 \mid i \geq 2\}\rangle$). Note that $Rx_1 \cong R/\mathrm{Ann}(x_1) = F[x_1, x_2, \ldots, x_n, ...]/\langle x_1^2, x_2, ..., x_n, ...\rangle$, and so Rx_1 is a completely cyclic R-module. Also one can easily check that Rx_i is a completely cyclic R-module for each $i \geq 2$ and so $\mathcal{M}$ is a direct sum of completely cyclic R-modules. It is easy to check that $\mathrm{Spec}(R) = \{\mathcal{M}_1, \bigoplus_{i=2}^{\infty} Rx_i\}$, and so every prime ideal of R is an infinite direct sum of completely cyclic modules.

Also, the following example shows that there exists a local ring $(R, \mathcal{M})$ with $\dim(R) = 0$ such that every maximal (prime) ideal is an infinite direct sum of uniserial completely cyclic modules.

Example 3.2. Let F be a field and R be the F-algebra with generators $\{x_i \mid i \in N\}$ subject to the relations $x_i x_j = 0$ for $i, j \in \mathbb{N}$ (*i.e.*, $R = F[\{X_i \mid i \in \mathbb{N}\}]/\langle X_i X_j \mid i, j \in \mathbb{N}\rangle$). Then the ring R is a non-Noetherian local ring with the maximal ideal $\mathcal{M} = \bigoplus_{i\in\mathbb{N}} Rx_i$. Since $Rx_i \cong R/\mathrm{Ann}(x_i) = F[x_1, x_2, \ldots, x_n, ...]/\langle x_1, x_2, ..., x_n, ...\rangle$, so for each i, Rx_i is a completely cyclic R-module. Also it is easy to check that $\mathrm{Spec}(R) = \{\mathcal{M}\}$ and so the only prime ideal of R is an infinite direct sum of completely cyclic modules.

The natural question arises "whether it is possible to deduce every ideal is a direct sum of completely cyclic modules, if one only assumes that every prime ideal is a direct sum of completely cyclic modules?" The following example shows that this question is not true in general.

Example 3.3. Let $R = \mathbb{Z}_2[\{X_i \mid 1 \leq i \leq n\}]/\langle X_i X_j \mid 1 \leq i \neq j \leq n\rangle$, then R is a commutative non-domain Artinian ring with $\dim(R) = 0$, $\mathcal{M} = Rx_1 \oplus \ldots \oplus Rx_n$ (where $x_i = X_i + \langle\{X_i X_j \mid 1 \leq i \neq j \leq n\}\rangle$) and $\mathrm{Spec}(R) = \{\mathcal{M}\}$. But there exists an ideal I of R that is not a direct sum of completely cyclic R-modules (see Theorem 3.9, [3]).

We know that if an R-module is a direct sum of completely cyclic R-modules then it is a direct sum of cyclic R-modules. The following example shows that the converse is not true in general.

Example 3.4. Let R be the subring of all sequences from the ring $\prod_{i\in\mathbb{N}} \mathbb{Z}_2$ that are eventually constant. Then R is a zero-dimensional Boolean ring with minimal prime ideals $P_i = \{\{a_n\} \in R \mid a_i = 0\}$ and $P_\infty = \{\{a_n\} \in R \mid a_n = 0 \text{ for large } n\}$ (See [1]). Clearly, each P_i is cyclic (in fact $P_i = Rv_i$ where $v_i = (1, 1, \cdots, 1, 0, 1, 1, \cdots)$) and $P_\infty = \bigoplus_{i\in\mathbb{N}} \mathbb{Z}_2 = \bigoplus_{i\in\mathbb{N}} Rw_i$ where $w_i = (0, 0, \cdots, 0, 1, 0, 0, \cdots)$. Thus every prime ideal of R is a direct sum of cyclic modules. But the factor ring $R/\mathrm{Ann}(v_1) = R/\mathrm{Ann}(0, 1, 1, 1, \cdots)$ is not a principal ideal ring (since prime ideal $P_\infty/\mathrm{Ann}(v_1)$ is not a principal ideal of $R/\mathrm{Ann}(v_1)$). Also, one can easily to see that if $P_1 = \bigoplus_{\lambda\in\Lambda} Rz_\lambda$ where Λ is an index set and $z_\lambda \in P_1$, then $|\Lambda| = 1$ and $P_1 = Rz_\lambda = Rv_1$.

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Received 23 June 2015

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