ON THE TOTAL IRREGULARITY STRENGTH OF DISJOINT UNION OF ARBITRARY GRAPHS

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We deal with the modifications of the well-known irregular assignments, namely vertex irregular total labelings, edge irregular total labelings and totally irregular total labelings of graphs. In the paper, we study the total vertex (edge) irregularity strength and total irregularity strength for disjoint union of arbitrary graphs and we establish the upper bounds for these invariants.

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1. INTRODUCTION

Let G be a connected, simple and undirected graph with vertex set $V(G)$ and edge set $E(G)$. A total labeling $f : V(G) \cup E(G) \to \{1, 2, \dots, k\}$ is called a *vertex irregular total k-labeling* of G if every two distinct vertices x and y in $V(G)$ satisfy $wt_f(x) \neq wt_f(y)$, where

$$wt_f(x) = f(x) + \sum_{xz \in E(G)} f(xz).$$

The *total vertex irregularity strength* of G, denoted by $\mathrm{tvs}(G)$, is the minimum k for which G has a vertex irregular total k-labeling. In [5] are given the bounds for the total vertex irregularity strength of a graph G with minimum degree $\delta(G)$ and maximum degree $\Delta(G)$ by the following form:

$$(1) \qquad \left\lceil \frac{|V(G)| + \delta(G)}{\Delta(G) + 1} \right\rceil \leq \mathrm{tvs}(G) \leq |V(G)| + \Delta(G) - 2\delta(G) + 1.$$

Moreover, there is proved that $\mathrm{tvs}(G) \leq |V(G)| - 1 - \lfloor (|V(G)| - 2)/(\Delta + 1) \rfloor$ for a graph with no component of order ≤ 2. Przybylo [19] proved that $\mathrm{tvs}(G) < 32|V(G)|/\delta(G) + 8$ in general and $\mathrm{tvs}(G) < 8|V(G)|/r + 3$ for r-regular graphs. This was then improved in [4] that $\mathrm{tvs}(G) \leq 3\left\lceil |V(G)|/\delta(G) \right\rceil + 1 \leq$

$3|V(G)|/\delta(G)+4$. Recently, Majerski and Przybylo in [12] based on a random ordering of the vertices proved that if $\delta(G) \geq n^{0.5}\ln n$ then $\mathrm{tvs}(G) \leq (2+o(1))|V(G)|/\delta(G)+4$. The exact values of the total vertex irregularity strength for several families of graphs can be found in [16, 17] and [18].

Furthermore, in [5] Bača, Jendroľ, Miller and Ryan defined the total labeling φ to be an *edge irregular total k-labeling* of the graph G if for every two different edges x_1x_2 and $x_1'x_2'$ of G one has $wt_\varphi(x_1x_2) \neq wt_\varphi(x_1'x_2')$, where

$$wt_\varphi(x_1x_2) = \varphi(x_1) + \varphi(x_1x_2) + \varphi(x_2).$$

The *total edge irregularity strength* of G, denoted by $\mathrm{tes}(G)$, is defined as the minimum k for which G has an edge irregular total k-labeling.

In [5] is proved that for any graph G with a non-empty edge set $E(G)$

$$\left\lceil \frac{|E(G)|+2}{3} \right\rceil \leq tes(G) \leq |E(G)|. \tag{2}$$

Ivančo and Jendroľ [9] posed a conjecture that for arbitrary graph G different from K_5,

$$\mathrm{tes}(G) = \max\left\{ \left\lceil \frac{|E(G)|+2}{3} \right\rceil, \left\lceil \frac{\Delta(G)+1}{2} \right\rceil \right\}.$$

This conjecture has been verified for complete graphs and complete bipartite graphs in [10] and [11], for the Cartesian, categorical and strong products of two paths in [1, 2, 14], for the categorical product of two cycles in [3], for generalized Petersen graphs in [8], for generalized prisms in [6], for corona product of a path with certain graphs in [15] and for large dense graphs with $(|E(G)|+2)/3 \leq (\Delta(G)+1)/2$ in [7].

Combining previous modifications of the irregularity strength, Marzuki, Salman and Miller [13] introduced a new irregular total k-labeling of a graph G called *totally irregular total* k-labeling, which is required to be at the same time vertex irregular total and also edge irregular total. The minimum k for which a graph G has a totally irregular total k-labeling is called the *total irregularity strength* of G and it is denoted by $\mathrm{ts}(G)$. In [13] there is proved that for every graph G,

$$ts(G) \geq \max\{tes(G), tvs(G)\}. \tag{3}$$

Ramdani and Salman in [20] determined the exact values of the total irregularity strength for several Cartesian product graphs. Namely, they proved that for $n \geq 3$, $\mathrm{ts}(P_n \square P_2) = n$, $\mathrm{ts}(C_n \square P_2) = n+1$ and $\mathrm{ts}(S_n \square P_2) = n+1$.

2. MAIN RESULTS

Let $\bigcup_{i=1}^{m} G_i$ denote the disjoint union of graphs $G_1, G_2, \ldots, G_m$, $m \geq 2$. Next theorem gives an upper bound of the total edge irregularity strength of the disjoint union of graphs.

THEOREM 2.1. *The total edge irregularity strength of disjoint union of graphs $G_1, G_2, \dots, G_m$, $m \geq 2$, is*

$$tes\left(\bigcup_{i=1}^{m} G_i\right) \leq \sum_{i=1}^{m} tes(G_i) - \left\lfloor \frac{m-1}{2} \right\rfloor.$$

Proof. Let G_i, $i = 1, 2, \dots, m$, be a graph of order p_i and size q_i. Let $tes(G_i) = t_i$ and $f_i : V(G_i) \cup E(G_i) \to \{1, 2, \dots, t_i\}$ be an edge irregular total t_i-labeling of G_i. We define $t_0 = 0$. Let $V(G_i) = \{v_{ia} : a = 1, 2, \dots, p_i\}$ and $E(G_i) = \{e_{ix} : x = 1, 2, \dots, q_i\}$, for every $i = 1, 2, \dots, m$. Define a total $\left(\sum_{i=1}^{m} t_i - \lfloor \frac{m-1}{2} \rfloor\right)$-labeling g of $\bigcup_{i=1}^{m} G_i$ as follows:

$$g(v_{ia}) = f_i(v_{ia}) + \sum_{s=0}^{i-1} t_s - \left\lfloor \tfrac{i}{2} \right\rfloor \qquad \text{if } v_{ia} \in V(G_i),$$

$$g(e_{ix}) = f_i(e_{ix}) + \sum_{s=0}^{i-1} t_s - \left\lfloor \tfrac{i-1}{2} \right\rfloor \qquad \text{if } e_{ix} \in E(G_i),$$

where $a = 1, 2, \dots, p_i$, $x = 1, 2, \dots, q_i$ and $i = 1, 2, \dots, m$.

Let $e_{ix} = v_{ia}v_{ib}$ be an edge in $E(G_i)$. We distinguish two cases.

Case 1. If i is odd then for edge-weight of $e_{ix} = v_{ia}v_{ib}$ under the labeling g we have

$$\begin{aligned} wt_g(e_{ix}) &= g(v_{ia}) + g(e_{ix}) + g(v_{ib}) \\ &= \left(f_i(v_{ia}) + \sum_{s=0}^{i-1} t_s - \tfrac{i-1}{2}\right) + \left(f_i(e_{ix}) + \sum_{s=0}^{i-1} t_s - \tfrac{i-1}{2}\right) \\ &+ \left(f_i(v_{ib}) + \sum_{s=0}^{i-1} t_s - \tfrac{i-1}{2}\right) = wt_{f_i}(e_{ix}) + 3\left(\sum_{s=0}^{i-1} t_s - \tfrac{i-1}{2}\right). \end{aligned}$$

Since, for a fixed i, $wt_{f_i}(e_{ix}) \neq wt_{f_i}(e_{iy})$ for every $x \neq y$ and $3\left(\sum_{s=0}^{i-1} t_s - \frac{i-1}{2}\right)$ is a constant, thus also $wt_g(e_{ix}) \neq wt_g(e_{iy})$.

Case 2. If i is even then for edge-weight of the edge $e_{ix} = v_{ia}v_{ib}$ under the labeling g we get

$$\begin{aligned} wt_g(e_{ix}) &= g(v_{ia}) + g(e_{ix}) + g(v_{ib}) \\ &= \left(f_i(v_{ia}) + \sum_{s=0}^{i-1} t_s - \frac{i}{2}\right) + \left(f_i(e_{ix}) + \sum_{s=0}^{i-1} t_s - \frac{i}{2} + 1\right) \\ &+ \left(f_i(v_{ib}) + \sum_{s=0}^{i-1} t_s - \frac{i}{2}\right) = wt_{f_i}(e_{ix}) + 3\left(\sum_{s=0}^{i-1} t_s - \frac{i}{2}\right) + 1. \end{aligned}$$

Again we can see that for a fixed i we have $wt_g(e_{ix}) \neq wt_g(e_{iy})$ for every $x \neq y$.

Now we are going to show that $wt_g(e_{ix}) < wt_g(e_{i+1y})$ for every $x = 1, 2, \dots, q_i$ and $y = 1, 2, \dots, q_{i+1}$. We distinguish two cases according to the parity of i.

Case 1. If i is odd then

$$\begin{aligned} wt_g(e_{ix}) &= wt_{f_i}(e_{ix}) + 3\left(\sum_{s=0}^{i-1} t_s - \frac{i-1}{2}\right) \\ &\leq 3t_i + 3\left(\sum_{s=0}^{i-1} t_s - \frac{i-1}{2}\right) = 3\left(\sum_{s=0}^{i} t_s - \frac{i-1}{2}\right) \end{aligned} \tag{4}$$

and

$$\begin{aligned} wt_g(e_{i+1y}) &= wt_{f_{i+1}}(e_{i+1y}) + 3\left(\sum_{s=0}^{i} t_s - \frac{i+1}{2}\right) + 1 \\ &\geq 3 + 3\left(\sum_{s=0}^{i} t_s - \frac{i+1}{2}\right) + 1 = 3\left(\sum_{s=0}^{i} t_s - \frac{i-1}{2}\right) + 1. \end{aligned} \tag{5}$$

Thus from (4) and (5) it follows that

$$wt_g(e_{ix}) \leq 3\left(\sum_{s=0}^{i} t_s - \frac{i-1}{2}\right) < 3\left(\sum_{s=0}^{i} t_s - \frac{i-1}{2}\right) + 1 \leq wt_g(e_{i+1y}).$$

Case 2. If i is even then

$$\begin{aligned} wt_g(e_{ix}) &= wt_{f_i}(e_{ix}) + 3\left(\sum_{s=0}^{i-1} t_s - \frac{i}{2}\right) + 1 \\ &\leq 3t_i + 3\left(\sum_{s=0}^{i-1} t_s - \frac{i}{2}\right) + 1 = 3\left(\sum_{s=0}^{i} t_s - \frac{i}{2}\right) + 1 \end{aligned} \tag{6}$$

and

$$wt_g(e_{i+1y}) = wt_{f_{i+1}}(e_{i+1y}) + 3\left(\sum_{s=0}^{i} t_s - \frac{i}{2}\right) \geq 3 + 3\left(\sum_{s=0}^{i} t_s - \frac{i}{2}\right). \tag{7}$$

So (6) and (7) give that

$$wt_g(e_{ix}) \leq 3\left(\sum_{s=0}^{i} t_s - \frac{i}{2}\right) + 1 < 3\left(\sum_{s=0}^{i} t_s - \frac{i}{2}\right) + 3 \leq wt_g(e_{i+1y}).$$

Hence, in $\bigcup_{i=1}^{m} G_i$ under the labeling g, there are no two edges of the same edge-weight. Therefore, g is an edge irregular total $\left(\sum_{i=1}^{m} t_i - \left\lfloor \frac{m-1}{2} \right\rfloor\right)$-labeling of $\bigcup_{i=1}^{m} G_i$. This concludes the proof. □

Next theorem provides an upper bound of the total vertex irregularity strength for disjoint union of regular graphs.

THEOREM 2.2. *Let G_i, $i = 1, 2, \dots, m$, be an r-regular graph. Then*

$$tvs\left(\bigcup_{i=1}^{m} G_i\right) \leq \sum_{i=1}^{m} tvs(G_i) - \left\lfloor \frac{m-1}{2} \right\rfloor.$$

Proof. Let $tvs(G_i) = t_i$ and $f_i : V(G_i) \cup E(G_i) \to \{1, 2, \dots, t_i\}$ be a vertex irregular total t_i-labeling of an r-regular graph G_i, $i = 1, 2, \dots, m$. We define $t_0 = 0$. Let $V(G_i) = \{v_{ia} : a = 1, 2, \dots, p_i\}$ and $E(G_i) = \{e_{ix} : x = 1, 2, \dots, q_i\}$, for every $i = 1, 2, \dots, m$. Define a total $\left(\sum_{i=1}^{m} t_i - \left\lfloor \frac{m-1}{2} \right\rfloor\right)$-labeling g of $\bigcup_{i=1}^{m} G_i$ in the following way:

$$g(v_{ia}) = f_i(v_{ia}) + \sum_{s=0}^{i-1} t_s - \left\lfloor \tfrac{i}{2} \right\rfloor \qquad \text{if } v_{ia} \in V(G_i),$$

$$g(e_{ix}) = f_i(e_{ix}) + \sum_{s=0}^{i-1} t_s - \left\lfloor \tfrac{i-1}{2} \right\rfloor \qquad \text{if } e_{ix} \in E(G_i),$$

where $a = 1, 2, \dots, p_i$, $x = 1, 2, \dots, q_i$ and $i = 1, 2, \dots, m$.

Let $e_{ia_1}, e_{ia_2}, \dots, e_{ia_r}$ be edges incident with the vertex v_{ia}.

Case 1. If i is odd then for vertex-weights under the labeling g we have

$$\begin{aligned} wt_g(v_{ia}) &= g(v_{ia}) + \sum_{h=1}^{r} g(e_{ia_h}) \\ &= \left(f_i(v_{ia}) + \sum_{s=0}^{i-1} t_s - \frac{i-1}{2}\right) + \sum_{h=1}^{r}\left(f_i(e_{ia_h}) + \sum_{s=0}^{i-1} t_s - \frac{i-1}{2}\right) \\ &= wt_{f_i}(v_{ia}) + (r+1)\left(\sum_{s=0}^{i-1} t_s - \frac{i-1}{2}\right). \end{aligned}$$

It is easy to see that $(r+1)\left(\sum_{s=0}^{i-1} t_s - \frac{i-1}{2}\right)$ is a constant for a fixed i and since $wt_{f_i}(v_{ia}) \neq wt_{f_i}(v_{ib})$ for every $a \neq b$, thus also $wt_g(v_{ia}) \neq wt_g(v_{ib})$, for $i = 1, 2, \dots, m$.

Case 2. If i is even then for vertex-weights we get

$$\begin{aligned} wt_g(v_{ia}) &= g(v_{ia}) + \sum_{h=1}^{r} g(e_{ia_h}) \\ &= \left(f_i(v_{ia}) + \sum_{s=0}^{i-1} t_s - \frac{i}{2}\right) + \sum_{h=1}^{r}\left(f_i(e_{ia_h}) + \sum_{s=0}^{i-1} t_s - \frac{i}{2} + 1\right) \end{aligned}$$

$$= wt_{f_i}(v_{ia}) + (r+1)\left(\sum_{s=0}^{i-1} t_s - \frac{i}{2}\right) + r.$$

Again we can see that $wt_g(v_{ia}) \neq wt_g(v_{ib})$ for every $a \neq b$ and fixed i, $i = 1, 2, \dots, m$.

Next we show that $wt_g(v_{ia}) < wt_g(v_{i+1b})$ for every $a \in \{1, 2, \dots, p_i\}$ and $b \in \{1, 2, \dots, p_{i+1}\}$.

Case 1. If i is odd then

(8)

$$\begin{aligned} wt_g(v_{ia}) &= wt_{f_i}(v_{ia}) + (r+1)\left(\sum_{s=0}^{i-1} t_s - \frac{i-1}{2}\right) \\ &\le (r+1)t_i + (r+1)\left(\sum_{s=0}^{i-1} t_s - \frac{i-1}{2}\right) = (r+1)\left(\sum_{s=0}^{i} t_s - \frac{i-1}{2}\right) \end{aligned}$$

and

(9)
$$\begin{aligned} wt_g(v_{i+1b}) &= wt_{f_{i+1}}(v_{i+1b}) + (r+1)\left(\sum_{s=0}^{i} t_s - \frac{i+1}{2}\right) + r \\ &\ge (r+1)\cdot 1 + (r+1)\left(\sum_{s=0}^{i} t_s - \frac{i+1}{2}\right) + r \\ &= (r+1)\left(\sum_{s=0}^{i} t_s - \frac{i-1}{2}\right) + r > (r+1)\left(\sum_{s=0}^{i} t_s - \frac{i-1}{2}\right). \end{aligned}$$

According to (8) and (9) we have that

$$wt_g(v_{ia}) \le (r+1)\left(\sum_{s=0}^{i} t_s - \frac{i-1}{2}\right) < wt_g(v_{i+1b}).$$

Case 2. If i is even then

(10)
$$\begin{aligned} wt_g(v_{ia}) &= wt_{f_i}(v_{ia}) + (r+1)\left(\sum_{s=0}^{i-1} t_s - \frac{i}{2}\right) + r \\ &\le (r+1)t_i + (r+1)\left(\sum_{s=0}^{i-1} t_s - \frac{i}{2}\right) + r \\ &= (r+1)\left(\sum_{s=0}^{i} t_s - \frac{i}{2}\right) + r \end{aligned}$$

and

$$(11) \qquad wt_g(v_{i+1b}) = wt_{f_{i+1}}(v_{i+1b}) + (r+1)\left(\sum_{s=0}^{i} t_s - \frac{i}{2}\right)$$

$$\geq (r+1)\cdot 1 + (r+1)\left(\sum_{s=0}^{i} t_s - \frac{i}{2}\right) > (r+1)\left(\sum_{s=0}^{i} t_s - \frac{i}{2}\right) + r.$$

From (10) and (11) it follows that

$$wt_g(v_{ia}) \leq (r+1)\left(\sum_{s=0}^{i} t_s - \frac{i}{2}\right) + r < wt_g(v_{i+1b}).$$

Thus, the labeling g has the required properties of vertex irregular total $\left(\sum_{i=1}^{m} t_i - \lfloor \frac{m-1}{2} \rfloor\right)$-labeling of $\bigcup_{i=1}^{m} G_i$. □

Using Theorems 2.1 and 2.2 we obtain the following result.

THEOREM 2.3. *Let G_i, $i = 1, 2, \ldots, m$, be an r-regular graph. Then*

$$ts\left(\bigcup_{i=1}^{m} G_i\right) \leq \sum_{i=1}^{m} ts(G_i) - \left\lfloor \frac{m-1}{2} \right\rfloor.$$

Proof. Let $ts(G_i) = t_i$ and $f_i : V(G_i) \cup E(G_i) \to \{1, 2, \ldots, t_i\}$ be a totally irregular total t_i-labeling of an r-regular graph G_i, for every $i = 1, 2, \ldots, m$. We put $t_0 = 0$. Let $V(G_i) = \{v_{ia} : a = 1, 2, \ldots, p_i\}$ and $E(G_i) = \{e_{ix} : x = 1, 2, \ldots, q_i\}$, for every $i = 1, 2, \ldots, m$. Define a total $\left(\sum_{i=1}^{m} t_i - \lfloor \frac{m-1}{2} \rfloor\right)$-labeling g of $\bigcup_{i=1}^{m} G_i$ as follows:

$$g(v_{ia}) = f_i(v_{ia}) + \sum_{s=0}^{i-1} t_s - \lfloor \tfrac{i}{2} \rfloor \qquad \text{if } v_{ia} \in V(G_i),$$

$$g(e_{ix}) = f_i(e_{ix}) + \sum_{s=0}^{i-1} t_s - \lfloor \tfrac{i-1}{2} \rfloor \qquad \text{if } e_{ix} \in E(G_i),$$

where $a = 1, 2, \ldots, p_i$, $x = 1, 2, \ldots, q_i$ and $i = 1, 2, \ldots, m$.

It follows from Theorems 2.1 and 2.2 that under the labeling g for every two different edges e and e' of $\bigcup_{i=1}^{m} G_i$ there is $wt_g(e) \neq wt_g(e')$ and for every two distinct vertices x and y of $\bigcup_{i=1}^{m} G_i$ there is $wt_g(x) \neq wt_g(y)$. Therefore, g is a totally irregular total $\left(\sum_{i=1}^{m} t_i - \lfloor \frac{m-1}{2} \rfloor\right)$-labeling of $\bigcup_{i=1}^{m} G_i$. □

Next theorem gives the exact value of the total edge irregularity strength for disjoint union of graphs G_i, $i = 1, 2, \ldots, m$, satisfying some certain conditions.

THEOREM 2.4. *Let G_i, $i = 1, 2, \ldots, m$, be a graph. If there is an edge irregular total $(tes(G_i))$-labeling of G_i such that the edge-weight function $wt_{f_i}(e_{ix}) : E(G_i) \to \{3, 4, \ldots, 3tes(G_i) - 1\}$ is a bijection for every $i = 1, 2, \ldots, m$, then*

$$tes\left(\bigcup_{i=1}^{m} G_i\right) = \sum_{i=1}^{m} tes(G_i) - m + 1.$$

Proof. Let G_i, $i = 1, 2, \ldots, m$, be a graph of order p_i and size q_i. Let $tes(G_i) = t_i$ and $f_i : V(G_i) \cup E(G_i) \to \{1, 2, \ldots, t_i\}$ be an edge irregular total t_i-labeling of G_i such that the edge-weight function wt_{f_i} is a bijection from $E(G_i)$ to $\{3, 4, \ldots, 3t_i - 1\}$, for every $i = 1, 2, \ldots, m$. For purposes of the proof let $t_0 = 0$. Thus the size of G_i is $q_i = 3t_i - 3$, $i = 1, 2 \ldots, m$. Therefore,

$$\left|E\left(\bigcup_{i=1}^{m} G_i\right)\right| = \sum_{i=1}^{m} q_i = \sum_{i=1}^{m} (3t_i - 3) = \sum_{i=1}^{m} 3t_i - 3m = 3\left(\sum_{i=1}^{m} t_i - m\right).$$

From (2) it follows that

$$(12) \quad tes\left(\bigcup_{i=1}^{m} G_i\right) \geq \left\lceil \frac{\left|E\left(\bigcup_{i=1}^{m} G_i\right)\right| + 2}{3} \right\rceil = \left\lceil \frac{3\left(\sum_{i=1}^{m} t_i - m\right) + 2}{3} \right\rceil = \sum_{i=1}^{m} t_i - m + 1.$$

For the converse, we define a suitable total $(\sum_{i=1}^{m} t_i - m + 1)$-labeling g of $\bigcup_{i=1}^{m} G_i$ as follows:

$$g(z_{ia}) = f_i(z_{ia}) + \sum_{s=0}^{i-1} t_s - i + 1 \quad \text{if } z_{ia} \in V(G_i) \cup E(G_i),$$

where $a = 1, 2, \ldots, p_i$, $x = 1, 2, \ldots, q_i$ and $i = 1, 2, \ldots, m$.

Let $e_{ix} = v_{ia}v_{ib}$ be an edge in $E(G_i)$ for $v_{ia}, v_{ib} \in V(G_i)$. For edge-weight of an edge $e_{ix} = v_{ia}v_{ib}$ in $E(G_i)$ under the labeling g we have

$$\begin{aligned} wt_g(e_{ix}) =& g(v_{ia}) + g(e_{ix}) + g(v_{ib}) \\ =& \left(f_i(v_{ia}) + \sum_{s=0}^{i-1} t_s - i + 1\right) + \left(f_i(e_{ix}) + \sum_{s=0}^{i-1} t_s - i + 1\right) \\ &+ \left(f_i(v_{ib}) + \sum_{s=0}^{i-1} t_s - i + 1\right) = wt_{f_i}(e_{ix}) + 3\left(\sum_{s=0}^{i-1} t_s - i + 1\right). \end{aligned}$$

Since for fixed i is $wt_{f_i}(e_{ix}) \neq wt_{f_i}(e_{iy})$ for every $x \neq y$ and $3\left(\sum_{s=0}^{i-1} t_s - i + 1\right)$ is also a constant, thus we get $wt_g(e_{ix}) \neq wt_g(e_{iy})$.

Now we show that $wt_g(e_{ix}) < wt_g(e_{i+1y})$, for every $x = 1, 2, \dots, q_i$, $y = 1, 2, \dots, q_{i+1}$ and $i = 1, 2, \dots, m-1$. So

$$wt_g(e_{ix}) = wt_{f_i}(e_{ix}) + 3\left(\sum_{s=0}^{i-1} t_s - i + 1\right) \tag{13}$$
$$\leq 3t_i - 1 + 3\left(\sum_{s=0}^{i-1} t_s - i + 1\right) = 3\left(\sum_{s=0}^{i} t_s - i + 1\right) - 1$$
$$= 3\left(\sum_{s=0}^{i} t_s - i\right) + 2$$

and

$$wt_g(e_{i+1y}) = wt_{f_{i+1}}(e_{i+1y}) + 3\left(\sum_{s=0}^{i} t_s - i\right) \geq 3 + 3\left(\sum_{s=0}^{i} t_s - i\right) \tag{14}$$
$$> 3\left(\sum_{s=0}^{i} t_s - i\right) + 2.$$

From (13) and (14) it follows that

$$wt_g(e_{ix}) \leq 3\left(\sum_{s=0}^{i} t_s - i\right) + 2 < wt_g(e_{i+1y}).$$

We can see that all vertex and edge labels are at most $\sum_{i=1}^{m} t_i - m + 1$ and the edge-weights are distinct for all pairs of distinct edges. Therefore, the labeling g is an edge irregular total $(\sum_{i=1}^{m} t_i - m + 1)$-labeling of $\bigcup_{i=1}^{m} G_i$. It proves that

$$tes\left(\bigcup_{i=1}^{m} G_i\right) \leq \sum_{i=1}^{m} t_i - m + 1. \tag{15}$$

Inequalities (12) and (15) imply the assertion. □

The exact value of the total vertex irregularity strength for disjoint union of arbitrary r-regular graphs G_i, $i = 1, 2, \dots, m$, satisfying some certain conditions determines the following theorem.

THEOREM 2.5. *Let G_i, $i = 1, 2, \dots, m$, be an r-regular graph. If there is a vertex irregular total $(tvs(G_i))$-labeling of G_i such that the vertex-weight function $wt_{f_i}(v_{ia}) : V(G_i) \to \{r+1, r+2, \dots, (r+1)tvs(G_i) - 1\}$ is a bijection for every $i = 1, 2, \dots, m$, then*

$$tvs\left(\bigcup_{i=1}^{m} G_i\right) = \sum_{i=1}^{m} tvs(G_i) - m + 1.$$

Proof. Let G_i, $i = 1, 2, \ldots, m$, be an r-regular graph of order p_i. Let $tvs(G_i) = t_i$ and $f_i : V(G_i) \cup E(G_i) \to \{1, 2, \ldots, t_i\}$ be a vertex irregular total t_i-labeling of G_i such that the vertex-weight function wt_{f_i} is a bijection from $V(G_i)$ to $\{r+1, r+2, \ldots, (r+1)t_i - 1\}$, for every $i = 1, 2, \ldots, m$. Thus $p_i = (r+1)(t_i - 1)$. Therefore,

$$\left| V\left(\bigcup_{i=1}^{m} G_i\right)\right| = \sum_{i=1}^{m} p_i = \sum_{i=1}^{m}(r+1)(t_i-1) = (r+1)(\sum_{i=1}^{m} t_i - m).$$

From (1) it follows that

$$tvs\left(\bigcup_{i=1}^{m} G_i\right) \geq \left\lceil \frac{(r+1)\left(\sum\limits_{i=1}^{m} t_i - m\right) + r}{r+1} \right\rceil = \sum_{i=1}^{m} t_i - m + 1. \tag{16}$$

For proving that $tvs\left(\bigcup_{i=1}^{m} G_i\right) \leq \sum_{i=1}^{m} t_i - m + 1$ we define a total $(\sum_{i=1}^{m} t_i - \ m+1)$-labeling g of $\bigcup_{i=1}^{m} G_i$ in the following way:

$$g(z_{ia}) = f_i(z_{ia}) + \sum_{s=0}^{i-1} t_s - i + 1 \quad \text{if } z_{ia} \in V(G_i) \cup E(G_i),$$

where $a = 1, 2, \ldots, p_i$, $x = 1, 2, \ldots, q_i$ and $i = 1, 2, \ldots, m$.

Observe that if $e_{ia_1}, e_{ia_2}, \ldots, e_{ia_r}$ are edges incident with the vertex $v_{ia} \in V(G_i)$ then under the total labeling g the vertex v_{ia}, $i = 1, 2, \ldots, m$, receives the weight

$$\begin{aligned}
wt_g(v_{ia}) =& g(v_{ia}) + \sum_{h=1}^{r} g(e_{ia_h}) \\
=& \left(f_i(v_{ia}) + \sum_{s=0}^{i-1} t_s - i + 1\right) + \sum_{h=1}^{r}\left(f_i(e_{ia_h}) + \sum_{s=0}^{i-1} t_s - i + 1\right) \\
=& f_i(v_{ia}) + \sum_{s=0}^{i-1} t_s - i + 1 + \sum_{h=1}^{r} f_i(e_{ia_h}) + r\left(\sum_{s=0}^{i-1} t_s - i + 1\right) \\
=& \ wt_{f_i}(v_{ia}) + (r+1)\left(\sum_{s=0}^{i-1} t_s - i + 1\right).
\end{aligned}$$

Since, for a fixed i, the expression $(r+1)\left(\sum_{s=0}^{i-1} t_s - i + 1\right)$ is a constant and $wt_{f_i}(v_{ia}) \neq wt_{f_i}(v_{ib})$ for every $a \neq b$, thus also $wt_g(v_{ia}) \neq wt_g(v_{ib})$.

Next we show that $wt_g(v_{ia}) < w_g(v_{i+1b})$ for every $a = 1, 2, \ldots, p_i$, $b = 1, 2, \ldots, p_{i+1}$ and $i = 1, 2, \ldots, m-1$. It means

$$wt_g(v_{ia}) = wt_{f_i}(v_{ia}) + (r+1)\left(\sum_{s=0}^{i-1} t_s - i + 1\right) \tag{17}$$

$$\leq (r+1)t_i - 1 + (r+1)\left(\sum_{s=0}^{i-1} t_s - i + 1\right)$$
$$= (r+1)\left(\sum_{s=0}^{i} t_s - i + 1\right) - 1.$$

and

$$wt_g(v_{i+1b}) = wt_{f_{i+1}}(v_{i+1b}) + (r+1)\left(\sum_{s=0}^{i} t_s - i\right) \tag{18}$$
$$\geq (r+1) + (r+1)\left(\sum_{s=0}^{i} t_s - i\right)$$
$$= (r+1)\left(\sum_{s=0}^{i} t_s - i + 1\right) > (r+1)\left(\sum_{p=1}^{i} t_p - i + 1\right) - 1.$$

According to inequalities (17) and (18) we have that $wt_g(v_{ia}) < w_g(v_{i+1b})$.

We can see that all vertex and edge labels are at most $\sum_{i=1}^{m} t_i - m + 1$ and the vertex-weights are different for all pairs of distinct vertices. In fact, the total labeling g has the required properties of vertex irregular total $(\sum_{i=1}^{m} t_i - m + 1)$-labeling of $\bigcup_{i=1}^{m} G_i$ and thus

$$tvs\left(\bigcup_{i=1}^{m} G_i\right) \leq \sum_{i=1}^{m} t_i - m + 1. \tag{19}$$

Combining (16) and (19) we obtain the equality. □

Using Theorems 2.4 and 2.5, we have the following result.

THEOREM 2.6. *Let G_i, $i = 1, 2, \ldots, m$, be an r-regular graph. If there is a totally irregular total $(ts(G_i))$-labeling of G_i such that the edge-weight function $wt_{f_i}(e_{ix}) : E(G_i) \to \{3, 4, \ldots, 3ts(G_i) - 1\}$ is a bijection for every $i = 1, 2, \ldots, m$ and the vertex-weight function $wt_{f_i}(v_{ia}) : V(G_i) \to \{r+1, r+2, \ldots, (r+1)ts(G_i) - 1\}$ is a bijection for every $i = 1, 2, \ldots, m$, then*

$$ts\left(\bigcup_{i=1}^{m} G_i\right) = \sum_{i=1}^{m} ts(G_i) - m + 1.$$

Proof. Let G_i, $i = 1, 2, \ldots, m$, be an r-regular graph of order p_i. Let $ts(G_i) = t_i$ and $f_i : V(G_i) \cup E(G_i) \to \{1, 2, \ldots, t_i\}$ be a totally irregular total t_i-labeling of G_i, for every $i = 1, 2, \ldots, m$. Let $t_0 = 0$.

As $wt_{f_i}(e_{ix}) : E(G_i) \to \{3, 4, \dots, 3ts(G_i) - 1\}$ is a bijection, for every $i = 1, 2, \dots, m$, then from (12) we have that

$$tes\left(\bigcup_{i=1}^{m} G_i\right) \geq \sum_{i=1}^{m} t_i - m + 1 \tag{20}$$

and from (3), we get

$$ts\left(\bigcup_{i=1}^{m} G_i\right) \geq tes\left(\bigcup_{i=1}^{m} G_i\right). \tag{21}$$

Thus (20) and (21) give

$$ts\left(\bigcup_{i=1}^{m} G_i\right) \geq \sum_{i=1}^{m} t_i - m + 1. \tag{22}$$

As $wt_{f_i}(v_{ia}) : V(G_i) \to \{r+1, r+2, \dots, (r+1)ts(G_i) - 1\}$ is a bijection, for every $i = 1, 2, \dots, m$, then from (16) we get

$$tvs\left(\bigcup_{i=1}^{m} G_i\right) \geq \sum_{i=1}^{m} t_i - m + 1 \tag{23}$$

and from (3) we have that

$$ts\left(\bigcup_{i=1}^{m} G_i\right) \geq tvs\left(\bigcup_{i=1}^{m} G_i\right). \tag{24}$$

Hence from (23) and (24) it follows

$$ts\left(\bigcup_{i=1}^{m} G_i\right) \geq \sum_{i=1}^{m} t_i - m + 1. \tag{25}$$

For the converse, we define a total $(\sum_{i=1}^{m} t_i - m + 1)$-labeling g of $\bigcup_{i=1}^{m} G_i$ as follows:

$$g(z_{ia}) = f_i(z_{ia}) + \sum_{s=0}^{i-1} t_s - i + 1 \quad \text{if } z_{ia} \in V(G_i) \cup E(G_i),$$

where $a = 1, 2, \dots, p_i$, $x = 1, 2, \dots, q_i$ and $i = 1, 2, \dots, m$.

From Theorem 2.4 and 2.5 it follows that the labeling g is a totally irregular total labeling of $\bigcup_{i=1}^{m} G_i$ which proves that $ts\left(\bigcup_{i=1}^{m} G_i\right) \leq \sum_{i=1}^{m} t_i - m + 1$. Combining with the lower bound (25), we conclude that

$$ts\left(\bigcup_{i=1}^{m} G_i\right) = \sum_{i=1}^{m} t_i - m + 1 = \sum_{i=1}^{m} ts(G_i) - m + 1. \qquad \square$$

As a consequence of Theorem 2.6 we obtain the exact value of the total irregularity strength for disjoint union of prisms $D_{n_i} = C_{n_i} \Box P_2$, $i = 1, 2, \dots, m$.

Corollary 2.7. *Let $m \geq 2$, $n_i \geq 3$, $i = 1, 2, \ldots, m$. Then the total irregularity strength for disjoint union of prisms $D_{n_i} = C_{n_i} \square P_2$ is*

$$ts\left(\bigcup_{i=1}^{m} D_{n_i}\right) = \sum_{i=1}^{m} n_i + 1.$$

Proof. Ramdani and Salman [20] constructed a totally irregular total $(ts(D_n))$-labeling $f : V(D_n) \cup E(D_n) \to \{1, 2, \ldots, ts(D_n)\}$ of D_n such that the edge-weight and vertex-weight functions satisfy the assumptions of Theorem 2.6 and they proved that $ts(D_n) = ts(C_n \square P_2) = n + 1$. Hence from Theorem 2.6 for disjoint union of prism D_{n_i}, $i = 1, 2, \ldots, m$, it follows that

$$ts\left(\bigcup_{i=1}^{m} D_{n_i}\right) = \sum_{i=1}^{m} ts(D_{n_i}) - m + 1. \tag{26}$$

As $ts(D_{n_i}) = n_i + 1$, for $i = 1, 2, \ldots, m$, then from (26) we get

$$ts\left(\bigcup_{i=1}^{m} D_{n_i}\right) = \sum_{i=1}^{m} ts(D_{n_i}) - m + 1 = \sum_{i=1}^{m} (n_i + 1) - m + 1 = \sum_{i=1}^{m} n_i + 1. \quad \square$$

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