ON $\mathfrak{aff}(1)$-RELATIVE COHOMOLOGY OF THE LIE ALGEBRA OF VECTOR FIELDS ON WEIGHTED DENSITIES ON $\mathbb{R}$

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Let $\mathrm{Vect}(\mathbb{R})$ be the Lie algebra of smooth vector fields on $\mathbb{R}$ and $\mathbb{F}_\lambda$ be the space of λ-densities on $\mathbb{R}$. $\mathrm{Vect}(\mathbb{R})$ acts on $\mathbb{F}_\lambda$ by Lie derivative. In this paper, we compute the first and the second differential $\mathfrak{aff}(1)-$relative cohomology of the Lie algebra $\mathrm{Vect}(\mathbb{R})$ with coefficients in the space $\mathbb{F}_\lambda$. Explicit cocycles spanning these cohomology spaces are given.

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1. INTRODUCTION

Let $\mathrm{Vect}(\mathbb{R})$ be the Lie algebra of all vector fields $X\frac{\mathrm{d}}{\mathrm{d}x}$ on $\mathbb{R}$, ($X \in C^\infty(\mathbb{R})$). For any $\lambda \in \mathbb{R}$, we define a structure of $\mathrm{Vect}(\mathbb{R})$-module over $C^\infty(\mathbb{R})$ by

$$\mathbb{L}^{\lambda}_{X\frac{\mathrm{d}}{\mathrm{d}x}}(f) = Xf' + \lambda X' f, \tag{1.1}$$

where f', X' are $\frac{\mathrm{d}f}{\mathrm{d}x}$, $\frac{\mathrm{d}X}{\mathrm{d}x}$.

The corresponding $\mathrm{Vect}(\mathbb{R})$-module is the space of weighted densities on $\mathbb{R}$ of weight λ with respect to the 1-form $\mathrm{d}x$, denoted by:

$$\mathbb{F}_\lambda = \left\{ f(\mathrm{d}x)^\lambda,\ f \in C^\infty(\mathbb{R}) \right\}, \quad (\lambda \in \mathbb{R}).$$

The space $\mathbb{F}_\lambda$ coincides with the space of vector fields, functions and differential 1-forms for $\lambda = -1$, 0 and 1, respectively. Obviously the adjoint $\mathrm{Vect}(\mathbb{R})$-module is isomorphic to $\mathbb{F}_{-1}$. The Lie algebra $\mathrm{Vect}(\mathbb{R})$ has a Lie subalgebra $\mathfrak{aff}(1) = \mathrm{Span}\big(\frac{\mathrm{d}}{\mathrm{d}x}, x\frac{\mathrm{d}}{\mathrm{d}x}\big)$.

Our purpose in this paper is to compute the spaces $\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}), \mathfrak{aff}(1); \mathbb{F}_\lambda)$, and $\mathrm{H}^2_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}), \mathfrak{aff}(1); \mathbb{F}_\lambda)$, where $\mathrm{H}^*_{\mathrm{diff}}$ denotes the differential $\mathfrak{aff}(1)$-relative cohomology, that is, only cochains given by differential operators are considered.

Let us mention that these results lead to the study of the $\mathfrak{aff}(1)$-trivial deformation of the standard embedding of the Lie algebra $\mathrm{Vect}(\mathbb{R})$ into the Lie algebra of pseudodiffential operators on $\mathbb{R}$.

2. RELATIVE COHOMOLOGY

Let us first recall some fundamental concepts from cohomology theory (see, *e.g.*, [4]). Let $\mathfrak{g}$ be a Lie algebra acting on a vector space V and let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. (If $\mathfrak{h}$ is omitted it is assumed to be $\{0\}$.) The space of $\mathfrak{h}$-relative n-cochains of $\mathfrak{g}$ with values in V is the $\mathfrak{g}$-module

$$C^n(\mathfrak{g},\mathfrak{h};V) := \mathrm{Hom}_{\mathfrak{h}}(\Lambda^n(\mathfrak{g}/\mathfrak{h});V).$$

The *coboundary operator* $\delta_n : C^n(\mathfrak{g},\mathfrak{h};V) \longrightarrow C^{n+1}(\mathfrak{g},\mathfrak{h};V)$ is a $\mathfrak{g}$-map satisfying $\delta_n \circ \delta_{n-1} = 0$. The kernel of δ_n, denoted $Z^n(\mathfrak{g},\mathfrak{h};V)$, is the space of $\mathfrak{h}$-relative *n-cocycles*, among them, the elements in the range of δ_{n-1} are called $\mathfrak{h}$-relative *n-coboundaries*. We denote $B^n(\mathfrak{g},\mathfrak{h};V)$ the space of n-coboundaries.

By definition, the n^{th} $\mathfrak{h}$-relative cohomolgy space is the quotient space

$$\mathrm{H}^n(\mathfrak{g},\mathfrak{h};V) = Z^n(\mathfrak{g},\mathfrak{h};V)/B^n(\mathfrak{g},\mathfrak{h};V).$$

We will only need the formula of δ_n (which will be simply denoted δ) in degrees $0,1$ and 2: for $v \in C^0(\mathfrak{g},\,\mathfrak{h};V) = V^{\mathfrak{h}}$, $\delta v(g) := g\cdot v$, where

$$V^{\mathfrak{h}} = \{v\in V \;\mid\; h\cdot v = 0 \quad \text{for all } h\in\mathfrak{h}\},$$

for $\Upsilon \in C^1(\mathfrak{g},\mathfrak{h};V)$,

$$\delta(\Upsilon)(g,\,h) := g\cdot\Upsilon(h) - h\cdot\Upsilon(g) - \Upsilon([g,\;h]) \quad \text{for any} \quad g,h\in\mathfrak{g}.$$

and for $\Omega \in C^2(\mathfrak{g},V)$,

$$\begin{aligned}\delta(\Omega)(x,y,z) :=\; & x.\Omega(y,z) - y.\Omega(x,z) + z.\Omega(x,y)\\ & -\Omega([x,y],z) + \Omega([x,z],y) - \Omega([y,z],x),\end{aligned} \tag{2.2}$$

where $x,y,z\in\mathfrak{g}$.

3. THE SPACE $\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}),\mathfrak{aff}(1);\mathbb{F}_\lambda)$

In this section, we consider the Lie algebra $\mathrm{Vect}(\mathbb{R})$ acting on $\mathbb{F}_\lambda$ and we compute the first $\mathfrak{aff}(1)$-relative cohomology space of $\mathrm{Vect}(\mathbb{R})$ with coefficients in $\mathbb{F}_\lambda$.

Our main result in this section is the following:

THEOREM 3.1. *The space* $\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}),\,\mathfrak{aff}(1),\,\mathbb{F}_\lambda)$ *has the following structure:*

$$\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}),\,\mathfrak{aff}(1),\,\mathbb{F}_\lambda) = \begin{cases}\mathbb{R}, & \lambda = 1,2\\ 0, & \textit{otherwise.}\end{cases}$$

The corresponding spaces $\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}),\mathfrak{aff}(1),\,\mathbb{F}_\lambda)$ *are spanned by the relative cohomology classes of the 1-cocycles*

$$c_1(f\frac{\mathrm{d}}{\mathrm{d}x}) = f''\mathrm{d}x, \quad c_2(f\frac{\mathrm{d}}{\mathrm{d}x}) = f'''\mathrm{d}x^2. \tag{3.3}$$

To prove Theorem 3.1 we need the following Lemma.

LEMMA 3.2. *Any* 1*-cocycle* $c \in Z^1_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}); \mathbb{F}_\lambda)$ *vanishing on* $\mathfrak{aff}(1)$ *is* $\mathfrak{aff}(1)$*-invariant.*

Proof. The 1-cocycle relation of c reads:

$$f\frac{\mathrm{d}}{\mathrm{d}x} \cdot c(g\frac{\mathrm{d}}{\mathrm{d}x}) - g\frac{\mathrm{d}}{\mathrm{d}x} \cdot c(f\frac{\mathrm{d}}{\mathrm{d}x}) - c([f\frac{\mathrm{d}}{\mathrm{d}x},\ g\frac{\mathrm{d}}{\mathrm{d}x}]) = 0, \tag{3.4}$$

where $f\frac{\mathrm{d}}{\mathrm{d}x}$, $g\frac{\mathrm{d}}{\mathrm{d}x} \in \mathrm{Vect}(\mathbb{R})$. Thus, if $c(f\frac{\mathrm{d}}{\mathrm{d}x}) = 0$ for all $f\frac{\mathrm{d}}{\mathrm{d}x} \in \mathfrak{aff}(1)$, the equation (3.4) becomes

$$f\frac{\mathrm{d}}{\mathrm{d}x} \cdot c(g\frac{\mathrm{d}}{\mathrm{d}x}) - c([f\frac{\mathrm{d}}{\mathrm{d}x},\ g\frac{\mathrm{d}}{\mathrm{d}x}]) = 0 \tag{3.5}$$

expressing the $\mathfrak{aff}(1)$-invariance of c. □

Proof of Theorem 3.1

Recall that the spaces $\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}); \mathbb{F}_\lambda)$ were computed by D.B. Fuchs in [5], the description is the following:

$$\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}); \mathbb{F}_\lambda) = \begin{cases} \mathbb{R}, & \lambda = 0, 1, 2 \\ 0, & \text{otherwise} \end{cases}$$

represented by the cocycles

$$c_0(f\frac{\mathrm{d}}{\mathrm{d}x}) = f', \quad c_1(f\frac{\mathrm{d}}{\mathrm{d}x}) = f''\mathrm{d}x, \quad c_2(f\frac{\mathrm{d}}{\mathrm{d}x}) = f'''\mathrm{d}x^2. \tag{3.6}$$

According to Lemma 3.2, we can easily deduce the space $\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}), \mathfrak{aff}(1); \mathbb{F}_\lambda)$. □

4. THE SPACE $\mathrm{H}^2_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}), \mathfrak{aff}(1); \mathbb{F}_\lambda)$

The following steps to compute the relative cohomology have been used intensively in [1–3, 6]. First, we classify $\mathfrak{aff}(1)$-invariant differential operators, then we isolate among them those that are 2-cocycles. To do that, we need the following Lemma.

LEMMA 4.1. *Any* 2*-cocycle* $C \in Z^2_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}); \mathbb{F}_\lambda)$ *vanishing on* $\mathfrak{aff}(1)$ *is* $\mathfrak{aff}(1)$*-invariant.*

Proof. The 2-cocycle condition reads as follows:

$$C([X,Y],Z) - L^\lambda_X\, C(Y,Z) + \circlearrowleft (X,Y,Z) = 0$$

for every $X, Y, Z \in \mathrm{Vect}(\mathbb{R})$ where $\circlearrowleft (X,Y,Z)$ denotes the summands obtained from the two written ones by the cyclic permutation of the symbols X, Y, Z. Now, if $X \in \mathfrak{aff}(1)$, then the equation above becomes

$$C([X,Y],Z) - C([X,Z],Y) = L^\lambda_X\, C(Y,Z).$$

This condition is nothing but the invariance property. □

4.1. $\mathfrak{aff}(1)$-INVARIANT DIFFERENTIAL OPERATORS

As our 2-cocycles vanish on $\mathfrak{aff}(1)$, we will investigate $\mathfrak{aff}(1)$-invariant bilinear differential operators that vanish on $\mathfrak{aff}(1)$.

PROPOSITION 4.2. *Any skew-symmetric bilinear differential operators* $C_\lambda : \mathrm{Vect}(\mathbb{R}) \wedge \mathrm{Vect}(\mathbb{R}) \to \mathbb{F}_\lambda$, *which is* $\mathfrak{aff}(1)$*-invariant and vanish on* $\mathfrak{aff}(1)$, *is as follows:*

$$C_n(X,Y) = \sum_{i=2}^{[\frac{n-1}{2}]} c_{i,n-i} \begin{vmatrix} f^i & g^i \\ f^{n-i} & g^{n-i} \end{vmatrix} \mathrm{d}x^{n-2}$$

for $X = f\frac{\mathrm{d}}{\mathrm{d}x}, Y = g\frac{\mathrm{d}}{\mathrm{d}x}$, $c_{i,n-i} \in \mathbb{R}$, *and* $n \geq 5$. $[k]$ *denotes the integer part of* k.

Proof. The generic form of any such a differential operator is (here $X = f\frac{\mathrm{d}}{\mathrm{d}x}, Y = g\frac{\mathrm{d}}{\mathrm{d}x} \in \mathrm{Vect}(\mathbb{R})$):

$$C_\lambda(X,Y) = \sum_{i+j\leq k} c_{i,j}\, f^{(i)}\, g^{(j)} \mathrm{d}x^\lambda,$$

where $c_{i,j} = -c_{j,i}$ and $f^{(i)}$ stands for $\frac{d^i f}{\mathrm{d}x^i}$.

The invariance property with respect to the vector field $X = \frac{\mathrm{d}}{\mathrm{d}x}$ with arbitrary Y implies that $c'_{i,j} = 0$. Therefore $c_{i,j}$ are constants. Now, the invariance property with respect to $X = x\frac{\mathrm{d}}{\mathrm{d}x}$ with arbitrary Y implies that $i + j = \lambda + 2$, so in particular λ is integer. □

4.2. $\mathfrak{aff}(1)$-RELATIVE COHOMOLOGY OF $\mathrm{Vect}(\mathbb{R})$

The main result of this section is the following

THEOREM 4.3. *We have*

$$\mathrm{H}^2_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}), \mathfrak{aff}(1); \mathbb{F}_\lambda) = \begin{cases} \mathbb{R} & \text{if } \lambda = 5, 7 \\ 0 & \text{otherwise.} \end{cases}$$

The corresponding spaces $\mathrm{H}^2_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}), \mathfrak{aff}(1); \mathbb{F}_\lambda)$ *are spanned by the cohomology classes of the following non-trivial 2-cocycles:*

$$\Omega_7(X_f, X_g) = \begin{vmatrix} f''' & g''' \\ f^{(IV)} & g^{(IV)} \end{vmatrix} \mathrm{d}x^5 \tag{4.7}$$

$$\Omega_9(X_f, X_g) = \left(2\begin{vmatrix} f''' & g''' \\ f^{(VI)} & g^{(VI)} \end{vmatrix} - 9\begin{vmatrix} f^{(IV)} & g^{(IV)} \\ f^{(V)} & g^{(V)} \end{vmatrix}\right) \mathrm{d}x^7 \tag{4.8}$$

for $X_f = f\frac{\mathrm{d}}{\mathrm{d}x}$, $X_g = g\frac{\mathrm{d}}{\mathrm{d}x}$.

Proof. Let Ω_λ be a 2-cocycle on $\mathrm{Vect}(\mathbb{R})$ vanishing on $\mathfrak{aff}(1)$, with values in $\mathbb{F}_\lambda$. By Lemma 4.1, up to a scalar factor, Ω_λ is a skew-symmetric bilinear differential operators $\mathfrak{aff}(1)$-invariant $C_\lambda : \mathrm{Vect}(\mathbb{R}) \wedge \mathrm{Vect}(\mathbb{R}) \to \mathbb{F}_\lambda$. Thus, by Proposition 4.2, we get the explicit formulae for Ω_λ:

$$C_n(X,Y) = \sum_{i=2}^{[\frac{n-1}{2}]} c_{i,n-i} \begin{vmatrix} f^i & g^i \\ f^{n-i} & g^{n-i} \end{vmatrix} \mathrm{d}x^{n-2}$$

for $X = f\frac{\mathrm{d}}{\mathrm{d}x}, Y = g\frac{\mathrm{d}}{\mathrm{d}x},\ c_{i,n-i} \in \mathbb{R},$ and $n \geq 5$.
So any corresponding coboundary is up to a scalar factor as follows

$$\delta(B_n)(X,Y) = \sum_{k=1}^{[\frac{n-1}{2}]} \left[\binom{n+1}{k+1} - \binom{n+1}{k}\right] \begin{vmatrix} f^{k+1} & g^{k+1} \\ f^{n-k+1} & g^{n-k+1} \end{vmatrix} \mathrm{d}x^{n-2}$$

for $X = f\frac{\mathrm{d}}{\mathrm{d}x}, Y = g\frac{\mathrm{d}}{\mathrm{d}x},\ c_{i,n-i} \in \mathbb{R}, n \geq 5,$ and where $\binom{x}{i} = \frac{x(x-1)\cdots(x-i+1)}{i!}$.

- For $n = 5$, the operator C_5 satisfies the $2-$cocycle condition. By the coboundary expression we annul the term $c_{2,5}$, and so, we get the $2-$cocycle Ω_7.
- For $n = 6$, the operator C_6 is not but a coboundary.
- For $n = 7$, the $2-$cocycle condition applied to the operator C_7 leades to the condition $14c_{2,7} - 9c_{3,6} - 2c_{4,5} = 0$. So, thanks to the coboundary expression we annul the term $c_{2,7}$ and we get the cocycle Ω_9.
- For $n \geq 8$, we annul the term $c_{2,n}$ as in the foregoing cases then we apply the $2-$cocycle condition, one get by collecting the terms in $f^{(i)}g^{(n-i-1)}h''$, for $i \in \{3, \cdots, [\frac{n-3}{2}]\}$ the conditions

$$(n-i)(n-i-3)c_{i,n-i} + (i+1)(i-2)c_{i+1,n-i-1} = 0. \tag{4.9}$$

Then:

1. If n is even, we collect the term in$f^{(\frac{n}{2}-1)}g^{(\frac{n}{2})}h''$, one get the supplementary condition $c_{\frac{n}{2}-1,\frac{n}{2}+1} = 0$, and so $\Omega_n = 0$.
2. If n is odd, we collect the term in $f^{(\frac{n-3}{2})}g^{(\frac{n-1}{2})}h'''$, one get the supplementary condition

(4.10)
$$\left(\binom{n-3}{\frac{n-3}{2}} - \binom{n-3}{\frac{n-5}{2}}\right)c_{3,n-3} + \left(\binom{\frac{n+3}{2}}{2} - \binom{\frac{n+3}{2}}{3}\right)c_{\frac{n-3}{2},\frac{n+3}{2}} + \left(\binom{\frac{n+1}{2}}{3} - \binom{\frac{n+1}{2}}{2}\right)c_{\frac{n-1}{2},\frac{n+1}{2}} = 0.$$

So, for $n \leq 15$ we get $\Omega_n = 0$, and for $n \geq 17$, the determinant of the linear system (4.9–4.10) is

$$\Delta_n = (-1)^{(\frac{n-3}{2})} \prod_{i=4}^{\frac{n-7}{2}} \frac{(n-i)(3-n+i)}{2}$$

$$\left[\begin{array}{c} \binom{n-3}{\frac{n-3}{2}} - \binom{n-3}{\frac{n-5}{2}} - \frac{(n+3)(n+1)(n-3)(n-6)(n-7)}{32} \\ + \frac{(n+5)(n+1)(n-1)^2(n-3)(n-6)(n-9)}{384} \end{array} \right]$$

which is not zero, and so $\Omega_n = 0$. $\square$

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