ON APPROXIMATE ISOMETRIES IN MODULAR SPACES

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Let X_ρ be a ρ–complete modular space. A mapping $f : X_\rho \to X_\rho$ is called an ε-isometry if $|\rho\,(f(x) - f(y)) - \rho\,(x - y)\,| \leq \varepsilon$ for all $x, y \in X_\rho$. By making use of a direct method, it is shown that there exists an isometry $I : X_\rho \to X_\rho$ and constants A, B such that if $f : X_\rho \to X_\rho$ is an ε-isometry, then $\rho(f(x) - I(x)) \leq A\rho(x) + B\varepsilon$, where $I(x)$ is ρ-limit sequence $\{2^{-n} f(2^n x)\}$ for any $x \in X_\rho$; thus answering a question of Hyers and Ulam about the stability of isometries on ρ–complete modular spaces.

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1. INTRODUCTION

An *isometry* is a distance-preserving map between metric spaces. For normed spaces E and F, a function $f : E \to F$ is called a ε-*isometry* if

$$|\,\|f(x) - f(y)\| - \|x - y\|\,| \leq \varepsilon$$

for all $x, y \in E$ and some $\varepsilon \geq 0$. The basic question is how close is f to an actual isometry. In [12] by making use of a direct method, D.H. Hyers and S.M. Ulam proved that the surjective isometries of a complete Euclidean space are stable:

> Let E be a complete abstract Euclidean space. Assume that $f : E \to F$ is a surjective ε-isometry and $f(0) = 0$. Then there exists a surjective isometry $I : E \to E$ such that for all $x \in E$

$$\|f(x) - I(x)\| \leq 10\varepsilon. \tag{1.1}$$

D.G. Bourgin [5], R.D. Bourgin [7] and P.M. Gruber [11] continued the study of stability problems for isometries. In 1983, after many partial results extending over almost four decades, Gevirtz [10] extended this theorem to arbitrary Banach spaces E and F with the better estimate 5ε in (1.1). Finally, Omladič and Šemrl [26] showed that 2ε is a sharp constant in (1.1) for general Banach spaces.

J.W. Fickett [9] by making use of a different method from the direct method of Hyers and Ulam proved the Hyers-Ulam-Rassias stability of isometries on a restricted domain. In [18], S.-M. Jung by applying the fixed point method, presented a short and simple proof for the Hyers-Ulam-Rassias stability of isometries of which domain is a normed space and range is a Banach space in which the parallelogram law holds true.

On the other hand, G. Dolinar [8] proved the superstability property for isometries. In fact, he proved that for $p > 1$ every surjective (ε, p)-isometry $f : E \to F$ between finite-dimensional real Banach spaces is an isometry, where a mapping $f : E \to F$ is called an (ε, p)-isometry if f satisfies the inequality

$$| \, \|f(x) - f(y)\| - \|x - y\| \, | \leq \varepsilon \, \|x - y\|^p$$

for some $\varepsilon > 0$ and for all $x, y \in E$.

For more general information on the stability property for isometries and related topics, refer to [1, 3, 4, 6], [13–17], [14, 21], [28–34] and [36].

In this paper, we obtain an approximation result for near isometries on modular spaces. The theory of modulars on linear spaces and the corresponding theory of modular linear spaces were founded by Nakano [25] and were intensively developed by Amemiya, Koshi, Shimogaki, Yamamuro [19, 37] and others. Further and the most complete development of these theories are due to Orlicz, Mazur, Musielak, Luxemburg, Turpin [22, 24, 35] and their collaborators. In the present time, the theory of modulars and modular spaces is extensively applied, in particular, in the study of various Orlicz spaces [27] and interpolation theory [20, 23], which in turn have broad applications [24]. The importance for applications consists in the richness of the structure of modular function spaces, that–besides being Banach spaces (or F–spaces in more general setting)– are equipped with modular equivalent of norm or metric notions.

Definition 1.1. Let X be an arbitrary vector space.
(a) A functional $\rho : X \to [0, \infty]$ is called a modular if for arbitrary $x, y \in X$,
(i) $\rho(x) = 0$ if and only if $x = 0$,
(ii) $\rho(\alpha x) = \rho(x)$ for every scaler α with $|\alpha| = 1$,
(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ if and only if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$,
(b) if (iii) is replaced by
(iii)′ $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ if and only if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$,
then we say that ρ is a convex modular.

A modular ρ defines a corresponding modular space, *i.e.*, the vector space X_ρ given by

$$X_\rho = \{x \in X : \quad \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}.$$

Let ρ be a convex modular, the modular space X_ρ can be equipped with a norm called the Luxemburg norm, defined by

$$\|x\|_\rho = \inf\left\{\lambda > 0 \quad ; \quad \rho\left(\frac{x}{\lambda}\right) \leq 1\right\}.$$

A function modular is said to satisfy the Δ_2–condition if there exists $\kappa > 0$ such that $\rho(2x) \leq \kappa\rho(x)$ for all $x \in X_\rho$.

Definition 1.2. Let $\{x_n\}$ and x be in X_ρ. Then
(i) the sequence $\{x_n\}$, with $x_n \in X_\rho$, is ρ–convergent to x and write $x_n \xrightarrow{\rho} x$ if $\rho(x_n - x) \to 0$ as $n \to \infty$.
(ii) The sequence $\{x_n\}$, with $x_n \in X_\rho$, is called ρ–Cauchy if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.
(iii) A subset $\mathcal{S}$ of X_ρ is called ρ–complete if and only if any ρ–Cauchy sequence is ρ–convergent to an element of $\mathcal{S}$.

The modular ρ has the Fatou property if and only if $\rho(x) \leq \liminf_{n\to\infty}\rho(x_n)$ whenever the sequence $\{x_n\}$ is ρ–convergent to x.

Remark 1.1. Note that ρ is an increasing function. Suppose $0 < a < b$, then property (iii) of Definition 1.1 with $y = 0$ shows that $\rho(ax) = \rho\left(\frac{a}{b}bx\right) \leq \rho(bx)$ for all $x \in X$. Moreover, if ρ is a convex modular on X and $|\alpha| \leq 1$, then $\rho(\alpha x) \leq \alpha\rho(x)$ and also $\rho(x) \leq \frac{1}{2}\rho(2x)$ for all $x \in X$.

A convex function φ defined on the interval $[0, \infty)$, nondecreasing and continuous for $\alpha \geq 0$ and such that $\varphi(0) = 0, \varphi(\alpha) > 0$ for $\alpha > 0$, $\varphi(\alpha) \to \infty$ as $\alpha \to \infty$, is called an Orlicz function. The Orlicz function φ satisfies the Δ_2–condition if there exists $\kappa > 0$ such that $\varphi(2\alpha) \leq \varphi(\alpha)$ for all $\alpha > 0$. Let (Ω, Σ, μ) be a measure space. Let us consider the space $L^0(\mu)$ consisting of all measurable real–valued (or complex–valued) functions on Ω. Define for every $f \in L^0(\mu)$ the Orlicz modular $\rho_\varphi(f)$ by the formula

$$\rho_\varphi(f) = \int_\Omega \varphi(|f|)d\mu.$$

The associated modular function space with respect to this modular is called an Orlicz space, and will be denoted by $L^\varphi(\Omega, \mu)$ or briefly L^φ. In other words,

$$L^\varphi = \{f \in L^0(\mu) \quad | \quad \rho_\varphi(\lambda f) \to 0 \text{ as } \lambda \to 0\}$$

or equivalently as

$$L^\varphi = \{f \in L^0(\mu) \quad | \quad \rho_\varphi(\lambda f) < \infty \text{ for some } \lambda > 0\}.$$

It is known that the Orlicz space L^φ is ρ_φ–complete. Moreover, $(L^\varphi, \|.\|_{\rho_\varphi})$ is a Banach space, where the Luxemburg norm $\|.\|_{\rho_\varphi}$ is defined as follows

$$\|f\|_{\rho_\varphi} = \inf\left\{\lambda > 0 \quad : \quad \int_\Omega \varphi\left(\frac{|f|}{\lambda}\right)d\mu \leq 1\right\}.$$

Moreover, if $\mathfrak{L}$ is the space of sequences $x = \{x_i\}_{i=1}^{\infty}$ with real or complex terms x_i, $\varphi = \{\varphi_i\}_{i=1}^{\infty}$, φ_i are Orlicz functions and $\varrho_\varphi(x) = \Sigma_{i=1}^{\infty}\varphi_i(|x_i|)$, we shall write ℓ^φ in place of L^φ. The space ℓ^φ is called the generalized Orlicz sequence space. The motivation for the study of modular spaces (and Orlicz spaces) and many examples are detailed in [23–25, 27].

2. MAIN RESULTS

Throughout this paper, we assume that ρ is a convex modular on X with the Fatou property such that satisfies the Δ_2–condition with $0 < \kappa < 2$. By making use of a direct method, we establish the stability of ρ–isometries in modular spaces.

THEOREM 2.1. *Let X_ρ be a ρ–complete modular space. Suppose $f : X_\rho \to X_\rho$ satisfies the condition $f(0) = 0$ and an inequality of the form*

$$|\rho\left(f(x) - f(y)\right) - \rho\left(x - y\right)| \le \varepsilon \tag{2.1}$$

for all $x, y \in X_\rho$ and for some $\varepsilon \ge 0$. Then the ρ-limit

$$I(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for any $x \in X_\rho$ and

$$\rho(I(x) - I(y)) = \rho(x - y) \tag{2.2}$$

for all $x, y \in X_\rho$. Moreover, $I(2x) = 2I(x)$ and

$$\rho(f(x) - I(x)) \le \tfrac{\kappa^2+2\kappa}{4-2\kappa}\rho(x) + \tfrac{3\kappa}{2}\varepsilon \tag{2.3}$$

for all $x \in X_\rho$.

Proof. Putting $y = 0$ and replacing x by $2x$ in (2.1), we get

$$\rho(f(2x)) \le \rho(2x) + \varepsilon \tag{2.4}$$

for all $x \in X$. Putting $y = 2x$ in (2.1), we get

$$\rho(f(x) - f(2x)) \le \rho(x) + \varepsilon \tag{2.5}$$

for all $x \in X$. Since ρ is convex modular which satisfies the Δ_2–condition, by (2.4) and (2.5) we have

$$\begin{aligned} \rho\left(f(x) - \frac{f(2x)}{2}\right) &\le \frac{1}{2}\rho(2\{f(x) - f(2x)\}) + \frac{1}{2}\rho\left(2\left\{\frac{f(2x)}{2}\right\}\right) \\ &\le \frac{\kappa}{2}\rho(f(x) - f(2x)) + \frac{\kappa}{4}\rho(f(2x)) \\ &\le \frac{\kappa}{2}(\rho(x) + \varepsilon) + \frac{\kappa}{4}(\rho(2x) + \varepsilon). \end{aligned} \tag{2.6}$$

By replacing x by $x/2$ in last inequality, we have

$$\rho\left(f(\frac{x}{2})-\frac{f(x)}{2}\right) \leq \frac{\kappa}{2}(\rho\left(\frac{x}{2}\right)+\varepsilon)+\frac{\kappa}{4}(\rho(x)+\varepsilon)$$

$$\leq \frac{\kappa}{4}\rho(x)+\frac{\kappa}{2}\varepsilon+\frac{\kappa}{4}\rho(x)+\frac{\kappa}{4}\varepsilon=\frac{\kappa}{2}\rho(x)+\frac{3\kappa}{4}\varepsilon, \tag{2.7}$$

for all $x \in X$. By replacing x by $x/2$ in (2.5), we have

$$\rho\left(f(\frac{x}{4})-\frac{f(x/2)}{2}\right) \leq \frac{\kappa}{2}\rho\left(\frac{x}{2}\right)+\frac{3\kappa}{4}\varepsilon \leq \frac{\kappa}{4}\rho\left(x\right)+\frac{3\kappa}{4}\varepsilon.$$

Therefore, by (2.7) and last inequality, we have

$$\rho\left(f(\frac{x}{4})-\frac{f(x)}{4}\right) \leq \frac{\kappa}{2}\rho\left(f(\frac{x}{4})-\frac{f(x/2)}{2}\right)+\frac{\kappa}{2}\rho\left(\frac{f(x/2)}{2}-\frac{f(x)}{4}\right)$$

$$\leq \frac{\kappa^2}{2^2}\rho(x)+\left(\frac{3\kappa}{4}\right)^2\varepsilon.$$

By mathematical induction, we can easily see that

$$\rho\left(f(\frac{x}{2^n})-\frac{f(x)}{2^n}\right) \leq \frac{\kappa^n}{2^n}\rho(x)+\left(\frac{3\kappa}{4}\right)^n\varepsilon, \tag{2.8}$$

for all $x \in X$ and $n \in \mathbb{N}$.

We will now present that $\{2^{-n}f(2^n x)\}$ is a ρ-Cauchy sequence. If n and p are any positive integers, then it follows from (2.8) that

$$\rho\left(\frac{f(2^n x)}{2^n}-\frac{f(2^{n+p}x)}{2^{n+p}}\right) \leq \frac{1}{2^n}\rho\left(f(\frac{2^{n+p}x}{2^p})-\frac{f(2^{n+p}x)}{2^p}\right)$$

$$\leq \frac{1}{2^n}\left(\frac{\kappa^p}{2^p}\rho(2^{n+p}x)+\left(\frac{3\kappa}{4}\right)^p\varepsilon\right)$$

$$= \left(\frac{\kappa}{2}\right)^n\left(\frac{\kappa^2}{2}\right)^p\rho(x)+\frac{1}{2^n}\left(\frac{3\kappa}{4}\right)^p\varepsilon$$

$$\longrightarrow 0 \quad \text{as} \quad n \to \infty,$$

for any $x \in X$, which implies that $\{2^{-n}f(2^n x)\}$ is a ρ-Cauchy sequence. Since X_ρ is ρ-complete, we can define a function $I : X_\rho \to X_\rho$ by ρ-limit sequence $\{2^{-n}f(2^n x)\}$.

By (2.1) we have

$$\rho\left(x-y\right)-\varepsilon \leq \rho\left(f(x)-f(y)\right) \leq \rho\left(x-y\right)+\varepsilon. \tag{2.9}$$

By replacing x and y by $2^n x$ and $2^n y$ in inequality (2.9), respectively, we have

$$\rho\left(f(2^n x)-f(2^n y)\right) \leq \rho\left(2^n(x-y)\right)+\varepsilon, \tag{2.10}$$

$$\rho\left(f(2^n x)-f(2^n y)\right) \geq \rho\left(2^n(x-y)\right)-\varepsilon. \tag{2.11}$$

Since ρ satisfies the Δ_2–condition, by (2.10) we obtain

$$\begin{aligned}\rho\left(\frac{f(2^n x)}{2^n}-\frac{f(2^n y)}{2^n}\right) &\leq \frac{1}{2^n}\rho\left(f(2^n x)-f(2^n y)\right)\\ &\leq \frac{1}{2^n}\rho\left(2^n(x-y)\right)+\frac{\varepsilon}{2^n}\\ &\leq \left(\frac{\kappa}{2}\right)^n\rho\left(x-y\right)+\frac{\varepsilon}{2^n}\leq\rho\left(x-y\right)+\frac{\varepsilon}{2^n},\end{aligned}\tag{2.12}$$

for all $x,y\in X_\rho$. On other hand, by (2.11) we have

$$\begin{aligned}2^n\rho\left(x-y\right)\ \leq\ \rho\left(2^n(x-y)\right) &\leq \rho\left(f(2^n x)-f(2^n y)\right)+\varepsilon\\ &\leq \rho\left(2^n\left\{\frac{f(2^n x)}{2^n}-\frac{f(2^n y)}{2^n}\right\}\right)+\varepsilon\\ &\leq \kappa^n\rho\left(\frac{f(2^n x)}{2^n}-\frac{f(2^n y)}{2^n}\right)+\varepsilon.\end{aligned}$$

Dividing by 2^n the last expression we get

$$\begin{aligned}\rho\left(x-y\right) &\leq \left(\frac{\kappa}{2}\right)^n\rho\left(\frac{f(2^n x)}{2^n}-\frac{f(2^n y)}{2^n}\right)+\frac{\varepsilon}{2^n}\\ &\leq \rho\left(\frac{f(2^n x)}{2^n}-\frac{f(2^n y)}{2^n}\right)+\frac{\varepsilon}{2^n},\end{aligned}\tag{2.13}$$

for all $x,y\in X_\rho$. Letting $n\to\infty$ in (2.13) and (2.13) yield I ia an ρ-isometry, that is I satisfies (2.2).

Now we show that $I(2x)=2I(x)$. Putting $y=0$ in (2.1), we get

$$\rho(f(x))\leq\rho(x)+\varepsilon,$$

for all $x\in X_\rho$. By (2.6) and the last expression, we get

$$\begin{aligned}\rho\left(2f(x)-\frac{f(2x)}{2}\right) &\leq \frac{1}{2}\rho(2\{f(x)\})+\frac{1}{2}\rho\left(2\left\{f(x)-\frac{f(2x)}{2}\right\}\right)\\ &\leq \frac{\kappa}{2}\rho(f(x))+\frac{\kappa}{2}\rho\left(f(x)-\frac{f(2x)}{2}\right)\\ &\leq \left(1+\frac{\kappa}{2}+\frac{\kappa^2}{4}\right)\rho(x)+\left(1+\frac{3\kappa}{4}\right)\varepsilon.\end{aligned}$$

By replacing x by $2^n x$ in last inequality and dividing by 2^n, we have

$$\begin{aligned}\rho\left(2\times\frac{f(2^n x)}{2^n}-\frac{f(2^n\times 2x)}{2^n}\right) &\leq \frac{1}{2^n}\left(1+\frac{\kappa}{2}+\frac{\kappa^2}{4}\right)\rho(2^n x)+\frac{1}{2^n}\left(1+\frac{3\kappa}{4}\right)\varepsilon\\ &\leq \left(\frac{\kappa}{2}\right)^n\left(1+\frac{\kappa}{2}+\frac{\kappa^2}{4}\right)\rho(x)+\frac{1}{2^n}\left(1+\frac{3\kappa}{4}\right)\varepsilon,\end{aligned}$$

for all $x\in X_\rho$. Taking the limit, we deduce that $I(2x)=2I(x)$.

By replacing x by $2x$ in (2.6) and dividing by 2, we have

$$\rho\left(\frac{f(2x)}{2}-\frac{f(4x)}{4}\right)\le\left(\frac{\kappa^2}{4}+\frac{\kappa^3}{8}\right)\rho(x)+\frac{3\kappa}{8}\varepsilon.$$

The last expression together with (2.6), yield

$$\begin{aligned}\rho\left(f(x)-\frac{f(4x)}{4}\right) &\le \frac{\kappa}{2}\rho\left(f(x)-\frac{f(2x)}{2}\right)+\frac{\kappa}{2}\rho\left(\frac{f(2x)}{2}-\frac{f(4x)}{4}\right)\\ &\le \left\{\left(\frac{\kappa^2}{4}+\frac{\kappa^3}{8}\right)\rho(x)+\frac{3\kappa}{8}\varepsilon\right\}+\left\{\left(\frac{\kappa}{2}+\frac{\kappa^2}{4}\right)\rho(x)+\frac{3\kappa}{4}\varepsilon\right\}\\ &\le \left(\frac{\kappa}{2}+\frac{\kappa^2}{2}+\frac{\kappa^3}{8}\right)\rho(x)+\frac{9\kappa}{16}\varepsilon\end{aligned}$$

By mathematical induction, we can easily see that

$$\begin{aligned}\rho\left(f(x)-\frac{f(2^nx)}{2^n}\right) &\le \left(\frac{\kappa}{2}+\frac{\kappa^2}{2}+\cdots+\frac{\kappa^n}{2^{n-1}}+\frac{\kappa^{n+1}}{2^{n+1}}\right)\rho(x)+\frac{3\kappa(2^n-1)}{2^{n+1}}\varepsilon\\ &\le \left(\frac{\kappa}{2}+\sum_{i=1}^{n}\frac{\kappa^{i+1}}{2^i}\right)\rho(x)+\frac{3\kappa(2^n-1)}{2^{n+1}}\varepsilon\\ &\le \frac{\kappa^2+2\kappa}{4-2\kappa}\,\rho(x)+\frac{3\kappa}{2}\,\varepsilon,\end{aligned}$$

for all $x\in X_\rho$. Taking the limit, we deduce that I satisfies (2.3). □

Before presenting a corollary in this concept, we first introduce some useful concepts: we fix a real number β with $0<\beta\le 1$ and let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. Let X be a linear space over $\mathbb{K}$. A real-valued function $\|\ .\ \|_\beta$ is called a *β-norm* on X if and only if it satisfies

($\beta N1$) $\|x\|_\beta=0$ if and only if $x=0$;

($\beta N2$) $\|\lambda x\|_\beta=|\lambda|^\beta.\ \|x\|$ for all $\lambda\in\mathbb{K}$ and all $x\in X$;

($\beta N3$) $\|x+y\|_\beta\le\|x\|_\beta+\|y\|_\beta$ for all $x,y\in X$.

The pair $(X,\|\ .\ \|_\beta)$ is called a *β-normed space* (see [2]). A β-Banach space is a complete β-normed space. Notice that if $0<\beta<1$ and $\kappa:=2^\beta$, then $0<\kappa<2$ and $\|2x\|_\beta\le\kappa\,\|x\|_\beta$.

COROLLARY 2.1. *Let X be a β-Banach space with $0<\beta<1$. Suppose $f:X\to X$ satisfies the condition $f(0)=0$ and an inequality of the form*

$$|\,\|f(x)-f(y)\|_\beta-\|x-y\|_\beta\,|\le\varepsilon$$

for all $x,y\in X$ and for some $\varepsilon\ge 0$. Then

$$I(x)=\lim_{n\to\infty}\frac{f(2^nx)}{2^n}$$

exists for any $x \in X$, such that $I(2x) = 2I(x)$ and $\|I(x) - I(y)\|_\beta = \|x - y\|_\beta$ for all $x, y \in X$. Moreover

$$\|f(x) - I(x)\|_\beta \le \frac{4^\beta + 2^{\beta+1}}{4 - 2^{\beta+1}} \|x\|_\beta + 3(2^{\beta-1})\varepsilon$$

for all $x \in X$.

Proof. Set $\rho(x) = \|x\|_\beta$ and $\kappa = 2^\beta$ and apply Theorem 2.1. □

REFERENCES

[1] J.A. Baker, *Isometries in normed spaces.* Amer. Math. Monthly **78** (1971), 655–658.
[2] V.K. Balachandran, *Topological Algebras.* Narosa Publishing House, New Delhi, Madras, Bombay, Calcutta, London, 1999.
[3] W. Benz and H. Berens, *A contribution to a theorem of Ulam and Mazur.* Aequationes Math. **34** (1987), 61–63.
[4] R. Bhatia and P. Šemrl, *Approximate isometries on Euclidean spaces.* Amer. Math. Monthly **104** (1997), 497–504.
[5] D.G. Bourgin, *Approximate isometries.* Bull. Amer. Math. Soc. **52** (1946), 704–714.
[6] D.G. Bourgin, *Approximately isometric and multiplicative transformations on continuous function rings.* Duke Math. J. **16** (1949), 385–397.
[7] R.D. Bourgin, *Approximate isometries on finite dimensional Banach spaces.* Trans. Amer. Math. Soc. **207** (1975), 309–328.
[8] G. Dolinar, *Generalized stability of isometries.* J. Math. Anal. Appl. **242** (2000), 39–56.
[9] J.W. Fickett, *Approximate isometries on bounded sets with an application to measure theory.* Studia Math. **72** (1981), 37–46.
[10] J. Gevirtz, *Stability of isometries on Banach spaces.* Proc. Amer. Math. Soc. **89** (1983), 633–636.
[11] P.M. Gruber, *Stability of isometries.* Trans. Amer. Math. Soc. **245** (1978), 263–277.
[12] D.H. Hyers and S.M. Ulam, *On approximate isometries.* Bull. Amer. Math. Soc. **51** (1945), 288–292.
[13] D.H. Hyers and S.M. Ulam, *Approximate isometries of the space of continuous functions.* Ann. of Math. **48** (1947), 285–289.
[14] S.-M. Jung, *Hyers–Ulam–Rassias Stability of Functional Equations in Nonlinear Analysis Series.* Springer Optimization and Its Applications **48**, Springer, 2011.
[15] S.-M. Jung, *Hyers–Ulam–Rassias stability of isometries on restricted domains.* Nonlinear Stud. **8** (2001), 125–134.
[16] S.-M. Jung, *Asymptotic properties of isometries.* J. Math. Anal. Appl. **276** (2002), 642–653.
[17] S.-M. Jung and B. Kim, *Stability of isometries on restricted domains.* J. Korean Math. Soc. **37** (2000), 125–137.
[18] S.-M. Jung and T.-S. Kim, *A fixed point approach to the stability of isometries.* J. Math. Anal. Appl. **329** (2007), 879–890.
[19] S. Koshi and T. Shimogaki, *On F-norms of quasi-modular spaces.* J. Fac. Sci. Hokkaido Univ. Ser. I **15** (1961), *3*, 202–218.
[20] M. Krbec, *Modular interpolation spaces.* Z. Anal. Anwendungen **1** (1982), 25–40.
[21] J. Lindenstrauss and A. Szankowski, *Nonlinear perturbations of isometries*, Astérisque **131** (1985), 357–371.

[22] W.A. Luxemburg, *Banach function spaces.* Ph. D. thesis, Delft University of technology, Delft, The Netherlands, 1959.

[23] L. Maligranda, *Orlicz Spaces and Interpolation.* In: Seminars in Math., Vol. **5**, Univ. of Campinas, Brazil, 1989.

[24] J. Musielak, *Orlicz Spaces and Modular Spaces.* In: Lecture Notes in Math. Vol. **1034**, Springer-verlag, Berlin, 1983.

[25] H. Nakano, *Modulared Semi-Ordered Linear Spaces.* In: Tokyo Math. Book Ser., Vol. **1**, Maruzen Co., Tokyo, 1950.

[26] M. Omladič and P. Šemrl, *On nonlinear perturbations of isometries.* Math. Ann. **303** (1995), 617–628.

[27] W. Orlicz, *Collected Papers*, Vols. **I, II**, PWN, Warszawa, 1988.

[28] Th.M. Rassias, *Properties of isometric mappings.* J. Math. Anal. Appl. **235** (1999), 108–121.

[29] Th.M. Rassias, *Isometries and approximate isometries.* Internat. J. Math. Math. Sci. **25** (2001), 73–91.

[30] Th.M. Rassias and C.S. Sharma, *Properties of isometries.* J. Natural Geom. **3** (1993), 1–38.

[31] P. Šemrl, *Hyers-Ulam-stability of isometries on Banach spaces.* Aequationes Math. **58** (1999), 157–162.

[32] F. Skof, *Sulle δ-isometrie negli spazi normati.* Rend. Mat. Ser. VII, Roma **10** (1990), 853–866.

[33] F. Skof, *On asymptotically isometric operators in normed spaces.* Istit. Lombardo Acad. Sci. Lett. Rend. A **131** (1997), 117–129.

[34] R.L. Swain, *Approximate isometries in bounded spaces.* Proc. Amer. Math. Soc. **2** (1951), 727–729.

[35] Ph. Turpin, *Fubini inequalities and bounded multiplier property in generalized modular spaces.* Comment. Math., Tomus specialis in honorem Ladislai Orlicz I (1978), 331–353.

[36] J. Väisälä, *Isometric approximation property of unbounded sets.* Results Math. **43** (2003), 359–372.

[37] S. Yamamuro, *On conjugate spaces of Nakano spaces.* Trans. Amer. Math. Soc. **90** (1959), 291–311.

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