SOME NEW CHARACTERIZATION OF SOLVABLE PST-GROUPS

G.R. REZAEEZADEH and S.E. MIRDAMADI

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Let H be a subgroup of a finite group G. We say that H is τSS-permutable (rep. τSS-normal) in G if H has a supplement K in G such that H permutes (resp. is normalized) with every Sylow subgroup X of K with $(|H|, |X^G|) \neq 1$. In this paper, the Structure of τSS-permutable subgroups, and finite groups in which τSS-permutability is a transitive relation are described. It is shown that a finite solvable group G is a PST-group if and only if every subgroup of $F^*(G)$ is τSS-permutable in G.

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1. INTRODUCTION

Throughout this paper, all the groups are considered to be finite. For a group G, we denote by $\pi(G)$ the set of prime numbers dividing $|G|$; H^G is the normal closure of H in G, that is, the intersection of all normal subgroups of G containing H; $F^*(G)$ the generalized Fitting subgroup of G, that is, the product of all normal quasinilpotent subgroup of G. Two subgroups H and K of a group G are said to be permutable if $HK = KH$. A subgroup H of a group G is said to be permutable (resp. S-permutable) in G if H permutes with all the subgroups (resp. Sylow subgroups) of G. A group G is called a T-group (resp. PT-group, PST-group) if normality (resp. permutability, S-permutability) is a transitive relation, that is if H and K are subgroups of G such that H is normal (resp. permutable, S-permutable) in K and K is normal (resp. permutable, S-permutable) in G, then H is normal (resp. permutable, S-permutable) in G. Let G be a group and $p \in \pi(G)$. A group G is said to be y_p-group when, for all p-subgroups H and K of G with $H \leq K$, H is S-permutable in $N_G(K)$.

By [11], a group G is a PST-group if and only if every subnormal subgroup of G is S-permutable in G. Agrawal [1] showed that a group G is a solvable PST-group if and only if the nilpotent residual L of G is a normal abelian Hall subgroup of G upon which G acts by conjugation as power automorphisms.

In particular, a solvable PST-group is supersolvable. The structure of PST-groups has been investigated by many authors, see for example [2, 4, 7].

A subgroup H of a group G is said to be τ-quasinormal in G if $HG_p = G_pH$ for every $G_p \in Syl_p(G)$, where p is prime number, with $(|H|, p) = 1$ and $\big(|H|, |(G_p)^G|\big) \neq 1$. A group G is called a TQT-group [14] if τ-quasinormality is a transitive relation in G.

A subgroup H of a group G is said to be SS-permutable (resp. NSS-permutable) [9, 12] in G if H has a supplement (resp. normal supplement) K in G such that H permutes with every Sylow subgroup of K.

In this case, K is called an SS-permutable supplement (resp. NSS-permutable supplement) of H in G. A group G is said to be SST-group (resp. NSST-group) [9] if SS-permutability (resp. NSS-permutability) is a transitive relation. The following results have been established by Chen and Guo.

Theorem 1.1 ([9] Theorem 1.5). *Let G be a group. Then the following statements are equivalent:*
(1) G is solvable and every subnormal subgroup of G is SS-permutable in G.
(2) G is solvable and every subnormal subgroup of G is NSS-permutable in G.
(3) Every subgroup of $F^*(G)$ is SS-permutable in G.
(4) Every subgroup of $F^*(G)$ is NSS-permutable in G.
(5) G is a solvable PST-group.

Theorem 1.2 ([9] Theorem 1.6). *Let G be a group. Then the following statements are equivalent:*
(1) Whenever $H \leq K$ are two p-subgroups of G with $p \in \pi(G)$, H is SS-permutable in $N_G(K)$.
(2) Whenever $H \leq K$ are two p-subgroups of G with $p \in \pi(G)$, H is NSS-permutable in $N_G(K)$.
(3) G is a solvable PST-group.

Theorem 1.3 ([9] Theorem 1.7). *Let G be a solvable group. Then the following statements are equivalent:*
(1) G is an SST-group.
(2) G is an NSST-group.
(3) Every subgroup of G is SS-permutable in G.
(4) Every subgroup of G is NSS-permutable in G.
(5) Every subgroup of G of prime power order is SS-permutable in G.
(6) Every subgroup of G of prime power order is NSS-permutable in G.

Definition 1.4. We say that a subgroup H of a group G is τSS-permutable (resp. τNSS-permutable) in G if H has a supplement (resp. normal supplement) K in G such that H permutes with every Sylow subgroup X of K with

$(|H|, |X^G|) \neq 1$. In this case, K is called a τSS-permutable supplement (resp. τNSS-permutable supplement) of H in G.

It is clear that every τNSS-permutable subgroup of a group G is a τSS-permutable subgroup in G, but the next example illustrates that the converse is not true.

Example 1. Let $G = \mathbb{A}_5$ and $H = \mathbb{A}_4$, where $\mathbb{A}_5$ and $\mathbb{A}_4$ denote the alternating group of degree 5 and 4, respectively. Clearly every 5-Sylow subgroup of G is a τSS-permutable supplement of H in G. If H is τNSS-permutable in G, then G is the only τNSS-permutable supplement of H in G. This implies that H permutes with every Sylow subgroup X of G with $(|X|, |H^G|) \neq 1$. Since $\mathbb{A}_5$ is simple, H is S-permutable in G. Hence H is subnormal in G, which is impossible. Therefore, H is not τNSS-permutable in G.

Definition 1.5. We say that a group G is a τSST-group (resp. a τNSST-group) if τSS-permutability (resp. τNSS-permutability) is a transitive relation.

A subgroup H of a group G is said to be seminormal (resp. S-seminormal) [8] in G if H is normalized by all subgroups (resp. Sylow subgroups) X of G such that $(|H|, |X|) = 1$. Beidlema and Raglad [8] showed that a subgroup H of a group G is seminormal in G if ad only if it is S-seminormal in G. They also showed that if a p-subgroup H of a group G is seminormal in G, then H is subnormal in G. A subgroup H of a group G is said to be τ-seminormal [3] in G if H is normalized by all Sylow subgroups X of G such that $(|H|, |X|) = 1$ ad $(|H|, |X^G|) \neq 1$. Ballester-Bolinches and Beidlema [3] showed that a finite solvable group G is a PST-group if and only if every subgroup of $\mathrm{F}(G)$ is τ-seminormal in G.

Definition 1.6. We say that a subgroup H of a group G is τSS-normal in G if H has a supplement K in G such that H is normalized by all Sylow subgroups X of K with $(|H|, |X^G|) \neq 1$. In this case, K is called a τSS-normal supplement of H in G

The aim of this paper is to complement some earlier characterizations for some classes of groups, such as PST-groups, by using the τSS-permutable and τSS-normal concepts. For example, we extend Theorems 1.1, 1.2, and 1.3 using the concept of τSS-permutable subgroup.

THEOREM 1.7. *Let G be a solvable τSST-group. Then G is a supersolvable group.*

THEOREM 1.8. *Let G be a group. Then the following statements are equivalent:*

(1) G is solvable and every subnormal subgroup of G is τSS-permutable in G.

(2) G is solvable and every subnormal subgroup of G is τNSS-permutable in G.
(3) Every subgroup of $F^*(G)$ is τSS-permutable in G.
(4) Every subgroup of $F^*(G)$ is τNSS-permutable in G.
(5) G is a solvable PST-group.

THEOREM 1.9. *Let G be a group. Then the following statements are equivalent:*
(1) Whenever $H \leq K$ are two p-subgroups of G with $p \in \pi(G)$, H is τSS-permutable in $N_G(K)$.
(2) Whenever $H \leq K$ are two p-subgroups of G with $p \in \pi(G)$, H is τNSS-permutable in $N_G(K)$.
(3) G is a solvable PST-group.

THEOREM 1.10. *Let G be a solvable group. Then the following statements are equivalent:*
(1) G is a τSST-group.
(2) G is a $\tau NSST$-group.
(3) Every subgroup of G is τSS-permutable in G.
(4) Every subgroup of G is τNSS-permutable in G.

THEOREM 1.11. *Let G be a group and for any $p \in \pi(G)$, every p-subgroup of G is τSS-normal in G. Then G is a PST-group.*

2. PRELIMINARIES

In this section, we give some results which are useful in the sequel. The following first lemma is easy to prove.

LEMMA 2.1. *Let N_1 and N_2 be subgroups of a group G and assume that $N_1N_2 \leq G$. If P_1 and P_2 are Sylow p-subgroups of N_1 and N_2 respectively, where $p \in \pi(G)$, and $P_1P_2 \leq N_1N_2$, then P_1P_2 is a Sylow p-subgroup of N_1N_2.*

LEMMA 2.2. *Suppose that a subgroup H of a group G is τSS-permutable (resp. τNSS-permutable) in G with a τSS-permutable supplement (resp. τNSS-permutable supplement) K, $L \leq G$ and $N \trianglelefteq G$. Then:*
(1) HN/N is τSS-permutable (resp. τNSS-permutable) in G/N.
(2) If N is a p-group, $N \leq L$, and L/N is τSS-permutable (resp. τNSS-permutable) in G/N, then L is τSS-permutable (resp. τNSS-permutable) in G.
(3) H is τ-quasinormal in G.
(4) If $H \leq O_p(G)$, for some $p \in \pi(G)$, then H is S-permutable in G.
(5) If N is nilpotent, then NK is an τSS-permutable supplement (resp. τNSS-permutable supplement) of H in G.

Proof. We only prove the statements for τSS-permutable subgroups. For τNSS-permutable subgroups, the statements can be handled similarly.
(1) It is clear that KN/N is a supplement of HN/N in G/N. Let A/N be a Sylow p-subgroup of KN/N, where p is a prime number, such that $(|HN/N|, |(A/N)^{G/N}|) \neq 1$. Note that A/N has the form XN/N, where X is a Sylow p-subgroup of KN. Further, by (VI. 4.7) in [10], there exist Sylow p-subgroups K_p, N_p, and Y of K, N, and G, respectively such that $Y = K_pN_p$. By Sylow's theorem, $XN/N = (YN/N)^{kN} = K_p^kN/N$, for some $k \in K$. Since $\left(K_p^kN/N\right)^{G/N} \leq ((K_p^k)^GN)/N$, by hypothesis we obtain that $HK_p^k = K_p^kH$. Therefore, we have:

$$\frac{HN}{N}\frac{XN}{N} = \frac{HK_p^kN}{N} = \frac{K_p^kHN}{N} = \frac{XN}{N}\frac{HN}{N}.$$

This shows that HN/N is τSS-permutable in G/N.
(2) Let B/N be a τSS-permutable supplement of L/N in G/N. Then B is a supplement of L in G. Let X be a Sylow q-subgroup of B such that $(|L|, |X^G|) \neq 1$, where $q \in \pi(G)$. Then XN/N is a Sylow q-subgroup of B/N. If $q \neq p$, we have that $L = N$ or $(|L/N|, |(XN/N)^{G/N}|) \neq 1$. Hence $(XN/N)(L/N) = (L/N)(XN/N)$ implies that $XL = LX$. Now suppose that $q = p$. If $(|L/N|, |(XN/N)^G|) = 1$, then $X = N$ or $L = N$. Hence $XL = LX$. Otherwise $(|L/N|, |(XN/N)^G|) \neq 1$ yields $XL = LX$. This completes the proof.
(3) Let X be a Sylow subgroup of G such that $(|H|, |X|) = 1$ and $\left(|H|, |X^G|\right) \neq 1$. Then there exists an element $h \in H$ such that $X^h \leq K$. It follows that $HX^h = X^hH$ and so $HX = XH$. Therefore, H is τ-quasinormal in G.
(4) By (3) and Lemma 2.2 in [13], we can obtain that H is S-permutable in G.
(5) Since N is nilpotent, for every $N_p \in Syl_p(N)$, where $p \in \pi(G)$, we have $N_p \trianglelefteq G$. By Lemma 2.1, for every $K_p \in Syl_p(K)$, $N_pK_p \in Syl_p(NK)$. Let X be a Sylow p-subgroup of NK such that $(|H|, |X^G|) \neq 1$. Then there exists an element $g \in NK$ such that $X = (N_pK_p)^g$ for some $N_p \in Syl_p(N)$ and $K_p \in Syl_p(K)$. Note that $((N_pK_p)^g)^G = N_pK_p^G$. If $(|H|, |K_p^G|) = 1$, then $K_p = 1$. Otherwise $(|H|, |K_p^G|) \neq 1$ and so $HK_p = K_pH$. Therefore, $XH = HX$ and this shows that NK is an τSS-permutable supplement of H in G. $\square$

Lemma 2.3 ([6] Lemma 2.1.3). *Let p be a prime and N a normal p-subgroup of a group G. Then all subgroups of N are S-permutable in G if and only if all chief factors of G below N are cyclic and G-isomorphic when regarded as modules over G.*

Proposition 2.4. *Suppose that a subgroup H of a group G is τSS-permutable*

(resp. τNSS-permutable) in G with a τSS-permutable supplement (resp. τNSS-permutable supplement) K. Then:
(1) If $H \leq L$, then H is τSS-permutable (resp. τNSS-permutable) in L.
(2) Every conjugate of K in G is τSS-permutable supplement (resp. τNSS-permutable supplement) of H in G.
(3) If H is a p-subgroup, where $p \in \pi(G)$, then $HK_p \in Syl_p(G)$ for every $K_p \in Syl_p(K)$.

Proof. We only prove the statements for τSS-permutable subgroups. For τNSS-permutable subgroups, the statements can be handled similarly.
(1) By Dedekind identity, we have $L = (HK) \cap L = H(K \cap L)$. This means that $(K \cap L)$ is a supplement of H in L. Now let X be a Sylow subgroup of $K \cap L$ such that $(|H|, |X^L|) \neq 1$. Then there exists $Y \in Syl(K)$ such that $X \leq Y$. Since $X \leq (Y^G \cap L) \trianglelefteq L$, we have $X^L \leq Y^G$ and so $HY = YH$. Consequently, $HY \cap L = H(Y \cap L) = HX$ and $L \cap (YH) = (L \cap Y)H = XH$. Therefore, $HX = XH$ and this shows that H is τSS-permutable in L.
(2) If $g \in G$, it is easy to see that $K^g H = G$. Now suppose that X is a Sylow subgroup of K^g such that $\left(|H|, |X^G|\right) \neq 1$. Then $X^{g^{-1}}$ is a Sylow subgroup of K and $(|H|, |(X^{g^{-1}})^G|) \neq 1$. Hence $X^{g^{-1}}H = HX^{g^{-1}}$ implies that $XH = HX$. Therefore, for every $g \in G$, K^g is a τSS-permutable supplement of H in G.
(3)Let $K_p \in Syl_p(K)$. Then $(|H|, |K_p^G|) \neq 1$ and so $HK_p = K_pH$. Now by Lemma 2.1, HK_p is a Sylow p-subgroup of G. $\square$

Lemma 2.5. *Suppose that a subgroup H of a group G is τSS-normal in G with an τSS-normal supplement K, $L \leq G$, and $N \trianglelefteq G$. Then:*
(1) HN/N is τSS-normal in G/N.
(2) if N is a p-group, $N \leq L$, and L/N is τSS-normal in G/N, then L is τSS-normal in G.
(3) If $H \leq L$, then H is τSS-normal in L.
(4) H is τ-seminormal in G.
(5) If $H \leq F(G)$, then H is normal in G.

Proof. Using ideas similar to those used in the proof of Proposition 2.4(1) and of Lemma 2.2(1,2,3), one can prove (1)-(4).
(5) Let Q be a Sylow q-subgroup of G with $q \in \pi$, where $\pi = \pi(G) - \pi(H)$. If $Q \nleq N_G(H)$, by hypothesis $Q^G \leq O_\pi(G)$. Hence $H \leq O_{\pi'}(G) \leq C_G(Q)$, a contradiction. Thus $Q \leq N_G(H)$. Note that for every Sylow p-subgroup P of K, $P \leq N_G(H)$, where $p \in \pi(H)$. Therefore, $K \leq N_G(H)$ and so $H \trianglelefteq G$. $\square$

Lemma 2.6. *Suppose that a subgroup H of a group G is τSS-normal in G with the τSS-normal supplement G and $L \leq G$. If $H \leq L$, then H is τSS-normal in L with the τSS-normal supplement L.*

Proof. The proof is similar to that of Proposition 2.4(1). □

LEMMA 2.7 ([14] Theorem 1.2). *Let G be a group. Then every subgroup of $F^*(G)$ is τ-quasinormal in G if and only if G is a solvable PST-group.*

PROPOSITION 2.8. *Let G be a nilpotent-by-abelian group and $H \leq G$. Then H is τSS-permutable in G if and only if H is τNSS-permutable in G.*

Proof. The sufficiency is clear. Let H be τSS-permutable in G with a τSS-permutable supplement K. Since G is nilpotent-by-abelian, $G' \leq F(G)$ and so $G/F(G)$ is abelian. By Lemma 2.2(5), $F(G)K$ is also an τSS-permutable supplement of H in G. As $F(G)K \trianglelefteq G$, $F(G)K$ is an τNSS-permutable supplement of H in G. Therefore, H is τNSS-permutable in G. □

3. PROOF OF THE MAIN RESULTS

Proof of Theorem 1.7. We only prove statement for τSS-permutable. Let N be a minimal normal subgroup of G. Note that G is a solvable group and so N is elementary abelian p-subgroup of G. By Lemma 2.2(1) and (2), G/N is also a τSST-group. By induction G/N is an supersolvable group. Now we only need to prove that N is simple. Since G is a τSST-group, all subgroups of N are τSS-permutable in G. By Lemma 2.2(4), all subgroups of N are S-permutable in G. By Lemma 2.3, N is cyclic and so N is simple. Therefore, G is supersolvable group. □

Proof of Theorem 1.8. It is clear that (2) implies (1), and (4) implies (3). Assume that G is a solvable PST-group. Then every subnormal subgroup of G is S-permutable in G and so τNSS-permutable in G. Therefore, (5) implies (2).

Now we show that (3) implies (5). Suppose that every subgroup of $F^*(G)$ is τSS-permutable in G. Then by Lemma 2.2(3), every subgroup of $F^*(G)$ is τ-quasinormal in G. Hence by Lemma 2.7, G is a solvable PST-group and thus (5) holds.

Finally, we prove that (1) implies (4). Assume that G is a solvable and every subnormal subgroup of G is τSS-permutable in G. Then $F^*(G) = F(G)$ by [10] (X. Corollary 13.7(d)). Therefore, every subgroup of $F(G)$ is τSS-permutable in G and so every subgroup of prime power order of $F(G)$ is τSS-permutable in G. By Lemma 2.2(4), every subgroup of prime power order of $F(G)$ is S-permutable and so every subgroup of $F(G)$ is S-permutable in G. Thus every subgroup of $F(G)$ is τNSS-permutable in G. □

Proof of Theorem 1.9. It is clear that (2) implies (1). Assume that (1) holds. By Lemma 2.2(4), whenever $H \leq K$ are two p-subgroup of G with $p \in \pi(G)$, H is S-permutable in $N_G(K)$. It follows from [5] (3, Theorem 4),

that G is a solvable PST-group and so (3) follows.
By [5] (3, Theorem 4), again, we also see that (3) implies (2). □

Proof of Theorem 1.10. ($1 \leftrightarrow 3$) Clearly (3) implies (1).
Now suppose that G is a τSST-group. Then every subnormal subgroup of G is τSS-permutable in G. By Theorem 1.8, G is a PST-group. Let L be the nilpotent residual of G. Then for any subgroup H of G, H is τSS-permutable in HL, because L is a normal abelian subgroup of G. Since HL is subnormal subgroup of G, we deduce that H is τSS-permutable in G.
($2 \leftrightarrow 4$) With a similar argument as above, we get that (2) is equivalent to (4).
($1 \leftrightarrow 2$) It is clear that (2) implies (1). Now suppose that G is a τSST-group. By Theorem 1.7 and Lemma 2.8, it easy to see that (1) implies (2). □

Proof of Theorem 1.11. Let G be a group and for any $p \in \pi(G)$, every p-subgroup of G is τSS-normal in G. By Theorem (4) in [5], it is enough to show that G is a Y_p-group for all primes $p \in \pi(G)$. Let $H \leq K$ be p-subgroup of G, where $p \in \pi(G)$. By hypothesis, H is τSS-normal in G and so by Lemma 2.5(3,5), $H \trianglelefteq N_G(K)$. Hence H is S-permutable in $N_G(K)$. This means that H is a Y_p-group. □

4. APPLICATIONS

In this section, we will establish some applications that involve the concepts of τ-quasinormal, τSS-normal, and τ-seminormal subgroups.

PROPOSITION 4.1. *A subgroup H of a group G is τ-quasinormal in G if and only if H has a supplement K in G such that H permutes with every Sylow subgroup X of K such that $(|H|, |X|) = 1$ and $(|H|, |X^G|) \neq 1$.*

Proof. In Lemma 2.2(3), the necessity is proved and the sufficiency is obvious. □

PROPOSITION 4.2. *Let G be a group. Then the following statements are equivalent:*
(1) Whenever $H \leq K$ are two p-subgroups of G with $p \in \pi(G)$, H is τ-quasinormal in $N_G(K)$.
(2) G is a solvable PST-group.

Proof. The proof follows by an argument similar to that in Theorem 1.9, except that here we use Lemma 2.1 in [14] instead of Lemma 2.2(4). □

PROPOSITION 4.3. *Let G be a solvable group. Then the following statements are equivalent:*
(1) G is a TQT-group.
(2) Every subgroup of G is τ-quasinormal in G.

Proof. For the proof one may use an argument similar to that used in the proof of Theorem 1.10, and also Theorem 1.2 in [14], which shows that if every subgroup of a group G is τ-quasinormal in G, then G is a PST-group. □

PROPOSITION 4.4. *Let G be a solvable group. If τSS-normality is a transitive relation in G, then G is a supersolvable group.*

Proof. Let N be a minimal normal subgroup of G. Note that G is a solvable group and so N is elementary abelian p-subgroup of G. By Lemma 2.5(1,2), τSS-normality is a transitive relation in G/N. By induction G/N is a supersolvable group. Since τSS-normality is a transitive relation, by Lemma 2.5(5), all subgroups of N are normal in G and so N is simple. Therefore, G is a supersolvable group. □

PROPOSITION 4.5. *Let G be a group and every subgroups of $F^*(G)$ is τSS-normal in G. Then G is a solvable PST-group.*

Proof. Assume that every subgroup of $F^*(G)$ is τSS-normal in G. Then every subgroup of $F^*(G)$ is τSS-permutable in G. Hence by Theorem 1.8, G is a solvable PST-group. □

PROPOSITION 4.6. *A subgroup H of a group G is τ-seminormal in G if and only if H has a supplement K in G such that H is normalized by all Sylow subgroups X of K such that $(|H|, |X|) = 1$ and $(|H|, |X^G|) \neq 1$.*

Proof. The necessity follows by Lemma 2.5(4), and the sufficiency is obvious. □

PROPOSITION 4.7. *Let G be a group and whenever $H \leq K$ are two p-subgroups of G with $p \in \pi(G)$, H is τSS-normal in $N_G(K)$, then G is a solvable PST-group.*

Proof. Assume that whenever $H \leq K$ are two p-subgroups of G with $p \in \pi(G)$, H is τSS-normal in $N_G(K)$. Then H is τSS-permutable in $N_G(K)$. Now by Theorem 1.9, G is a solvable PST-group. □

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Shahrekord University,
Faculty of Mathematical Sciences,
P.O. Box 115, Shahrekord, Iran
rezaeezadeh@sci.sku.ac.ir

Shahrekord University,
Faculty of Mathematical Sciences,
P.O. Box 115, Shahrekord, Iran
ebrahimmirdamadi@stu.sku.ac.ir