# PERIODICITY AND J-CLEAN-LIKE RINGS

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A ring $R$ is periodic provided that for any $a \in R$ there exist distinct elements $m, n \in \mathbb{N}$ such that $a^m = a^n$. We shall prove that periodicity is inherited by all generalized matrix rings. A ring $R$ is called strongly periodic if for any $a \in R$ there exists a potent $p \in R$ such that $a - p$ is in its Wedderburn radical and $ap = pa$. A ring $R$ is J-clean-like if for any $a \in R$ there exists a potent $p \in R$ such that $a - p$ is in its Jacobson radical. Furthermore, we completely determine the connections between strongly periodic rings and periodic rings. The relations among J-clean-like rings and these rings are also obtained.

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## 1. INTRODUCTION

A ring $R$ is periodic provided that for any $a \in R$ there exist distinct elements $m, n \in \mathbb{N}$ such that $a^m = a^n$. Examples of periodic rings are finite rings and Boolean rings. There are many interesting problems related to periodic rings. We explore, in this article, the periodicity of a type of generalized matrix rings. An element $p \in R$ is potent if $p = p^m$ for some $m \geq 2$. For later convenience we state here some elementary characterizations of periodic rings:

THEOREM 1.1. *Let $R$ be a ring. Then the following are equivalent:*

(1) *$R$ is periodic.*
(2) *For any $a \in R$, there exists some $m \geq 2$ such that $a^m = a^{m+1}f(a)$ for some $f(t) \in \mathbb{Z}[t]$.*
(3) *For any $a \in R$, there exists some $m \geq 2$ such that $a - a^m \in R$ is nilpotent.*
(4) *For any $a \in R$, there exists a potent $p \in R$ such that $a - p \in R$ is nilpotent and $ap = pa$.*

Here, the equivalences of all items are stated in [1, Lemma 2], [5, Proposition 2], [19, Theorem 3], and the simple implication from (3) to (2). A Morita context $(A, B, M, N, \psi, \varphi)$ consists of two rings $A$ and $B$, two bimodules

${}_AN_B$ and ${}_BM_A$, and a pair of bimodule homomorphisms $\psi : N \bigotimes_B M \to A$ and $\varphi : M \bigotimes_A N \to B$ which satisfy the following associativity: $\psi(n \bigotimes m)n' = n\varphi(m \bigotimes n')$ and $\varphi(m \bigotimes n)m' = m\psi(n \bigotimes m')$ for any $m, m' \in M, n, n' \in N$. These conditions ensure that the set $T$ of generalized matrices $\begin{pmatrix} a & n \\ m & b \end{pmatrix}; a \in A, b \in B, m \in M, n \in N$ will form a ring with addition defined componentwise and with multiplication defined by

$$\begin{pmatrix} a_1 & n_1 \\ m_1 & b_1 \end{pmatrix} \begin{pmatrix} a_2 & n_2 \\ m_2 & b_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + \psi(n_1 \bigotimes m_2) & a_1n_2 + n_1b_2 \\ m_1a_2 + b_1m_2 & \varphi(m_1 \bigotimes n_2) + b_1b_2 \end{pmatrix},$$

called the ring of the Morita context (cf. [20]). The class of rings of the Morita contexts is a type of generalized matrix rings. For instances, all $2 \times 2$ matrix rings and all triangular matrix rings.

Let $T$ be the ring of a Morita context $(A, B, M, N, \psi, \varphi)$. We prove, in Section 2, that if $im(\psi)$ and $im(\varphi)$ are nilpotent, then $A$ and $B$ are periodic if and only if so is $T$. This provides a large new class of periodic rings for generalized matrix rings.

It is an attractive problem to express an element in a ring as the sum of idempotents and units (cf. [4], [6], [8] and [9]). We say that a ring $R$ is clean provided that every element in $R$ is the sum of an idempotent and a unit. Such rings have been extensively studied in recent years, see [7] and [21]. This motivates us to combine periodic rings with clean rings together, and investigate further properties of related rings.

For a ring $R$ the Wedderburn radical is denoted by $P(R)$, *i.e.*, $P(R)$ is the sum of all nilpotent ideals of $R$. We now introduce a new type of rings. A ring $R$ is said to be strongly periodic provided that for any $a \in R$ there exists a potent $p \in R$ such that $a - p \in P(R)$ and $ap = pa$. Strongly periodic rings form a subclass of periodic rings. We shall prove that a ring $R$ is strongly periodic if and only if for any $a \in R$ there exists a potent $p \in R$ such that $a - p \in P(R)$, and determine completely the connections between these ones and periodic rings. A ring is 2-primary provided that its Wedderburn radical coincides with the set of nilpotent elements of the ring. It is proved that a ring $R$ is strongly periodic if and only if $R$ is a 2-primary periodic ring. From this, we show that the strong periodicity will be inherited by generalized matrix rings.

Replacing the Wedderburn radical $P(R)$ by the Jacobson radical $J(R)$, we introduce a type of rings which behave like that of periodic rings. We say that a ring $R$ is J-clean-like provided that for any $a \in R$ there exists a potent $p \in R$ such that $a - p \in J(R)$. This is a natural generalization of J-clean rings

[6]. Many properties of periodic rings are extended to these ones. We shall characterize J-clean-like rings and obtain the relations among these rings.

Throughout, all rings are associative with an identity. $M_n(R)$ will denote the ring of all $n \times n$ matrices over $R$ with an identity $I_n$. $N(R)$ stands for the set of all nilpotent elements in $R$. $C(R)$ denote the center of $R$. $P(R)$ and $J(R)$ denote the Wedderburn radical and Jacobson radical of $R$, respectively.

## 2. PERIODIC RINGS

The purpose of this section is to investigate the periodicity for Morita contexts. The following lemma is known [17, Lemma 3.1.23], and we include a simple proof for the sake of completeness.

LEMMA 2.1. *A ring $R$ is periodic if and only if for any $a, b \in R$, there exists an $n \in \mathbb{N}$ such that $a - a^n, b - b^n \in N(R)$.*

*Proof.* $\Longleftarrow$ For any $a \in R$, we can find $n \in \mathbb{N}$ such that $a - a^n \in N(R)$. This implies that $R$ is periodic, by Theorem 1.1.

$\Longrightarrow$ Suppose that $R$ is periodic. For any $a, b \in R$, we can find $p, q, s, t \in R$ $(p < q, s < t)$ such that $a^p = a^q$ and $b^s = b^t$. Hence, $a^{ps} = a^{qs}$ and $b^{ps} = b^{pt}$. This implies that

$$a^{ps} = a^{ps}a^{(q-p)s} = a^{ps}a^{2(q-p)s} = \cdots = a^{ps}a^{(t-s)p(q-p)s}.$$

Likewise, we get $b^{ps} = b^{ps}b^{(q-p)s(t-s)p}$. Choose $k = ps$ and $l = ps + (t-s)p(q-p)s$. Then $a^k = a^l, b^k = b^l$ $(k < l)$. Thus, $a^k = a^l = a^{(l-k)+k} = \cdots = a^{k(l-k)+k}$, and so $a^k = \big(a^k\big)^{l-k+1}$. This implies that $\big(a^{k(l-k)}\big)^2 = \big(a^{k(l-k)+k}\big)\big(a^{k(l-k)-k}\big) = a^k\big(a^{k(l-k)-k}\big) = a^{k(l-k)}$. Choose $n = k(l-k)$. Then $\big(a - a^{n+1}\big)^n = a^n\big(1-a^n\big)^n = a^n\big(1 - a^n\big) = 0$. Thus, $a - a^n \in N(R)$. Likewise, $b - b^n \in N(R)$. Therefore, we complete the proof. □

THEOREM 2.2. *Let $T$ be the ring of a Morita context $(A, B, M, N, \psi, \varphi)$. If $im(\psi)$ and $im(\varphi)$ are nilpotent, then $A$ and $B$ are periodic if and only if so is $T$.*

*Proof.* Suppose $A$ and $B$ are periodic. For any $\begin{pmatrix} a & n \\ m & b \end{pmatrix} \in T$, as in the proof of Lemma 2.1, there exists a $k \in \mathbb{N}$ such that $a - a^k \in N(A)$ and $b - b^k \in N(B)$. Hence,

$$\begin{pmatrix} a & n \\ m & b \end{pmatrix} - \begin{pmatrix} a & n \\ m & b \end{pmatrix}^k = \begin{pmatrix} a - a^k + c & * \\ * & b - b^k + d \end{pmatrix},$$

where $c \in \text{im}(\psi)$ and $d \in \text{im}(\varphi)$. Write $(a-a^k)^l = 0$ and $(b-b^k)^l = 0$. By hypothesis, $\text{im}(\psi)$ and $\text{im}(\varphi)$ are nilpotent ideals of $A$ and $B$, respectively. Say $\big(\text{im}(\psi)\big)^s = 0$ and $\big(\text{im}(\varphi)\big)^t = 0$. Choose $p = \max(s,t)$ and $q = p(l+1)$. Then

$$\big(a-a^k+c\big)^q = 0 \text{ and } \big(b-b^k+d\big)^q = 0.$$

Obviously,

$$\begin{pmatrix} a-a^k+c & * \\ * & b-b^k+d \end{pmatrix}^{q+1} \in \begin{pmatrix} \text{im}(\psi) & N \\ M & \text{im}(\varphi) \end{pmatrix}.$$

Set $NM := \text{im}(\psi)$ and $MN := \text{im}(\varphi)$. We see that

$$\begin{pmatrix} NM & N \\ M & MN \end{pmatrix}^2 \subseteq \begin{pmatrix} NM & (NM)N \\ (MN)M & MN \end{pmatrix}.$$

For any $l \in \mathbb{N}$, by induction, one easily checks that

$$\begin{pmatrix} NM & N \\ M & MN \end{pmatrix}^{2l} \subseteq \begin{pmatrix} NM & (NM)N \\ (MN)M & MN \end{pmatrix}^l \subseteq \begin{pmatrix} (NM)^l & (NM)^lN \\ (MN)^lM & (MN)^l \end{pmatrix}.$$

Choose $j = 2p(q+1)$. As $(NM)^p = (MN)^p = 0$, we get

$$\begin{pmatrix} a-a^k+c & * \\ * & b-b^k+d \end{pmatrix}^j = \begin{pmatrix} 0 & t \\ s & 0 \end{pmatrix}$$

for some $s \in M, t \in N$. Hence,

$$\begin{pmatrix} 0 & t \\ s & 0 \end{pmatrix}^2 = \begin{pmatrix} \psi(t \bigotimes s) & 0 \\ 0 & \varphi(s \bigotimes t) \end{pmatrix},$$

and so

$$\left(\begin{pmatrix} a & n \\ m & b \end{pmatrix} - \begin{pmatrix} a & n \\ m & b \end{pmatrix}^k\right)^{2jp} = 0.$$

Accordingly, $T$ is periodic, by Theorem 1.1. The converse is obvious. □

Let $R$ be a ring, and let $s \in C(R)$. Let $M_{(s)}(R) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in R\}$, where the operations are defined as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a+a' & b+b' \\ c+c' & d+d' \end{pmatrix},$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} aa'+sbc' & ab'+bd' \\ ca'+dc' & scb'+dd' \end{pmatrix}.$$

Then $M_{(s)}(R)$ is a ring with the identity $\begin{pmatrix} 1_R & 0 \\ 0 & 1_R \end{pmatrix}$. Recently, the strong cleanness of such type generalized matrix rings was studied in [21]. For the periodicity of such rings, we derive

COROLLARY 2.3. *Let $R$ be periodic, and let $s \in N(R) \bigcap C(R)$. Then $M_{(s)}(R)$ is periodic.*

*Proof.* Let $\psi : R \otimes R \to R, n \otimes m \mapsto snm$ and $\varphi : R \otimes R \to R, m \otimes n \mapsto smn$. Then $M_s(R) = (R, R, R, R, \psi, \varphi)$. As $s \in N(R) \bigcap C(R)$, we see that $\mathrm{im}(\varphi)$ and $\mathrm{im}(\psi) \subseteq J(R)$ are nilpotent, and we are through by Theorem 2.2. $\square$

As a consequence, a ring $R$ is periodic if and only if so is the trivial Morita context $M_{(0)}(R)$. Choosing $s = 0 \in R$, we are through from Corollary 2.3. Given a ring $R$ and an $R$-$R$-bimodule $M$, the trivial extension of $R$ by $M$ is the ring $T(R, M) = R \oplus M$ with the usual addition and the following multiplication: $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2)$.

COROLLARY 2.4. *Let $R$ be a ring, and let $M$ be a $R$-$R$-bimodule. Then the following are equivalent:*

(1) *$R$ is periodic.*
(2) *$T(R, M)$ is periodic.*

*Proof.* (1) $\Rightarrow$ (2) Let $R$ be a periodic ring and let $S = \begin{pmatrix} R & M \\ 0 & R \end{pmatrix}$. It is obvious by Theorem 2.2 that $S$ is periodic. Clearly, $T(R, M)$ is a subring of $S$, and so proving (2).

(2) $\Rightarrow$ (1) Let $T(R, M)$ be a periodic ring. As $R$ is isomorphic to a subring of $T(R, M)$, and so $R$ is periodic. $\square$

*Example* 2.5. Let $R$ be periodic, let

$$A = B = \begin{pmatrix} R & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & R \end{pmatrix}, M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & R & 0 \end{pmatrix} \text{ and } N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ R & R & 0 \end{pmatrix},$$

and let $\psi : N \bigotimes_B M \to A, \psi(n \otimes m) = nm$ and $\phi : M \bigotimes_A N \to B, \phi(m \otimes n) = mn$. Then $T = (A, B, M, N, \psi, \phi)$ is a Morita context with zero pairings, *i.e.*, $T$ is a trivial Morita context. Hence, $\mathrm{im}(\psi)$ and $\mathrm{im}(\varphi)$ are nilpotent. Clearly, $A$ and $B$ are both periodic. In light of Theorem 2.2, $T$ is periodic.

Let $R$ be a ring, and let $\alpha$ be an endomorphism of $R$. Let $T_n(R, \alpha)$ be the set of all upper triangular matrices over the rings $R$. For any $(a_{ij}), (b_{ij}) \in T_n(R, \alpha)$, we define $(a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$, and $(a_{ij})(b_{ij}) = (c_{ij})$ where $c_{ij} = \sum\limits_{k=i}^{n} a_{ik}\alpha^{k-i}(b_{kj})$. Then $T_n(R, \alpha)$ is a ring under the preceding addition and multiplication (cf. [14]). Clearly, $T_n(R, \alpha)$ will be $T_n(R)$ only when $\alpha$ is the identity morphism.

Lemma 2.6. *Let $R$ be periodic, and let $\alpha : R \to R$ be an endomorphism. Then*

(1) *$R[[x,\alpha]]/(x^n)$ is periodic.*
(2) *$T_n(R,\alpha)$ is periodic for all $n \in \mathbb{N}$.*

*Proof.* (1) For any $f(x) \in R[[x]]/(x^n)$, there exists an $m \in \mathbb{N}$ such that $f(0) - f^m(0) \in N(R)$. Hence, $f(x) - f^m(x) \in N\big(R[[x]]/(x^n)\big)$. According to Theorem 1.1, $R[[x]]/(x^n)$ is periodic.

(2) For any $(a_{ij}) \in T_n(R,\alpha)$, as in the proof of Lemma 2.1, we can find an $m \in \mathbb{N}$ such that $a_{ii} - a_{ii}^m \in N(R)$ for each $i$. Thus, $(a_{ij}) - (a_{ij})^m \in N\big(T_n(R,\alpha)\big)$, as required. □

We are now ready to prove:

Theorem 2.7. *Let $R$ be periodic. Then $M_{(x^m)}\big(R[[x]]/(x^n)\big)$ is periodic for all $1 \le m \le n$.*

*Proof.* Choose $\alpha = 1$. Then $R[[x]]/(x^n)$ is periodic, by Lemma 2.6. Choose $s = x^m (1 \le m \le n)$. Then $s \in N\big(R[[x]]/(x^n)\big) \bigcap C\big(R[[x]]/(x^n)\big)$. Applying Corollary 2.3 to $R[[x]]/(x^n)$, $M_{(x^m)}\big(R[[x]]/(x^n)\big)$ is periodic, as asserted. □

Corollary 2.8. *Let $R$ be a finite ring. Then $M_{(x^m)}\big(R[[x]]/(x^n)\big)$ is periodic for all $1 \le m \le n$.*

*Proof.* Since every finite ring is periodic, we complete the proof by Theorem 2.7. □

## 3. STRONGLY PERIODIC RINGS

A ring $R$ is potent if for any $a \in R$, there exists some $n \ge 2$ such that $a = a^n$. An ideal $I$ of a ring $R$ is locally nilpotent if, every finitely generated subring of elements belonging to $I$ is nilpotent. Clearly, an ideal $I$ of a ring $R$ is locally nilpotent if and only if $RxR$ is nilpotent for any $x \in I$. Recall that $J(R)$ consists of all $x \in R$ such that $1 + RxR$ is included in the set of units of $R$. We now derive

Theorem 3.1. *Let $R$ be a ring. Then the following are equivalent:*

(1) *$R$ is strongly periodic.*
(2) *$R$ is periodic and $N(R)$ is a locally nilpotent ideal of $R$.*
(3) *$R/J(R)$ is potent, every potent lifts modulo $J(R)$ and $J(R)$ is locally nilpotent.*

*Proof.* (1) $\Rightarrow$ (2) Clearly, $R$ is periodic. Let $x \in N(R)$. Then we can find a potent $p \in R$ such that $w := x - p \in P(R)$. Write $x^n = 0$ for some $n \in \mathbb{N}$. Then $p^n = (x-w)^n \in P(R)$. This shows that $p \in R$ is nilpotent, and so $p = 0$; hence, $x = w \in P(R)$. We infer that $N(R) = P(R)$ is an ideal of $R$.

For any $x \in P(R)$, we claim that $RxR$ is nilpotent. As $P(R)$ is the sum of all nilpotent ideals of $R$, we can find nilpotent ideals $I_1, \cdots, I_m$ of $R$ such that $x \in I_1 + \cdots I_m$. Clearly, $I_1 + \cdots I_m$ is a nilpotent ideal. It follows from $RxR \subseteq I_1 + \cdots I_m$ that $RxR$ is nilpotent. Thus, $N(R)$ is locally nilpotent.

(2) $\Rightarrow$ (1) Let $x \in N(R)$. As $N(R)$ is locally nilpotent, $RxR$ is nilpotent. Write $(RxR)^m = 0 (m \in \mathbb{N})$. Then $RxR \subseteq P(R)$. This implies that $x \in P$; hence, $N(R) \subseteq P(R)$. The implication is true, by Theorem 1.1.

(1) $\Rightarrow$ (3) For any $a \in R$ there exists some potent $p \in R$ such that $a - p \in P(R) \subseteq J(R)$. Hence, $\overline{a} = \overline{p}$ in $R/J(R)$. Therefore $R/J(R)$ is potent.

Let $x \in J(R)$. Then there exists a potent $p \in R$ such that $x - p \in P(R)$; hence, $p = x - (x - p) \in J(R)$. Write $p = p^m (m \geq 2)$. then $p(1 - p^{m-1}) = 0$, and so $p = 0$. Hence, $x \in P(R)$. By the preceding discussion, $RxR$ is nilpotent, and therefore $J(R)$ is locally nilpotent.

(3) $\Rightarrow$ (1) Let $a \in R$. Then $a - a^n \in J(R)$ for some $n \geq 2$. As $J(R)$ is locally nilpotent, it is nilpotent, and so $a - a^n \in N(R)$. In view of Theorem 1.1, $R$ is periodic. Let $x \in N(R)$. Then $\overline{x} \in R/J(R)$ is potent; hence, $\overline{x} = \overline{0}$ in $R/J(R)$. That is, $x \in J(R)$. By hypothesis, $J(R)$ is locally nilpotent; hence, $RxR$ is nilpotent. As in the proof in (2) $\Rightarrow$ (1), we see that $x \in P(R)$. Thus, $N(R) \subseteq P(R)$.

For any $a \in R$, there exists a potent $p \in R$ and a $w \in N(R)$ such that $a = p + w$ and $pw = wp$, by Theorem 1.1. By the preceding discussion, $w \in P(R)$. This proving (1). □

Corollary 3.2. *A ring $R$ is strongly periodic if and only if for any $a \in R$ there exists a potent $p \in R$ such that $a - p \in P(R)$.*

*Proof.* $\Longrightarrow$ This is trivial.

$\Longleftarrow$ For any $a \in R$ there exists potent $p \in R$ such that $a - p \in P(R) \subseteq J(R)$. Hence, $R/J(R)$ is potent. For any $x \in J(R)$, there exists a potent $q \in R$ such that $x - q \in P(R)$. Hence, $q = x - (x - q) \in J(R)$. Write $q = q^m (m \geq 2)$. Then $q(1 - q^{m-1}) = 0$, and so $q = 0$. We infer that $x \in P(R)$. As in the proof in Theorem 3.1, $RxR$ is nilpotent, and so $J(R)$ is locally nilpotent. This result follows, by using Theorem 3.1. □

A ring $R$ is a 2-primary ring $R$ if its Wedderburn radical coincides with the set of all nilpotent elements, *i.e.* $N(R) = P(R)$. A ring $R$ is weakly periodic provided that for any $a \in R$ there exists a potent $p \in R$ such that $a - p \in N(R)$ [17]. We now derive

Theorem 3.3. *A ring $R$ is strongly periodic if and only if $R$ is a 2-primary weakly periodic ring.*

*Proof.* $\Longrightarrow$ Clearly, $R$ is weakly periodic. For any $a \in N(R)$, there exists a potent $p \in R$ such that $w := a - p \in P(R)$. Hence, $p = a - w$. Write $a^m = 0 (m \in \mathbb{N})$. Then $p^m \in P(R)$, and so $p \in N(R)$. This implies that $p = 0$, and so $a = w \in P(R)$. Thus, $N(R) = P(R)$, and so $R$ is 2-primary.

$\Longleftarrow$ Let $a \in R$. Since $R$ is weakly periodic, there exists a potent $p \in R$ such that $a - p \in N(R)$. As $R$ is 2-primary, $N(R) \subseteq P(R)$, we get $a - p \in P(R)$. Therefore, we complete the proof, by Corollary 3.2. □

A ring $R$ is called strongly 2-primal provided that $R/I$ is 2-primal for all ideals $I$ of $R$.

Corollary 3.4. *A ring $R$ is strongly periodic if and only if the following two conditions hold:*

(1) *$R$ is a weakly periodic ring with locally nilpotent $J(R)$;*
(2) *Every prime ideal of $R$ is completely prime.*

*Proof.* $\Longrightarrow$ (1) is obvious. Clearly, $R/P(R)$ is potent. As is well known, every potent ring is commutative (cf. [10, Theorem 1 in Chapter X]), and so $R/P(R)$ is commutative. In view of [15, Proposition 1.2], every prime ideal of $R$ is completely prime.

$\Longleftarrow$ In view of [15, Proposition 1.2], $R$ is strongly 2-primal, and then it is 2-primal. As $J(R)$ is locally nilpotent, we show that $R$ is 2-primary. This completes the proof, in terms of Theorem 3.3. □

A ring $R$ is called nil-semicommutative if $ab = 0$ in $R$ implies that $aRb = 0$ for every $a, b \in N(R)$ (see [16]). For instance, every semicommutative ring (*i.e.*, $ab = 0$ in $R$ implies that $aRb = 0$) is nil-semicommutative.

Corollary 3.5. *Every nil-semicommutative weakly periodic ring is strongly periodic.*

*Proof.* One easily checks that every nil-semicommutative ring is 2-primary, so the result follows from Theorem 3.3. □

We note that strongly periodic rings may not be nil-semicommutative as the following shows.

*Example* 3.6. Let $\mathbb{Z}_2$ be the field of integral modulo 2, and let

$$R_n = \{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{1n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in \mathbb{Z}_2 \}$$

with $3 \le n \in \mathbb{N}$. Let $R = \big( \bigoplus_{n=3}^{\infty} R_n, 1 \big)$ be the subalgebra of $\prod_{n=3}^{\infty} R_n$ over $\mathbb{Z}_2$ generated by $\bigoplus_{n=3}^{\infty} R_n$ and 1. We note that $P(R) = \bigoplus_{n=3}^{\infty} P(R_n)$. Hence, $R/P(R) \cong \big( \bigoplus_{n=3}^{\infty} F_n, 1 \big)$, the subalgebra of $\prod_{n=3}^{\infty} F_n$ over $\mathbb{Z}_2$ generated by $\bigoplus_{n=3}^{\infty} F_n$ and $1_{\prod_{n=3}^{\infty} F_n}$, where $F_n = \mathbb{Z}_2$ for all $n = 3, 4, \cdots$. This implies that $R/P(R)$ is reduced. For any $a \in N(R)$, $\overline{a} \in R/P(R)$ is nilpotent, and so $\overline{a} = \overline{0}$. That is, $a \in P(R)$. Therefore $R$ is 2-primary. As $R_n$ is a finite ring for each $n$, we see that it is periodic. We infer that $R$ is periodic, and so it is weakly periodic. In light of Theorem 3.3, $R$ is strongly periodic. We claim that $R_4$ is not nil-semicommutative. Choose

$$a = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } b = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $a^2 = b^2 = 0$, and so $a, b \in N(R_4)$. Furthermore, $ab = 0$, while $axb \neq 0$. Thus, $R_4$ is not nil-semicommutative. Therefore $R$ is not nil-semicommutative, and we are done.

Theorem 3.7. *Let $T$ be the ring of a Morita context $(A, B, M, N, \psi, \varphi)$. If im($\psi$) and im($\varphi$) are nilpotent, then $A$ and $B$ are strongly periodic if and only if so is $T$.*

*Proof.* Suppose $A$ and $B$ are strongly periodic. Then $A$ and $B$ are 2-primary, by Theorem 3.3. Further, they are periodic. In view of Theorem 2.2, $T$ is periodic. It suffices to prove that $T$ is 2-primary.

Let $\begin{pmatrix} a & n \\ m & b \end{pmatrix} \in T$ be nilpotent. Then we can find some $c \in \text{im}(\psi)$ and $d \in \text{im}(\varphi)$ such that $a^k + c = 0$ and $b^l + d = 0$ for some $k, l \in \mathbb{N}$. This implies that $a \in N(A)$ and $b \in N(B)$. As $A$ is 2-primary, $a \in P(A)$. Analogously to the proof in Theorem 3.1, we see that $AaA$ is nilpotent. Likewise, $BbB$ is nilpotent. Clearly,

$$T\begin{pmatrix} a & n \\ m & b \end{pmatrix}T \subseteq \begin{pmatrix} AaA+\mathrm{im}(\psi) & N \\ M & BbB+\mathrm{im}(\varphi) \end{pmatrix}.$$

As the sum of two nilpotent ideal of a ring is nilpotent, we see that $AaA+\mathrm{im}(\psi)$ and $BbB+\mathrm{im}(\varphi)$ are nilpotent ideals of $A$ and $B$, respectively. Similarly to the proof of Theorem 2.2, we see that $\begin{pmatrix} AaA+\mathrm{im}(\psi) & N \\ M & BbB+\mathrm{im}(\varphi) \end{pmatrix}$ is a nilpotent ideal of $T$. Hence, $T\begin{pmatrix} a & n \\ m & b \end{pmatrix}T$ is nilpotent, and so $\begin{pmatrix} a & n \\ m & b \end{pmatrix} \in P(T)$. Thus, $T$ is 2-primary, and so $T$ is strongly periodic, by Theorem 3.3.

Conversely, assume that $T$ is strongly periodic. Then $A$ is periodic. Let $a \in N(R)$. Then $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in N(T)$. By virtue of Theorem 3.3, $T$ is 2-primary; hence, $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in P(T)$. As in the proof in Theorem 3.1, we see that $T\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}T$ is nilpotent. Then $AaA$ is nilpotent, and so $a \in P(R)$. It follows that $A$ is 2-primary. Therefore $A$ is strongly periodic, by Theorem 3.3. Likewise, $B$ is strongly periodic, as required. □

COROLLARY 3.8. *Let $R$ be strongly periodic, and let $s \in N(R)\bigcap C(R)$. Then $M_{(s)}(R)$ is strongly periodic.*

*Proof.* As in the proof of Corollary 2.3, we have $M_s(R) = (R,R,R,R,\psi,\varphi)$ where $\mathrm{im}(\varphi)$ and $\mathrm{im}(\psi)$ are nilpotent. This completes the proof, by Theorem 3.7. □

*Example* 3.9. Consider the Morita context $R = \begin{pmatrix} \mathbb{Z}_4 & \mathbb{Z}_4 \\ 2\mathbb{Z}_4 & \mathbb{Z}_4 \end{pmatrix}$, where the context products are the same as the product in $\mathbb{Z}_4$. Then we claim that $R$ is strongly periodic. Since $R$ is finite, it is periodic, and then we are done by Theorem 3.7.

As a consequence, a ring $R$ is strongly periodic if and only if so is the trivial Morita context $M_{(0)}(R)$. Now we exhibit the useful characterizations of strongly periodic rings as follows.

THEOREM 3.10. *Let $R$ be a ring. Then the following are equivalent:*

(1) *$R$ is strongly periodic.*
(2) *$R/P(R)$ is potent.*
(3) *For any $a \in R$, there exists a prime $m \geq 2$ such that $a - a^m \in P(R)$.*
(4) *For any $a \in R$, $a = eu+w$, where $e = e^2 \in R, u^m = 1$ $(m \in \mathbb{N}), w \in P(R)$ and $e,u,w$ commutate.*

*Proof.* (1) $\Rightarrow$ (2) This is obvious.

(2) $\Rightarrow$ (3) Luh's Theorem states that a ring $S$ is potent if and only if for any $x \in S$ there exists a prime $n$ such that $x = x^n$ (cf. [13]). Let $a \in R$. Since $R/J(R)$ is potent, we have a prime $m \geq 2$ such that $\overline{a} = \overline{a^m}$ in $R/P(R)$. Therefore, $a - a^m \in P(R)$.

(3) $\Rightarrow$ (4) Let $a \in R$. Then we have a prime $n \geq 2$ such that $a - a^n \in P(R) \subseteq N(R)$. By Theorem 1.1, $R$ is periodic. Let $x \in N(R)$. Then $\overline{x} \in R/P(R)$ is potent; whence, $\overline{x} = \overline{0}$ in $R/P(R)$. Thus, $x \in P(R)$, and so $N(R) \subseteq P(R)$. By [7, Proposition 13.1.18], $a = eu + w$, where $e = e^2 \in R, u \in U(R), w \in P(R)$ and $e, u, w$ commutate. Write $u^k = u^{k+m}$ for some $m, k \in \mathbb{N}$. Then $u^m = 1$, as desired,

(4) $\Rightarrow$ (1) For any $a \in R$, $a = eu + w$, where $e = e^2 \in R, u^m = 1$ $(m \in \mathbb{N}), w \in P(R)$ and $e, u, w$ commutate. Set $p = eu$. Then $p = eu^{m+1} = p^{m+1}$, *i.e.*, $p \in R$ is potent. Thus, $R$ is strongly periodic. $\square$

Corollary 3.11. *Every subring of a strongly periodic ring is strongly periodic.*

*Proof.* Let $R$ be strongly periodic, and let $S \subseteq R$. For any $a \in S$, there exists some $n \geq 2$ such that $a - a^n \in P(R)$ in terms of Theorem 3.10. Hence, $(R(a - a^n)R)^m = 0$ for some $m \in \mathbb{N}$. This forces that $(S(a - a^n)S)^m = 0$. Therefore $a - a^n \in P(S)$. By using Theorem 3.10 again, $S$ is strongly periodic, as needed. $\square$

For example, if $R$ is the finite subdirect product of strongly periodic rings, then Corollary 3.11 shows that $R$ is strongly periodic.

*Example* 3.12. Let $F = GF(q)$ be a Galois field and let $V$ be an infinite dimensional left vector space over $F_p$ with $\{v_1, v_2, \cdots\}$ a basis. For the endomorphism ring $A = End_F(V)$, define $A_1 = \{f \in A \mid rank(f) < \infty$ and $f(v_i) = a_1v_1 + \cdots + a_iv_i$ for $i = 1, 2, \cdots$ with $a_j \in F_p\}$ and let $R$ be the $F$-algebra of $A$ generated by $A_1$ and $1_A$. Then $R$ is strongly periodic. As in the proof of [15, Example 1.1], $R/P(R) \cong \{(a_1, \cdots, a_n, b, b, \cdots) \mid a_i, b \in F$ and $n = 1, 2, \cdots\}$. As $F = GF(q)$, we see that $x = x^q$ for all $x \in F$, and then $R/P(R)$ is potent. According to Theorem 3.10, $R$ is strongly periodic.

Lemma 3.13. *Let $I$ be a nilpotent ideal of a ring $R$. If $R/I$ is strongly periodic, then so is $R$.*

*Proof.* Let $a \in R$. Then there exists some $n \geq 2$ such that $\overline{a - a^n} \in P\big(R/I\big)$. Hence, $\big(R(a - a^n)R\big)^m \subseteq I$. As $I$ is nilpotent, $\big(R(a - a^n)R\big)^{mn} = 0$. This shows that $a - a^n \in P(R)$. Therefore $R$ is strongly periodic, by Theorem 3.9. $\square$

THEOREM 3.14. *Let $I$ be an ideal of a ring $R$. Then the following are equivalent:*

(1) *$R/I$ is strongly periodic.*

(2) *$R/I^n$ is strongly periodic for all $n \in \mathbb{N}$.*

(2) *$R/I^n$ is strongly periodic for some $n \in \mathbb{N}$.*

*Proof.* (1) $\Rightarrow$ (2) Clearly, $R/I \cong (R/I^n)/(I/I^n)$. Since $(I/I^n)^n = 0$, proving (2) by Lemma 3.13.

(2) $\Rightarrow$ (3) This is trivial.

(3) $\Rightarrow$ (1) For any $\overline{a} \in R/I$, we see that $a + I^n \in R/I^n$. By hypothesis, there exists a potent $\overline{p} \in R/I^n$ such that $\overline{a-p} \in P(R/I^n)$. Write $\overline{p} = \overline{p}^m$ for some $m \geq 2$. Then $p - p^m \in I^n \subseteq I$, and so $\overline{p} \in R/I$ is potent. Obviously, $(R/I^n)\overline{(a-p)}(R/I^n)$ is nilpotent, and then $\big(R(a-p)R\big)^s \subseteq I^n \subseteq I$ for some $s \in \mathbb{N}$. We infer that $(R/I)\overline{(a-p)}(R/I)$ is nilpotent. As in the proof of Theorem 3.1, we infer that $\overline{a-p} \in P(R/I)$, as required. $\square$

Recall that a ring $R$ is an abelian ring if every idempotent in $R$ is central. A ring $R$ is strongly $\pi$-regular if for any $a \in R$ there exists $n \in \mathbb{N}$ such that $a^n \in a^{n+1}R$. Obviously, every periodic ring is strongly $\pi$-regular. We now derive

LEMMA 3.15. *Every abelian periodic ring of bounded index is strongly periodic.*

*Proof.* Let $R$ be an abelian periodic ring of bounded index. Then $R$ is strongly $\pi$-regular. Badawi's Theorem states that the set of all nilpotent elements of an abelian strongly $\pi$-regular ring is an ideal [2]. Thus, $N(R)$ forms an ideal of $R$. This completes the proof, by Theorem 3.1. $\square$

Let $n \geq 2$ be a fixed integer. A ring $R$ is said to be generalized $n$-like provided that for any $a, b \in R$, $(ab)^n - ab^n - a^n b + ab = 0$ (cf. [18]). It is proved that every generalized 3-like ring is commutative ( [18, Theorem 3]). We now derive

THEOREM 3.16. *Every generalized $n$-like ring is strongly periodic.*

*Proof.* Let $R$ be a generalized $n$-like ring, and let $a \in R$. Then $a^{2n} - 2a^{n+1} + a^2 = 0$, and so $(a - a^n)^2 = 0$. Thus, $a - a^n \in N(R)$. Accordingly, $R$ is periodic by Theorem 1.1. In light of [18, Lemma 2], $R$ is abelian. If $a^m = 0$, then $a^2(1 - a^n) = 0$, and so $a^2 = 0$. Thus, $R$ is of bounded index 2. Therefore $R$ is strongly periodic, by Lemma 3.15. $\square$

Let $R = \{ \begin{pmatrix} x & y & z \\ 0 & x^2 & 0 \\ 0 & 0 & x \end{pmatrix} \mid x, y, z \in GF(4)\}$. Then for each $a \in R$, $a^7 = a$

or $a^7 = a^2 = 0$. Therefore $R$ is a generalized 7-like ring. By Theorem 3.16, $R$ is strongly periodic. In this case, $R$ is abelian but not commutative (cf. [18, Example 2]).

## 4. J-CLEAN-LIKE RINGS

We now consider J-clean-like Morita contexts and extend Theorem 2.2 as follows.

THEOREM 4.1. *Let $T$ be the ring of a Morita context $(A, B, M, N, \psi, \varphi)$ with $\mathrm{im}(\psi) \subseteq J(A)$ and $\mathrm{im}(\varphi) \subseteq J(B)$. If $A$ and $B$ are J-clean-like, then so is $T$.*

*Proof.* Let $\begin{pmatrix} a & n \\ m & b \end{pmatrix} \in T$. Then we have potent $p \in A$ and $q \in B$ such that $a - p \in J(A)$ and $b - q \in J(B$. Hence

$$\begin{pmatrix} a & n \\ m & b \end{pmatrix} - \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} = \begin{pmatrix} a-p & n \\ m & b-q \end{pmatrix}.$$

Let $\begin{pmatrix} c & s \\ t & d \end{pmatrix} \in T$. As $1_A - (a-p)c - \psi(n \bigotimes t) \in U(A)$ and $1_B - (b-q)d - \varphi(m \bigotimes s) \in U(B)$, it follows by [20, Lemma 3.1] that

$$\begin{aligned} & 1_T - \begin{pmatrix} a-p & n \\ m & b-q \end{pmatrix} \begin{pmatrix} c & s \\ t & d \end{pmatrix} \\ = & \begin{pmatrix} 1_A - (a-p)c - \psi(n \bigotimes t) & * \\ * & 1_B - (b-q)d - \varphi(m \bigotimes s) \end{pmatrix} \in U(T). \end{aligned}$$

Hence, $\begin{pmatrix} a-p & n \\ m & b-q \end{pmatrix} \in J(T)$, and therefore $T$ is J-clean-like. □

As a consequence, we deduce that the $n \times n$ lower (upper) triangular matrix ring over a J-clean-like ring is J-clean-like.

COROLLARY 4.2. *Let $R$ be J-clean-like, and let $s \in J(R) \bigcap C(R)$. Then $M_{(s)}(R)$ is J-clean-like.*

*Proof.* As in the proof of Corollary 2.3, $M_{(s)}(R)$ can be regarded as the ring of a Morita context $(R, R, R, R, \psi, \varphi)$ with $\mathrm{im}(\psi) \subseteq J(R)$ and $\mathrm{im}(\varphi) \subseteq J(R)$. According to Theorem 4.1, $M_{(s)}(R)$ is $J$-clean-like. □

COROLLARY 4.3. *Let $R$ be a J-clean-like ring. Then $M_{(x)}(R[[x]])$ is J-clean-like.*

*Proof.* For any $f(x) \in R[[x]]$, we can find a potent $p \in R$ such that $f(0) - p \in J(R)$. Hence, $f(x) = p + \big(f(x) - p\big)$. One easily checks that $f(x) - p \in J\big(R[[x]]\big)$. Thus, $R[[x]]$ is J-clean-like. Choose $s = x$. Applying Corollary 4.2 to $R[[x]]$, $M_{(x)}(R[[x]])$ is $J$-clean-like. □

Analogously, if $R$ is a J-clean-like ring then so is $M_{(x^m)}\big(R[[x]]/(x^n)\big)$ for all $1 \leq m \leq n$.

Proposition 4.4. *A ring $R$ is strongly periodic if and only if the following two conditions hold simultaneously:*

(1) *$R$ is J-clean-like;*
(2) *$J(R)$ is locally nilpotent.*

*Proof.* $\Longrightarrow$ Suppose $R$ is strongly periodic. As $P(R) \subseteq J(R)$, $R$ is J-clean-like. Let $x \in J(R)$. Then there exists a potent $p \in R$ such that $x - p \in P(R)$; hence, $p = x - (x - p) \in J(R)$. This shows that $p = 0$, and so $x \in P(R)$. As in the proof of Theorem 3.1, $RxR$ is nilpotent. As the sum of finite nilpotent ideal is nilpotent, we prove that $J(R)$ is locally nilpotent, as required.

$\Longleftarrow$ Let $x \in J(R)$. Since $J(R)$ is locally nilpotent, $RxR$ is nilpotent. As in the proof of Theorem 3.1, we get $x \in P(R)$. Hence, $J(R) \subseteq P(R)$. This completes the proof, by (1). □

Recall that a ring $R$ is J-clean provided the for any $a \in R$ there exists an idempotent $e \in R$ such that $a - e \in J(R)$ (cf. [6]). This following result explains the relation between J-clean rings and J-clean-like rings.

Proposition 4.5. *A ring $R$ is J-clean if and only if the following two conditions hold:*

(1) *$R$ is J-clean-like;*
(2) *$J(R) = \{x \in R \mid 1 - x \in U(R)\}$.*

*Proof.* $\Longrightarrow$ Clearly, $R$ is J-clean-like. It is easy to check that $J(R) \subseteq \{x \in R \mid 1 - x \in U(R)\}$. If $1 - x \in U(R)$, then there exists an idempotent $e \in R$ such that $w := x - e \in J(R)$. Hence, $1 - e = (1 - x) + w = (1 - x)\big(1 + (1 - x)^{-1}w\big) \in U(R)$. This shows that $1 - e = 1$, and so $e = 0$. Therefore $x \in J(R)$, and so $J(R) \supseteq \{x \in R \mid 1 - x \in U(R)\}$, as required.

$\Longleftarrow$ For any $a \in R$ there exists a potent $p \in R$ such that $(a-1)-p \in J(R)$. Write $p = p^m (m \geq 2)$. Then $p^{m-1} \in R$ is an idempotent. Set $e = 1 - p^{m-1}$ and $u = p - 1 + p^{m-1}$. Then $e = e^2 \in R$ and $u^{-1} = p^{m-1} - 1 + p^{m-1}p^{m-2}$. Further, $p = e + u$. This shows that $a - 1 = p + (a - 1 - p) = e + u + (a - 1 - p)$. Hence, $a = e + \big(u + (a - p)\big)$. As $1 - \big(u + (a - p)\big) = -u - (a - 1 - p) = -u\big(1 - u^{-1}(a - 1 - p)\big) \in U(R)$, we see that $u + (a - p) \in J(R)$. Therefore $R$ is J-clean, as asserted. □

*Example* 4.6. Let $R = \begin{pmatrix} \mathbb{Z}_3 & \mathbb{Z}_3 \\ 0 & \mathbb{Z}_3 \end{pmatrix}$. Then $R$ is J-clean-like, while it is not J-clean. For any $\begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \in R$, we see that $\begin{pmatrix} a & c \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}$ is the sum of a potent element in $R$ and an element in $J(R)$, hence that $R$ is J-clean-like. As $R/J(R) \cong \mathbb{Z}_3$ is not Boolean, we conclude that $R$ is not J-clean.

An element $p \in R$ is $J$-potent provided that there exists some $n \geq 2$ such that $p - p^n \in J(R)$. We say that every potent element lifts modulo $J(R)$ if for any $J$-potent $p \in R$ there exists a potent $q \in R$ such that $p - q \in J(R)$.

LEMMA 4.7. *A ring $R$ is J-clean-like if and only if the following two conditions hold:*
(1) *$R/J(R)$ is potent;*
(2) *Every potent element lifts modulo $J(R)$.*

*Proof.* $\Longrightarrow$ This is obvious.
$\Longleftarrow$ Let $a \in R$. Then $\overline{a} \in R/J(R)$ is potent. By hypothesis, we can find a potent $p \in R$ such that $a - p \in J(R)$. Accordingly, $R$ is J-clean-like. □

Recall that a ring $R$ is right (left) quasi-duo provided that every maximal right (left) ideal is a two-sided ideal. As is well known, every right (left) duo ring (*i.e.*, every right (left) ideal is two-sided) is right (left) quasi-duo. We come now to the main result of this section.

THEOREM 4.8. *A ring $R$ is J-clean-like if and only if the following three conditions hold:*
(1) *$R/J(R)$ is periodic;*
(2) *$R$ is right (left) quasi-duo;*
(3) *Every potent element lifts modulo $J(R)$.*

*Proof.* $\Longrightarrow$ In view of Lemma 4.7, $R/J(R)$ is potent, and so it is periodic. Let $M$ be a maximal right ideal of $R$, and let $r \in R$. Then $J(R) \subseteq M$, and that $M/J(R)$ is a maximal right of $R/J(R)$. As is well known, every potent ring is commutative, and so $rx + J(R) \in M/J(R)$ for any $x \in M$. Write $rx + J(R) = y + J(R)$ for a $y \in M$. hence, $rx - y \in J(R) \subseteq M$. This shows that $rx \in M$; hence, $rM \subseteq M$. Therefore $M$ is a two-sided ideal, and then $R$ is right quasi-duo. Likewise, $R$ is left quasi-duo. (3) is obvious, by Lemma 4.7.
$\Longleftarrow$ Let $a \in R$. By (1), there exists a $p \in R$ such that $\overline{a-p} \in N\big(R/J(R)\big)$, $p - p^n \in J(R) (n \geq 2)$, by Theorem 1.1. By (3), we may assume that $p = p^n$. Set $w = a - p$. Then $\overline{w}^m = 0$. Since $R$ is right (left) quasi-duo, as in [7, Corollary 3.4.7], we see that $ex - xe \in J(R)$ for any idempotent $e \in R$ and any

element $x \in R$. This means that $R/J(R)$ is abelian. Similarly to the proof of [7, COROLLARY 1.3.15], $R/J(R)$ is reduced. Hence, $\overline{w} = \overline{0}$, and then $w \in J(R)$. Therefore $a - p \in J(R)$, as desired. $\square$

As a consequence of Corollary 3.5, every right (left) duo periodic ring is strongly periodic. Further, we derive

Corollary 4.9. *A ring $R$ is strongly periodic if and only if*
(1) *$R$ is periodic;*
(2) *$R$ is right (left) quasi-duo;*
(3) *$J(R)$ is locally nilpotent.*

*Proof.* $\Longrightarrow$ Clearly, $R$ is periodic. It follows from Proposition 4.4 that $R$ is J-clean-like and $J(R)$ is locally nilpotent. Thus, $R$ is right (left) quasi-duo, by Theorem 4.8.

$\Longleftarrow$ Since $R$ is periodic, $R/J(R)$ is periodic. Thus, $R$ is J-clean-like, by Theorem 4.8. By (3), $J(R) = P(R)$, and the result follows. $\square$

*Example* 4.10. Let $R = \mathbb{Z}_{(5)}$. Then $R$ is right (left) quasi-duo, $R/J(R)$ is periodic, while $R$ is not $J$-clean-like.

*Proof.* Let $R = \mathbb{Z}_{(5)}$. Then $J(R) = 5R$. Hence, $R/J(R) \cong \mathbb{Z}_5$ is a finite field. Thus, $R/J(R)$ is periodic. Suppose every potent element lifts modulo $J(R)$. Clearly, $2 - 2^5 \in J(R)$. Hence, $\overline{2} \in R/J(R)$ is potent. Thus, we can find a potent $w \in R$ such that $2 - w \in J(R)$. Write $w = \frac{m}{n}$, where $(m, n) = 1, 5 \nmid n$ and $w = w^s (s \geq 2)$. Then $w(1 - w^{s-1}) = 0$, and so $w = 0$ or $w^{s-1} = 1$. If $w = 0$, then $2 \in J(R)$, a contradiction. If $w^{s-1} = 1$, then $\frac{m^{s-1}}{n^{s-1}} = 1$; whence, $m = \pm n$. This implies that $w = \pm 1$; hence, $2 - w = 1, 3 \notin J(R)$, a contradiction. Therefore $R$ is not J-clean-like, by Lemma 4.7. $\square$

Lemma 4.11. *Let $R$ be J-clean-like. Then $N(R) \subseteq J(R)$.*

*Proof.* Let $x \in N(R)$. Then $x^m = 0$ for some $m \geq 2$. Moreover, there exists a potent $p \in R$ such that $w := x - p \in J(R)$. Write $p = p^n$ for some $n \geq 2$. Then $p = p^n = (p^n)^n = p^{n^2} = (p^n)^{n^2} = p^{n^3} = \cdots = p^{n^m}$. Clearly, $n^m = \big(1+(n-1)\big)^m \geq m(n-1) \geq m$, and so $x^{n^m} = 0$. As $x^{n^m} - p^{n^m} \in J(R)$, we have $x = p + w = p^{n^m} + w \in J(R)$. Therefore $x \in J(R)$, hence the result. $\square$

Lemma 4.12. *Let $R$ be a ring. Then the following are equivalent:*
(1) *$R$ is a periodic ring in which every nilpotent is contained in $J(R)$.*
(2) *$R$ is J-clean-like and $J(R)$ is nil.*

*Proof.* (1) $\Rightarrow$ (2) Suppose $R$ is a periodic ring with $N(R) \subseteq J(R)$. Let $x \in J(R)$. Then we have $m, n \in \mathbb{N}$ such that $x^m = x^n (n > m)$. Hence, $x^m(1 - x^{n-m}) = 0$, and so $x^m = 0$. This shows that $J(R)$ is nil. Let $a \in R$. In view of Theorem 1.1, there exists a potent $p \in R$ such that $a - p \in N(R)$. By hypothesis, $a - p \in J(R)$. Therefore $R$ is J-clean-like.

(2) $\Rightarrow$ (1) For any $a \in R$, there exists a potent $p \in R$ such that $w := a - p \in J(R)$. Hence, $a = p + w$ and $p = p^n$ for some $n \geq 2$. Thus, $a^n = p^n + v$ for a $v \in J(R)$. This implies that $a - a^n = w - v \in J(R) \subseteq N(R)$. Therefore $R$ is periodic, by Theorem 1.1. In light of Lemma 4.11, every nilpotent of $R$ is contained in $J(R)$, as desired. □

THEOREM 4.13. *Let $R$ be a ring. If for any sequence of elements $\{a_i\} \subseteq R$ there exists a $k \in \mathbb{N}$ and $n_1, \cdots, n_k \geq 2$ such that $(a_1 - a_1^{n_1}) \cdots (a_k - a_k^{n_k}) = 0$, then $R$ is J-clean-like.*

*Proof.* For any $a \in R$, we have a $k \in \mathbb{N}$ and $n_1, \cdots, n_k \geq 2$ such that $(a - a^{n_1}) \cdots (a - a^{n_k}) = 0$. This implies that $a^k = a^{k+1} f(a)$ for some $f(t) \in \mathbb{Z}[t]$. In view of Theorem 1.1, $R$ is periodic.

Clearly, $R/J(R)$ is isomorphic to a subdirect product of some primitive rings $R_i$. Case 1. There exists a subring $S_i$ of $R_i$ which admits an epimorphism $\phi_i : S_i \to M_2(D_i)$ where $D_i$ is a division ring. Case 2. $R_i \cong M_{k_i}(D_i)$ for a division ring $D_i$. Clearly, the hypothesis is inherited by all subrings, all homomorphic images of $R$, we claim that, for any sequence of elements $\{a_i\} \subseteq M_2(D_i)$ there exists $s \in \mathbb{N}$ and $m_1, \cdots, m_s \geq 2$ such that $(a_1 - a_1^{m_1}) \cdots (a_s - a_s^{m_s}) = 0$. Choose $a_i = e_{12}$ if $i$ is odd and $a_i = e_{21}$ if $i$ is even. Then $(a_1 - a_1^{m_1})(a_2 - a_2^{m_2}) \cdots (a_s - a_s^{m_s}) = a_1 a_2 \cdots a_s \neq 0$, a contradiction. Thus, Case I do not happen. Further, in Case II, $k_i = 1$ for all $i$. This shows that each $R_i$ is reduced, and then so is $R/J(R)$. If $a \in N(R)$, we have some $n \in \mathbb{N}$ such that $a^n = 0$, and thus $\overline{a}^n = 0$ is $R/J(R)$. Hence, $\overline{a} \in J\big(R/J(R)\big) = 0$. This implies that $a \in J(R)$, and so $N(R) \subseteq J(R)$. Therefore $R$ is J-clean-like, by Lemma 4.12. □

Recall that a subset $I$ of a ring $R$ is left (resp., right) $T$-nilpotent in case for every sequence $a_1, a_2, \cdots$ in $I$ there is an $n$ such that $a_1 \cdots a_n = 0$ (resp., $a_n \cdots a_1 = 0$). Every nilpotent ideal is left and right $T$-nilpotent. The Jacobson radical $J(R)$ of a ring $R$ is left (resp., right) $T$-nilpotent if and only if for any nonzero left (resp., right) $R$-module $M$, $J(R)M \neq M$(resp., $MJ(R) \neq M$).

COROLLARY 4.14. *Let $R$ be a ring. If $R/J(R)$ is potent and $J(R)$ is left (resp., right) $T$-nilpotent, then $R$ is J-clean-like.*

*Proof.* We may assume $R/J(R)$ is potent and $J(R)$ is left $T$-nilpotent. For every sequence $a_1, a_2, \cdots, a_m, \cdots$ in $R$, there exists some $n_i \in \mathbb{N}$ such that $a_i - a_i^{n_i} \in J(R)$ for all $i$. We choose $b_1 = a_1 - a_1^{n_1}, b_2 = (1 - b_1)^{-1}\big(a_2 - a_2^{n_2}\big), b_3 = (1 - b_2)^{-1}\big(a_3 - a_3^{n_3}\big), \cdots, b_m = (1 - b_{m-1})^{-1}\big(a_m - a_m^{n_m}\big), \cdots$. By hypothesis, we can find some $k \in \mathbb{N}$ such that $b_1(1 - b_1)b_2(1 - b_2) \cdots b_{k-1}(1 - b_{k-1}) = 0$.

Hence, $b_1(1-b_1)b_2(1-b_2)\cdots b_{k-1}(1-b_{k-1})b_k = 0$. This shows that $(a_1 - a_1^{n_1})\cdots(a_s - a_k^{n_k}) = 0$. Therefore $R$ is J-clean-like, by Theorem 4.13. □

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