A ring $R$ is periodic provided that for any $a \in R$ there exist distinct elements $m, n \in \mathbb{N}$ such that $a^m = a^n$. We shall prove that periodicity is inherited by all generalized matrix rings. A ring $R$ is called strongly periodic if for any $a \in R$ there exists a potent $p \in R$ such that $a - p$ is in its Wedderburn radical and $ap = pa$. A ring $R$ is J-clean-like if for any $a \in R$ there exists a potent $p \in R$ such that $a - p$ is in its Jacobson radical. Furthermore, we completely determine the connections between strongly periodic rings and periodic rings. The relations among J-clean-like rings and these rings are also obtained.

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Key words: periodic ring, strongly periodic ring, J-clean-like ring, generalized matrix ring.

1. INTRODUCTION

A ring $R$ is periodic provided that for any $a \in R$ there exist distinct elements $m, n \in \mathbb{N}$ such that $a^m = a^n$. Examples of periodic rings are finite rings and Boolean rings. There are many interesting problems related to periodic rings. We explore, in this article, the periodicity of a type of generalized matrix rings. An element $p \in R$ is potent if $p = p^m$ for some $m \geq 2$. For later convenience we state here some elementary characterizations of periodic rings:

**Theorem 1.1.** Let $R$ be a ring. Then the following are equivalent:

1. $R$ is periodic.
2. For any $a \in R$, there exists some $m \geq 2$ such that $a^m = a^{m+1} f(a)$ for some $f(t) \in \mathbb{Z}[t]$.
3. For any $a \in R$, there exists some $m \geq 2$ such that $a - a^m \in R$ is nilpotent.
4. For any $a \in R$, there exists a potent $p \in R$ such that $a - p \in R$ is nilpotent and $ap = pa$.

Here, the equivalences of all items are stated in [1, Lemma 2], [5, Proposition 2], [19, Theorem 3], and the simple implication from (3) to (2). A Morita context $(A, B, M, N, \psi, \varphi)$ consists of two rings $A$ and $B$, two bimodules...
$A N_B$ and $B M_A$, and a pair of bimodule homomorphisms $\psi : N \otimes_M B \to A$ and $\varphi : M \otimes_A N \to B$ which satisfy the following associativity: $\psi(n \otimes_B m)n' = n\varphi(m \otimes_A n')$ and $\varphi(m \otimes_A n)m' = m\psi(n \otimes_B m')$ for any $m, m', n, n' \in M, n, n' \in N$.

These conditions ensure that the set $T$ of generalized matrices $\begin{pmatrix} a & n \\ m & b \end{pmatrix}$; $a \in A, b \in B, m \in M, n \in N$ will form a ring with addition defined componentwise and with multiplication defined by

$$\begin{pmatrix} a_1 & n_1 \\ m_1 & b_1 \end{pmatrix} \begin{pmatrix} a_2 & n_2 \\ m_2 & b_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + \psi(n_1 \otimes m_2) & a_1 n_2 + n_1 b_2 \\ m_1 a_2 + b_1 m_2 & \varphi(m_1 \otimes n_2) + b_1 b_2 \end{pmatrix},$$

called the ring of the Morita context (cf. [20]). The class of rings of the Morita contexts is a type of generalized matrix rings. For instances, all $2 \times 2$ matrix rings and all triangular matrix rings.

Let $T$ be the ring of a Morita context $(A, B, M, N, \psi, \varphi)$. We prove, in Section 2, that if $\text{im}(\psi)$ and $\text{im}(\varphi)$ are nilpotent, then $A$ and $B$ are periodic if and only if so is $T$. This provides a large new class of periodic rings for generalized matrix rings.

It is an attractive problem to express an element in a ring as the sum of idempotents and units (cf. [4], [6], [8] and [9]). We say that a ring $R$ is clean provided that every element in $R$ is the sum of an idempotent and a unit. Such rings have been extensively studied in recent years, see [7] and [21]. This motivates us to combine periodic rings with clean rings together, and investigate further properties of related rings.

For a ring $R$ the Wedderburn radical is denoted by $P(R)$, i.e., $P(R)$ is the sum of all nilpotent ideals of $R$. We now introduce a new type of rings. A ring $R$ is said to be strongly periodic provided that for any $a \in R$ there exists a potent $p \in R$ such that $a - p \in P(R)$ and $ap = pa$. Strongly periodic rings form a subclass of periodic rings. We shall prove that a ring $R$ is strongly periodic if and only if for any $a \in R$ there exists a potent $p \in R$ such that $a - p \in P(R)$, and determine completely the connections between these ones and periodic rings. A ring is 2-primary provided that its Wedderburn radical coincides with the set of nilpotent elements of the ring. It is proved that a ring $R$ is strongly periodic if and only if $R$ is a 2-primary periodic ring. From this, we show that the strong periodicity will be inherited by generalized matrix rings.

Replacing the Wedderburn radical $P(R)$ by the Jacobson radical $J(R)$, we introduce a type of rings which behave like that of periodic rings. We say that a ring $R$ is $J$-clean-like provided that for any $a \in R$ there exists a potent $p \in R$ such that $a - p \in J(R)$. This is a natural generalization of J-clean rings.
[6]. Many properties of periodic rings are extended to these ones. We shall characterize J-clean-like rings and obtain the relations among these rings.

Throughout, all rings are associative with an identity. \( M_n(R) \) will denote the ring of all \( n \times n \) matrices over \( R \) with an identity \( I_n \). \( N(R) \) stands for the set of all nilpotent elements in \( R \). \( C(R) \) denote the center of \( R \). \( P(R) \) and \( J(R) \) denote the Wedderburn radical and Jacobson radical of \( R \), respectively.

\[ \square \]

### 2. PERIODIC RINGS

The purpose of this section is to investigate the periodicity for Morita contexts. The following lemma is known [17, Lemma 3.1.23], and we include a simple proof for the sake of completeness.

**Lemma 2.1.** A ring \( R \) is periodic if and only if for any \( a, b \in R \), there exists an \( n \in \mathbb{N} \) such that \( a - a^n, b - b^n \in N(R) \).

**Proof.** \( \Longleftarrow \) For any \( a \in R \), we can find \( n \in \mathbb{N} \) such that \( a - a^n \in N(R) \). This implies that \( R \) is periodic, by Theorem 1.1.

\( \Longrightarrow \) Suppose that \( R \) is periodic. For any \( a, b \in R \), we can find \( p, q, s, t \in R \) \((p < q, s < t)\) such that \( a^p = a^q \) and \( b^s = b^t \). Hence, \( a^{pq} = a^{qs} \) and \( b^{ps} = b^{pt} \). This implies that

\[
a^{ps} = a^{ps} a^{(q-p)s} = a^{ps} a^{2(q-p)s} = \cdots = a^{ps} a^{(t-s)p(q-p)s}.
\]

Likewise, we get \( b^{ps} = b^{ps} b^{(q-p)s(t-s)p} \). Choose \( k = ps \) and \( l = ps + (t-s)p(q-p)s \). Then \( a^k = a^l, b^k = b^l \) \((k < l)\). Thus, \( a^k = a^l = a^{(l-k)+k} = \cdots = a^{k(l-k)+k} \), and so \( a^k = (a^k)^{l-k+1} \). This implies that \( (a^{k(l-k)})^2 = (a^{k(l-k)+k})(a^{k(l-k)-k}) = a^k(a^{k(l-k)-k}) = a^{k(l-k)} \). Choose \( n = k(l-k) \). Then \( (a-a^{n+1})^n = a^n(1-a^n)^n = a^n(1-a^n) = 0 \). Thus, \( a - a^n \in N(R) \). Likewise, \( b - b^n \in N(R) \). Therefore, we complete the proof. \( \square \)

**Theorem 2.2.** Let \( T \) be the ring of a Morita context \((A, B, M, N, \psi, \varphi)\). If \( \text{im}(\psi) \) and \( \text{im}(\varphi) \) are nilpotent, then \( A \) and \( B \) are periodic if and only if so is \( T \).

**Proof.** Suppose \( A \) and \( B \) are periodic. For any \( \begin{pmatrix} a & n \\ m & b \end{pmatrix} \in T \), as in the proof of Lemma 2.1, there exists a \( k \in \mathbb{N} \) such that \( a - a^k \in N(A) \) and \( b - b^k \in N(B) \). Hence,

\[
\begin{pmatrix} a & n \\ m & b \end{pmatrix} - \begin{pmatrix} a & n \\ m & b \end{pmatrix}^k = \begin{pmatrix} a - a^k + c & * \\ * & b - b^k + d \end{pmatrix},
\]

where \( c \) and \( d \) are from the nilpotent elements of \( A \) and \( B \), respectively.
where \( c \in \text{im}(\psi) \) and \( d \in \text{im}(\varphi) \). Write \( (a - a^k)^l = 0 \) and \( (b - b^k)^l = 0 \). By hypothesis, \( \text{im}(\psi) \) and \( \text{im}(\varphi) \) are nilpotent ideals of \( A \) and \( B \), respectively. Say \( (\text{im}(\psi))^s = 0 \) and \( (\text{im}(\varphi))^t = 0 \). Choose \( p = \max(s, t) \) and \( q = p(l + 1) \). Then
\[
(a - a^k + c)^q = 0 \text{ and } (b - b^k + d)^q = 0.
\]
Obviously,
\[
\begin{pmatrix}
a - a^k + c & * \\
* & b - b^k + d
\end{pmatrix}^{q+1} \in \begin{pmatrix}
\text{im}(\psi) & N \\
M & \text{im}(\varphi)
\end{pmatrix}.
\]
Set \( NM := \text{im}(\psi) \) and \( MN := \text{im}(\varphi) \). We see that
\[
\begin{pmatrix}
NM & N \\
M & MN
\end{pmatrix}^2 \subseteq \begin{pmatrix}
NM & (NM)N \\
(MN)M & MN
\end{pmatrix}.
\]
For any \( l \in \mathbb{N} \), by induction, one easily checks that
\[
\begin{pmatrix}
NM & N \\
M & MN
\end{pmatrix}^{2l} \subseteq \begin{pmatrix}
NM & (NM)N \\
(MN)M & MN
\end{pmatrix}^l \subseteq \begin{pmatrix}
(NM)^l & (NM)^lN \\
(MN)^lM & (MN)^l
\end{pmatrix}.
\]
Choose \( j = 2p(q + 1) \). As \( (NM)^p = (MN)^p = 0 \), we get
\[
\begin{pmatrix}
a - a^k + c & * \\
* & b - b^k + d
\end{pmatrix}^j = \begin{pmatrix} 0 & t \\
s & 0
\end{pmatrix}
\]
for some \( s \in M, t \in N \). Hence,
\[
\begin{pmatrix} 0 & t \\
s & 0
\end{pmatrix}^2 = \begin{pmatrix}
\psi(t \otimes s) & 0 \\
0 & \varphi(s \otimes t)
\end{pmatrix},
\]
and so
\[
\left( \begin{pmatrix} a & n \\ m & b \end{pmatrix} - \begin{pmatrix} a & n \\ m & b \end{pmatrix} \right)^k = \begin{pmatrix} 1_R & 0 \\ 0 & 1_R \end{pmatrix}.
\]
Accordingly, \( T \) is periodic, by Theorem 1.1. The converse is obvious. \( \square \)

Let \( R \) be a ring, and let \( s \in C(R) \). Let \( M_s(R) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, b, c, d \in R \} \), where the operations are defined as follows:
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a + a' & b + b' \\ c + c' & d + d' \end{pmatrix},
\]
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} aa' + sbc & ab' + bd' \\ ca' + dc' & scb + dd' \end{pmatrix}.
\]
Then \( M_s(R) \) is a ring with the identity \( \begin{pmatrix} 1_R & 0 \\ 0 & 1_R \end{pmatrix} \). Recently, the strong cleanness of such type generalized matrix rings was studied in [21]. For the periodicity of such rings, we derive
Corollary 2.3. Let $R$ be periodic, and let $s \in N(R) \cap C(R)$. Then $M_s(R)$ is periodic.

Proof. Let $\psi : R \otimes R \to R$, $r \otimes m \mapsto srm$ and $\varphi : R \otimes R \to R$, $m \otimes n \mapsto smn$. Then $M_s(R) = (R, R, R, R, \psi, \varphi)$. As $s \in N(R) \cap C(R)$, we see that $\text{im}(\varphi)$ and $\text{im}(\psi) \subseteq J(R)$ are nilpotent, and we are through by Theorem 2.2.

As a consequence, a ring $R$ is periodic if and only if so is the trivial Morita context $M(0)(R)$. Choosing $s = 0 \in R$, we are through from Corollary 2.3. Given a ring $R$ and an $R$-$R$-bimodule $M$, the trivial extension of $R$ by $M$ is the ring $T(R, M) = R \oplus M$ with the usual addition and the following multiplication: $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2)$.

Corollary 2.4. Let $R$ be a ring, and let $M$ be a $R$-$R$-bimodule. Then the following are equivalent:

(1) $R$ is periodic.
(2) $T(R, M)$ is periodic.

Proof. (1) $\Rightarrow$ (2) Let $R$ be a periodic ring and let $S = \left(\begin{array}{cc} R & M \\ 0 & R \end{array}\right)$. It is obvious by Theorem 2.2 that $S$ is periodic. Clearly, $T(R, M)$ is a subring of $S$, and so proving (2).

(2) $\Rightarrow$ (1) Let $T(R, M)$ be a periodic ring. As $R$ is isomorphic to a subring of $T(R, M)$, and so $R$ is periodic. □

Example 2.5. Let $R$ be periodic, let

\[
A = B = \begin{pmatrix} R & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & R \end{pmatrix},
M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & R & 0 \end{pmatrix},
\text{ and } N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ R & R & 0 \end{pmatrix},
\]

and let $\psi : N \otimes M \to A$, $\psi(n \otimes m) = nm$ and $\phi : M \otimes N \to B$, $\phi(m \otimes n) = mn$. Then $T = (A, B, M, N, \psi, \phi)$ is a Morita context with zero pairings, i.e., $T$ is a trivial Morita context. Hence, $\text{im}(\psi)$ and $\text{im}(\varphi)$ are nilpotent. Clearly, $A$ and $B$ are both periodic. In light of Theorem 2.2, $T$ is periodic.

Let $R$ be a ring, and let $\alpha$ be an endomorphism of $R$. Let $T_n(R, \alpha)$ be the set of all upper triangular matrices over the rings $R$. For any $(a_{ij}), (b_{ij}) \in T_n(R, \alpha)$, we define $(a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$, and $(a_{ij})(b_{ij}) = (c_{ij})$ where $c_{ij} = \sum_{k=i}^{n} a_{ik} \alpha^{k-i}(b_{kj})$. Then $T_n(R, \alpha)$ is a ring under the preceding addition and multiplication (cf. [14]). Clearly, $T_n(R, \alpha)$ will be $T_n(R)$ only when $\alpha$ is the identity morphism.
Lemma 2.6. Let $R$ be periodic, and let $\alpha : R \rightarrow R$ be an endomorphism. Then

1. $R[[x,\alpha]]/(x^n)$ is periodic.
2. $T_n(R,\alpha)$ is periodic for all $n \in \mathbb{N}$.

Proof. (1) For any $f(x) \in R[[x]]/(x^n)$, there exists an $m \in \mathbb{N}$ such that $f(0) - f^m(0) \in N(R)$. Hence, $f(x) - f^m(x) \in N(R[[x]]/(x^n))$. According to Theorem 1.1, $R[[x]]/(x^n)$ is periodic.

(2) For any $(a_{ij}) \in T_n(R,\alpha)$, as in the proof of Lemma 2.1, we can find an $m \in \mathbb{N}$ such that $a_{ii} - a_{ii}^m \in N(R)$ for each $i$. Thus, $(a_{ij}) - (a_{ij})^m \in N(T_n(R,\alpha))$, as required. □

We are now ready to prove:

Theorem 2.7. Let $R$ be periodic. Then $M_{(x_m)}(R[[x]]/(x^n))$ is periodic for all $1 \leq m \leq n$.

Proof. Choose $\alpha = 1$. Then $R[[x]]/(x^n)$ is periodic, by Lemma 2.6. Choose $s = x^m (1 \leq m \leq n)$. Then $s \in N(R[[x]]/(x^n)) \cap C(R[[x]]/(x^n))$. Applying Corollary 2.3 to $R[[x]]/(x^n)$, $M_{(x_m)}(R[[x]]/(x^n))$ is periodic, as asserted. □

Corollary 2.8. Let $R$ be a finite ring. Then $M_{(x_m)}(R[[x]]/(x^n))$ is periodic for all $1 \leq m \leq n$.

Proof. Since every finite ring is periodic, we complete the proof by Theorem 2.7. □

3. STRONGLY PERIODIC RINGS

A ring $R$ is potent if for any $a \in R$, there exists some $n \geq 2$ such that $a = a^n$. An ideal $I$ of a ring $R$ is locally nilpotent if, every finitely generated subring of elements belonging to $I$ is nilpotent. Clearly, an ideal $I$ of a ring $R$ is locally nilpotent if and only if $RxR$ is nilpotent for any $x \in I$. Recall that $J(R)$ consists of all $x \in R$ such that $1 + RxR$ is included in the set of units of $R$. We now derive

Theorem 3.1. Let $R$ be a ring. Then the following are equivalent:

1. $R$ is strongly periodic.
2. $R$ is periodic and $N(R)$ is a locally nilpotent ideal of $R$.
3. $R/J(R)$ is potent, every potent lifts modulo $J(R)$ and $J(R)$ is locally nilpotent.
Proof. (1) ⇒ (2) Clearly, $R$ is periodic. Let $x \in N(R)$. Then we can find a potent $p \in R$ such that $w := x - p \in P(R)$. Write $x^n = 0$ for some $n \in \mathbb{N}$. Then $p^n = (x - w)^n \in P(R)$. This shows that $p \in R$ is nilpotent, and so $p = 0$; hence, $x = w \in P(R)$. We infer that $N(R) = P(R)$ is an ideal of $R$.

For any $x \in P(R)$, we claim that $RxR$ is nilpotent. As $P(R)$ is the sum of all nilpotent ideals of $R$, we can find nilpotent ideals $I_1, \ldots, I_m$ of $R$ such that $x \in I_1 + \cdots + I_m$. Clearly, $I_1 + \cdots + I_m$ is a nilpotent ideal. It follows from $RxR \subseteq I_1 + \cdots + I_m$ that $RxR$ is nilpotent. Thus, $N(R)$ is locally nilpotent. Hence, $N(R)$ is nilpotent, and so $J$ and therefore $p$ hence, $p$.

(2) ⇒ (1) Let $x \in N(R)$. As $N(R)$ is locally nilpotent, $RxR$ is nilpotent. Write $(RxR)^m = 0 (m \in \mathbb{N})$. Then $RxR \subseteq P(R)$. This implies that $x \in P$; hence, $N(R) \subseteq P(R)$. The implication is true, by Theorem 1.1.

(1) ⇒ (3) For any $a \in R$ there exists some potent $p \in R$ such that $a - p \in P(R) \subseteq J(R)$. Hence, $\bar{a} = \bar{p}$ in $R/J(R)$. Therefore $R/J(R)$ is potent.

Let $x \in J(R)$. Then there exists a potent $p \in R$ such that $x - p \in P(R)$; hence, $p = x - (x - p) \in J(R)$. Write $p = p^m (m \geq 2)$. Then $p(1 - p^{m-1}) = 0$, and so $p = 0$. Hence, $x \in P(R)$. By the preceding discussion, $RxR$ is nilpotent, and therefore $J(R)$ is locally nilpotent.

(3) ⇒ (1) Let $a \in R$. Then $a - a^n \in J(R)$ for some $n \geq 2$. As $J(R)$ is locally nilpotent, it is nilpotent, and so $a - a^n \in N(R)$. In view of Theorem 1.1, $R$ is periodic. Let $x \in N(R)$. Then $\bar{x} \in R/J(R)$ is potent; hence, $\bar{x} = \bar{0}$ in $R/J(R)$. That is, $x \in J(R)$. By hypothesis, $J(R)$ is locally nilpotent; hence, $RxR$ is nilpotent. As in the proof in (2) ⇒ (1), we see that $x \in P(R)$. Thus, $N(R) \subseteq P(R)$.

For any $a \in R$, there exists a potent $p \in R$ and a $w \in N(R)$ such that $a = p + w$ and $pw = wp$, by Theorem 1.1. By the preceding discussion, $w \in P(R)$. This proving (1). □

Corollary 3.2. A ring $R$ is strongly periodic if and only if for any $a \in R$ there exists a potent $p \in R$ such that $a - p \in P(R)$.

Proof. ⇒ This is trivial.

⇐ For any $a \in R$ there exists potent $p \in R$ such that $a - p \in P(R) \subseteq J(R)$. Hence, $R/J(R)$ is potent. For any $x \in J(R)$, there exists a potent $q \in R$ such that $x - q \in P(R)$. Hence, $q = x - (x - q) \in J(R)$. Write $q = q^m (m \geq 2)$. Then $q(1 - q^{m-1}) = 0$, and so $q = 0$. We infer that $x \in P(R)$. As in the proof in Theorem 3.1, $RxR$ is nilpotent, and so $J(R)$ is locally nilpotent. This result follows, by using Theorem 3.1. □

A ring $R$ is a 2-primary ring $R$ if its Wedderburn radical coincides with the set of all nilpotent elements, i.e. $N(R) = P(R)$. A ring $R$ is weakly periodic provided that for any $a \in R$ there exists a potent $p \in R$ such that $a - p \in N(R)$ [17]. We now derive
Theorem 3.3. A ring $R$ is strongly periodic if and only if $R$ is a 2-
primary weakly periodic ring.

Proof. $\implies$ Clearly, $R$ is weakly periodic. For any $a \in N(R)$, there exists
a potent $p \in R$ such that $w := a - p \in P(R)$. Hence, $p = a - w$. Write
$a^m = 0 (m \in \mathbb{N})$. Then $p^m \in P(R)$, and so $p \in N(R)$. This implies that $p = 0,$
and so $a = w \in P(R)$. Thus, $N(R) = P(R)$, and so $R$ is 2-primary.

$\iff$ Let $a \in R$. Since $R$ is weakly periodic, there exists a potent $p \in R$
such that $a - p \in N(R)$. As $R$ is 2-primary, $N(R) \subseteq P(R)$, we get $a - p \in P(R)$. Therefore, we complete the proof, by Corollary 3.2. $\Box$

A ring $R$ is called strongly 2-primal provided that $R/I$ is 2-primal for all
ideals $I$ of $R$.

Corollary 3.4. A ring $R$ is strongly periodic if and only if the following
two conditions hold:

1) $R$ is a weakly periodic ring with locally nilpotent $J(R)$;
2) Every prime ideal of $R$ is completely prime.

Proof. $\implies$ (1) is obvious. Clearly, $R/P(R)$ is potent. As is well known,
every potent ring is commutative (cf. [10, Theorem 1 in Chapter X]), and so
$R/P(R)$ is commutative. In view of [15, Proposition 1.2], every prime ideal of
$R$ is completely prime.

$\iff$ In view of [15, Proposition 1.2], $R$ is strongly 2-primal, and then it
is 2-primary. As $J(R)$ is locally nilpotent, we show that $R$ is 2-primary. This
completes the proof, in terms of Theorem 3.3. $\Box$

A ring $R$ is called nil-semicommutative if $ab = 0$ in $R$ implies that $aRb = 0$
for every $a, b \in N(R)$ (see [16]). For instance, every semicommutative ring ($i.e.$,
$ab = 0$ in $R$ implies that $aRb = 0$) is nil-semicommutative.

Corollary 3.5. Every nil-semicommutative weakly periodic ring is strongly
periodic.

Proof. One easily checks that every nil-semicommutative ring is 2-primary, so the result follows from Theorem 3.3. $\Box$

We note that strongly periodic rings may not be nil-semicommutative as the following shows.

Example 3.6. Let $\mathbb{Z}_2$ be the field of integral modulo 2, and let
Periodicity and J-Clean-Like Rings

\[ R_n = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{1n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in \mathbb{Z}_2 \} \]

with \( 3 \leq n \in \mathbb{N} \). Let \( R = \bigoplus_{n=3}^{\infty} R_n \) be the subalgebra of \( \prod_{n=3}^{\infty} R_n \) over \( \mathbb{Z}_2 \) generated by \( \bigoplus_{n=3}^{\infty} R_n \) and 1. We note that \( P(R) = \bigoplus_{n=3}^{\infty} P(R_n) \). Hence, \( R/P(R) \cong \bigoplus_{n=3}^{\infty} F_n \), the subalgebra of \( \prod_{n=3}^{\infty} F_n \) over \( \mathbb{Z}_2 \) generated by \( \bigoplus_{n=3}^{\infty} F_n \) and 1, where \( F_n = \mathbb{Z}_2 \) for all \( n = 3, 4, \cdots \). This implies that \( R/P(R) \) is reduced. For any \( a \in N(R) \), \( \tilde{a} \in R/P(R) \) is nilpotent, and so \( \tilde{a} = \tilde{0} \). That is, \( a \in P(R) \). Therefore \( R \) is 2-primary. As \( R_n \) is a finite ring for each \( n \), we see that it is periodic. We infer that \( R \) is periodic, and so it is weakly periodic. In light of Theorem 3.3, \( R \) is strongly periodic. We claim that \( R_4 \) is not nil-semicommutative. Choose

\[
\begin{align*}
a &= \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
x &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \\
b &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\end{align*}
\]

Then \( a^2 = b^2 = 0 \), and so \( a, b \in N(R_4) \). Furthermore, \( ab = 0 \), while \( axb \neq 0 \). Thus, \( R_4 \) is not nil-semicommutative. Therefore \( R \) is not nil-semicommutative, and we are done.

**Theorem 3.7.** Let \( T \) be the ring of a Morita context \((A, B, M, N, \psi, \varphi)\). If \( \text{im}(\psi) \) and \( \text{im}(\varphi) \) are nilpotent, then \( A \) and \( B \) are strongly periodic if and only if so is \( T \).

**Proof.** Suppose \( A \) and \( B \) are strongly periodic. Then \( A \) and \( B \) are 2-primary, by Theorem 3.3. Further, they are periodic. In view of Theorem 2.2, \( T \) is periodic. It suffices to prove that \( T \) is 2-primary.

Let \( \begin{pmatrix} a & n \\ m & b \end{pmatrix} \in T \) be nilpotent. Then we can find some \( c \in \text{im}(\psi) \) and \( d \in \text{im}(\varphi) \) such that \( a^k + c = 0 \) and \( b^l + d = 0 \) for some \( k, l \in \mathbb{N} \). This implies that \( a \in N(A) \) and \( b \in N(B) \). As \( A \) is 2-primary, \( a \in P(A) \). Analogously to the proof in Theorem 3.1, we see that \( AaA \) is nilpotent. Likewise, \( BbB \) is...
nilpotent. Clearly,
\[
T \left( \begin{array}{cc} a & n \\ m & b \end{array} \right) T \subseteq \left( \begin{array}{cc} AaA + \text{im}(\psi) & N \\ M & BbB + \text{im}(\varphi) \end{array} \right).
\]
As the sum of two nilpotent ideal of a ring is nilpotent, we see that \( AaA + \text{im}(\psi) \) and \( BbB + \text{im}(\varphi) \) are nilpotent ideals of \( A \) and \( B \), respectively. Similarly to the proof of Theorem 2.2, we see that \( AaA + \text{im}(\psi) \) and \( BbB + \text{im}(\varphi) \) are nilpotent ideals of \( A \) and \( B \), respectively. Similarly to the proof of Theorem 2.2, we see that \( (AaA + \text{im}(\psi) \cap M BbB + \text{im}(\varphi)) \) is a nilpotent ideal of \( T \). Hence, \( T \left( \begin{array}{cc} a & n \\ m & b \end{array} \right) T \) is nilpotent, and so \( \left( \begin{array}{cc} a & n \\ m & b \end{array} \right) \in P(T) \). Thus, \( T \) is 2-primary, and so \( T \) is strongly periodic, by Theorem 3.3.

Conversely, assume that \( T \) is strongly periodic. Then \( A \) is periodic. Let \( a \in N(R) \). Then \( \left( \begin{array}{cc} a & 0 \\ 0 & 0 \end{array} \right) \in N(T) \). By virtue of Theorem 3.3, \( T \) is 2-primary; hence, \( \left( \begin{array}{cc} a & 0 \\ 0 & 0 \end{array} \right) \in P(T) \). As in the proof in Theorem 3.1, we see that \( T \left( \begin{array}{cc} a & 0 \\ 0 & 0 \end{array} \right) T \) is nilpotent. Then \( AaA \) is nilpotent, and so \( a \in P(R) \). It follows that \( A \) is 2-primary. Therefore \( A \) is strongly periodic, by Theorem 3.3. Likewise, \( B \) is strongly periodic, as required. \( \square \)

**Corollary 3.8.** Let \( R \) be strongly periodic, and let \( s \in N(R) \cap C(R) \). Then \( M_{(s)}(R) \) is strongly periodic.

**Proof.** As in the proof of Corollary 2.3, we have \( M_{s}(R) = (R, R, R, R, \psi, \varphi) \) where \( \text{im}(\varphi) \) and \( \text{im}(\psi) \) are nilpotent. This completes the proof, by Theorem 3.7. \( \square \)

**Example 3.9.** Consider the Morita context \( R = \left( \begin{array}{cc} \mathbb{Z}_4 & \mathbb{Z}_4 \\ 2\mathbb{Z}_4 & \mathbb{Z}_4 \end{array} \right) \), where the context products are the same as the product in \( \mathbb{Z}_4 \). Then we claim that \( R \) is strongly periodic. Since \( R \) is finite, it is periodic, and then we are done by Theorem 3.7.

As a consequence, a ring \( R \) is strongly periodic if and only if so is the trivial Morita context \( M_{(0)}(R) \). Now we exhibit the useful characterizations of strongly periodic rings as follows.

**Theorem 3.10.** Let \( R \) be a ring. Then the following are equivalent:

1. \( R \) is strongly periodic.
2. \( R/P(R) \) is potent.
3. For any \( a \in R \), there exists a prime \( m \geq 2 \) such that \( a - a^m \in P(R) \).
4. For any \( a \in R \), \( a = eu + w \), where \( e = e^2 \in R, w^m = 1 \) (\( m \in \mathbb{N} \)), \( w \in P(R) \) and \( e, u, w \) commutate.
Therefore, $x \in \mathbb{N}$ for any $x \in S$ there exists a prime $n$ such that $x = x^n$ (cf. [13]). Let $a \in R$. Since $R/J(R)$ is potent, we have a prime $m \geq 2$ such that $\overline{a} = \overline{a^m}$ in $R/P(R)$. Therefore, $a - a^m \in P(R)$.

(3) $\Rightarrow$ (4) Let $a \in R$. Then we have a prime $n \geq 2$ such that $a - a^n \in P(R) \subseteq N(R)$. By Theorem 1.1, $R$ is periodic. Let $x \in N(R)$. Then $\overline{x} \in R/P(R)$ is potent; whence, $\overline{x} = 0$ in $R/P(R)$. Thus, $x \in P(R)$, and so $N(R) \subseteq P(R)$. By [7, Proposition 13.1.18], $a = eu + w$, where $e = e^2 \in R, u \in U(R), w \in P(R)$ and $e, u, w$ commutate. Write $u^k = u^{k+m}$ for some $m, k \in \mathbb{N}$. Then $u^m = 1$, as desired,

(4) $\Rightarrow$ (1) For any $a \in R$, $a = eu + w$, where $e = e^2 \in R, u^m = 1 (m \in \mathbb{N}), w \in P(R)$ and $e, u, w$ commutate. Set $p = eu$. Then $p = eu^{m+1} = p^{m+1}$, i.e., $p \in R$ is potent. Thus, $R$ is strongly periodic. □

Corollary 3.11. Every subring of a strongly periodic ring is strongly periodic.

Proof. Let $R$ be strongly periodic, and let $S \subseteq R$. For any $a \in S$, there exists some $n \geq 2$ such that $a - a^n \in P(R)$ in terms of Theorem 3.10. Hence, $(R(a - a^n)R)^m = 0$ for some $m \in \mathbb{N}$. This forces that $(S(a - a^n)S)^m = 0$. Therefore $a - a^n \in P(S)$. By using Theorem 3.10 again, $S$ is strongly periodic, as needed. □

For example, if $R$ is the finite subdirect product of strongly periodic rings, then Corollary 3.11 shows that $R$ is strongly periodic.

Example 3.12. Let $F = GF(q)$ be a Galois field and let $V$ be an infinite dimensional left vector space over $F_p$ with $\{v_1, v_2, \cdots \}$ a basis. For the endomorphism ring $A = End_F(V)$, define $A_1 = \{f \in A \mid rank(f) < \infty \text{ and } f(v_i) = a_1v_1 + \cdots + a_iv_i \text{ for } i = 1, 2, \cdots \text{ with } a_j \in F_p\}$ and let $R$ be the $F$-algebra of $A$ generated by $A_1$ and $1_A$. Then $R$ is strongly periodic. As in the proof of [15, Example 1.1], $R/P(R) \cong \{(a_1, \cdots, a_n, b, b, \cdots) \mid a_i, b \in F \text{ and } n = 1, 2, \cdots\}$. As $F = GF(q)$, we see that $x = x^q$ for all $x \in F$, and then $R/P(R)$ is potent. According to Theorem 3.10, $R$ is strongly periodic.

Lemma 3.13. Let $I$ be a nilpotent ideal of a ring $R$. If $R/I$ is strongly periodic, then so is $R$.

Proof. Let $a \in R$. Then there exists some $n \geq 2$ such that $\overline{a - a^n} \in P(R/I)$. Hence, $(R(a - a^n)R)^m \subseteq I$. As $I$ is nilpotent, $(R(a - a^n)R)^{mn} = 0$. This shows that $a - a^n \in P(R)$. Therefore $R$ is strongly periodic, by Theorem 3.9. □
Theorem 3.14. Let $I$ be an ideal of a ring $R$. Then the following are equivalent:

1. $R/I$ is strongly periodic.
2. $R/I^n$ is strongly periodic for all $n \in \mathbb{N}$.
3. $R/I^n$ is strongly periodic for some $n \in \mathbb{N}$.

Proof. (1) $\Rightarrow$ (2) Clearly, $R/I \cong (R/I^n)/(I/I^n)$. Since $(I/I^n)^n = 0$, proving (2) by Lemma 3.13.

(2) $\Rightarrow$ (3) This is trivial.

(3) $\Rightarrow$ (1) For any $a \in R/I$, we see that $a + I^n \in R/I^n$. By hypothesis, there exists a potent $\overline{p} \in R/I^n$ such that $a - \overline{p} \in P(R/I^n)$. Write $\overline{p} = \overline{p}^m$ for some $m \geq 2$. Then $p - p^m \in I^n \subseteq I$, and so $\overline{p} \in R/I$ is potent. Obviously, $(R/I^n)(a - p)(R/I^n)$ is nilpotent, and then $(R(a - p)R)^s \subseteq I^n \subseteq I$ for some $s \in \mathbb{N}$. We infer that $(R/I)(a - \overline{p})(R/I)$ is nilpotent. As in the proof of Theorem 3.1, we infer that $\overline{a - p} \in P(R/I)$, as required. □

Recall that a ring $R$ is an abelian ring if every idempotent in $R$ is central. A ring $R$ is strongly $\pi$-regular if for any $a \in R$ there exists $n \in \mathbb{N}$ such that $a^n \in a^{n+1}R$. Obviously, every periodic ring is strongly $\pi$-regular. We now derive

Lemma 3.15. Every abelian periodic ring of bounded index is strongly periodic.

Proof. Let $R$ be an abelian periodic ring of bounded index. Then $R$ is strongly $\pi$-regular. Badawi’s Theorem states that the set of all nilpotent elements of an abelian strongly $\pi$-regular ring is an ideal [2]. Thus, $N(R)$ forms an ideal of $R$. This completes the proof, by Theorem 3.1. □

Let $n \geq 2$ be a fixed integer. A ring $R$ is said to be generalized $n$-like provided that for any $a, b \in R$, $(ab)^n - ab^n - a^n b + ab = 0$ (cf. [18]). It is proved that every generalized 3-like ring is commutative ( [18, Theorem 3]). We now derive

Theorem 3.16. Every generalized $n$-like ring is strongly periodic.

Proof. Let $R$ be a generalized $n$-like ring, and let $a \in R$. Then $a^{2n} - 2a^{n+1} + a^2 = 0$, and so $(a - a^n)^2 = 0$. Thus, $a - a^n \in N(R)$. Accordingly, $R$ is periodic by Theorem 1.1. In light of [18, Lemma 2], $R$ is abelian. If $a^m = 0$, then $a^2(1 - a^n) = 0$, and so $a^2 = 0$. Thus, $R$ is of bounded index 2. Therefore $R$ is strongly periodic, by Lemma 3.15. □

Let $R = \{ \begin{pmatrix} x & y & z \\ 0 & x^2 & 0 \\ 0 & 0 & x \end{pmatrix} | x, y, z \in GF(4) \}$. Then for each $a \in R$, $a^7 = a$
or \( a^7 = a^2 = 0 \). Therefore \( R \) is a generalized 7-like ring. By Theorem 3.16, \( R \) is strongly periodic. In this case, \( R \) is abelian but not commutative (cf. [18, Example 2]).

4. J-CLEAN-LIKE RINGS

We now consider J-clean-like Morita contexts and extend Theorem 2.2 as follows.

**Theorem 4.1.** Let \( T \) be the ring of a Morita context \((A, B, M, N, \psi, \varphi)\) with \( \text{im}(\psi) \subseteq J(A) \) and \( \text{im}(\varphi) \subseteq J(B) \). If \( A \) and \( B \) are J-clean-like, then so is \( T \).

**Proof.** Let \( \begin{pmatrix} a & n \\ m & b \end{pmatrix} \in T \). Then we have potent \( p \in A \) and \( q \in B \) such that \( a - p \in J(A) \) and \( b - q \in J(B) \). Hence

\[
\begin{pmatrix} a & n \\ m & b \end{pmatrix} - \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} = \begin{pmatrix} a - p & n \\ m & b - q \end{pmatrix}.
\]

Let \( \begin{pmatrix} c & s \\ t & d \end{pmatrix} \in T \). As \( 1_A - (a - p)c - \psi(n \otimes t) \in U(A) \) and \( 1_B - (b - q)d - \varphi(m \otimes s) \in U(B) \), it follows by [20, Lemma 3.1] that

\[
1_T - \begin{pmatrix} a - p & n \\ m & b - q \end{pmatrix} \begin{pmatrix} c & s \\ t & d \end{pmatrix} = \begin{pmatrix} 1_A - (a - p)c - \psi(n \otimes t) & * \\ * & 1_B - (b - q)d - \varphi(m \otimes s) \end{pmatrix} \in U(T).
\]

Hence, \( \begin{pmatrix} a - p & n \\ m & b - q \end{pmatrix} \in J(T) \), and therefore \( T \) is J-clean-like. \( \Box \)

As a consequence, we deduce that the \( n \times n \) lower (upper) triangular matrix ring over a J-clean-like ring is J-clean-like.

**Corollary 4.2.** Let \( R \) be J-clean-like, and let \( s \in J(R) \cap C(R) \). Then \( M_{(s)}(R) \) is J-clean-like.

**Proof.** As in the proof of Corollary 2.3, \( M_{(s)}(R) \) can be regarded as the ring of a Morita context \((R, R, R, R, \psi, \varphi)\) with \( \text{im}(\psi) \subseteq J(R) \) and \( \text{im}(\varphi) \subseteq J(R) \). According to Theorem 4.1, \( M_{(s)}(R) \) is J-clean-like. \( \Box \)

**Corollary 4.3.** Let \( R \) be a J-clean-like ring. Then \( M_{(x)}(R[[x]]) \) is J-clean-like.
Proof. For any \( f(x) \in R[[x]] \), we can find a potent \( p \in R \) such that \( f(0) - p \in J(R) \). Hence, \( f(x) = p + (f(x) - p) \). One easily checks that \( f(x) - p \in J(R[[x]]) \). Thus, \( R[[x]] \) is J-clean-like. Choose \( s = x \). Applying Corollary 4.2 to \( R[[x]] \), \( M(x)(R[[x]]) \) is J-clean-like.

Analogously, if \( R \) is a J-clean-like ring then so is \( M(x^m)(R[[x]]/(x^n)) \) for all \( 1 \leq m \leq n \).

**Proposition 4.4.** A ring \( R \) is strongly periodic if and only if the following two conditions hold simultaneously:

1. \( R \) is J-clean-like;
2. \( J(R) \) is locally nilpotent.

**Proof.** \( \implies \) Suppose \( R \) is strongly periodic. As \( P(R) \subseteq J(R) \), \( R \) is J-clean-like. Let \( x \in J(R) \). Then there exists a potent \( p \in R \) such that \( x - p \in P(R) \); hence, \( p = x - (x - p) \in J(R) \). This shows that \( p = 0 \), and so \( x \in P(R) \). As in the proof of Theorem 3.1, \( RxR \) is nilpotent. As the sum of finite nilpotent ideal is nilpotent, we prove that \( J(R) \) is locally nilpotent, as required.

\( \iff \) Let \( x \in J(R) \). Since \( J(R) \) is locally nilpotent, \( RxR \) is nilpotent. As in the proof of Theorem 3.1, we get \( x \in P(R) \). Hence, \( J(R) \subseteq P(R) \). This completes the proof, by (1).

Recall that a ring \( R \) is J-clean provided the for any \( a \in R \) there exists an idempotent \( e \in R \) such that \( a - e \in J(R) \) (cf. [6]). This following result explains the relation between J-clean rings and J-clean-like rings.

**Proposition 4.5.** A ring \( R \) is J-clean if and only if the following two conditions hold:

1. \( R \) is J-clean-like;
2. \( J(R) = \{ x \in R \mid 1 - x \in U(R) \} \).

**Proof.** \( \implies \) Clearly, \( R \) is J-clean-like. It is easy to check that \( J(R) \subseteq \{ x \in R \mid 1 - x \in U(R) \} \). If \( 1 - x \in U(R) \), then there exists an idempotent \( e \in R \) such that \( w := x - e \in J(R) \). Hence, \( 1 - e = (1 - x) + w = (1 - x)(1 + (1 - x)^{-1}w) \in U(R) \). This shows that \( 1 - e = 1 \), and so \( e = 0 \). Therefore \( x \in J(R) \), and so \( J(R) \supseteq \{ x \in R \mid 1 - x \in U(R) \} \), as required.

\( \iff \) For any \( a \in R \) there exists a potent \( p \in R \) such that \( (a-1)-p \in J(R) \). Write \( p = p^m(m \geq 2) \). Then \( p^{m-1} \in R \) is an idempotent. Set \( e = 1 - p^{m-1} \) and \( u = p - 1 + p^{m-1} \). Then \( e = e^2 \) \( e \in R \) and \( u^{-1} = p^{m-1} - 1 + p^{m-1}p^{m-2} \). Further, \( p = e + u \). This shows that \( a - 1 = p + (a - 1 - p) = e + u + (a - 1 - p) \). Hence, \( a = e + (u + (a - p)) \). As \( 1 - (u + (a - p)) = -u - (a - 1 - p) = -u(1 - u^{-1}(a - 1 - p)) \in U(R) \), we see that \( u + (a - p) \in J(R) \). Therefore \( R \) is J-clean, as asserted.
Example 4.6. Let \( R = \begin{pmatrix} \mathbb{Z}_3 & \mathbb{Z}_3 \\ 0 & \mathbb{Z}_3 \end{pmatrix} \). Then \( R \) is J-clean-like, while it is not J-clean. For any \( \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \in R \), we see that \( \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \) is the sum of a potent element in \( R \) and an element in \( J(R) \), hence that \( R \) is J-clean-like. As \( R/J(R) \cong \mathbb{Z}_3 \) is not Boolean, we conclude that \( R \) is not J-clean.

An element \( p \in R \) is \( J \)-potent provided that there exists some \( n \geq 2 \) such that \( p - p^n \in J(R) \). We say that every potent element lifts modulo \( J(R) \) if for any \( J \)-potent \( p \in R \) there exists a potent \( q \in R \) such that \( p - q \in J(R) \).

Lemma 4.7. A ring \( R \) is J-clean-like if and only if the following two conditions hold:

1. \( R/J(R) \) is potent;
2. Every potent element lifts modulo \( J(R) \).

Proof. \( \implies \) This is obvious.

\( \impliedby \) Let \( a \in R \). Then \( \overline{a} \in R/J(R) \) is potent. By hypothesis, we can find a potent \( p \in R \) such that \( a - p \in J(R) \). Accordingly, \( R \) is J-clean-like. \( \square \)

Recall that a ring \( R \) is right (left) quasi-duo provided that every maximal right (left) ideal is a two-sided ideal. As is well known, every right (left) duo ring \( (i.e., \) every right (left) ideal is two-sided) is right (left) quasi-duo. We come now to the main result of this section.

Theorem 4.8. A ring \( R \) is J-clean-like if and only if the following three conditions hold:

1. \( R/J(R) \) is periodic;
2. \( R \) is right (left) quasi-duo;
3. Every potent element lifts modulo \( J(R) \).

Proof. \( \implies \) In view of Lemma 4.7, \( R/J(R) \) is potent, and so it is periodic. Let \( M \) be a maximal right ideal of \( R \), and let \( r \in R \). Then \( J(R) \subseteq M \), and that \( M/J(R) \) is a maximal right of \( R/J(R) \). As is well known, every potent ring is commutative, and so \( rx + J(R) \in M/J(R) \) for any \( x \in M \). Write \( rx + J(R) = y + J(R) \) for a \( y \in M \). Hence, \( rx - y \in J(R) \subseteq M \). This shows that \( rx \in M \); hence, \( rM \subseteq M \). Therefore \( M \) is a two-sided ideal, and then \( R \) is right quasi-duo. Likewise, \( R \) is left quasi-duo. \( (3) \) is obvious, by Lemma 4.7.

\( \impliedby \) Let \( a \in R \). By (1), there exists a \( p \in R \) such that \( \overline{a - p} \in N(R/J(R)) \), \( p - p^n \in J(R) \, (n \geq 2) \), by Theorem 1.1. By (3), we may assume that \( p = p^n \). Set \( w = a - p \). Then \( \overline{w}^m = 0 \). Since \( R \) is right (left) quasi-duo, as in \( [7, \text{Corollary 3.4.7}] \), we see that \( e \overline{x} - \overline{x} e \in J(R) \) for any idempotent \( e \in R \) and any
element $x \in R$. This means that $R/J(R)$ is abelian. Similarly to the proof of [7, COROLLARY 1.3.15], $R/J(R)$ is reduced. Hence, $w = 0$, and then $w \in J(R)$. Therefore $a - p \in J(R)$, as desired. \qed

As a consequence of Corollary 3.5, every right (left) duo periodic ring is strongly periodic. Further, we derive

**Corollary 4.9.** A ring $R$ is strongly periodic if and only if

1. $R$ is periodic;
2. $R$ is right (left) quasi-duo;
3. $J(R)$ is locally nilpotent.

**Proof.** $\implies$ Clearly, $R$ is periodic. It follows from Proposition 4.4 that $R$ is J-clean-like and $J(R)$ is locally nilpotent. Thus, $R$ is right (left) quasi-duo, by Theorem 4.8.

$\iff$ Since $R$ is periodic, $R/J(R)$ is periodic. Thus, $R$ is J-clean-like, by Theorem 4.8. By (3), $J(R) = P(R)$, and the result follows. \qed

**Example 4.10.** Let $R = \mathbb{Z}_{(5)}$. Then $R$ is right (left) quasi-duo, $R/J(R)$ is periodic, while $R$ is not J-clean-like.

**Proof.** Let $R = \mathbb{Z}_{(5)}$. Then $J(R) = 5R$. Hence, $R/J(R) \cong \mathbb{Z}_5$ is a finite field. Thus, $R/J(R)$ is periodic. Suppose every potent element lifts modulo $J(R)$. Clearly, $2 - 2^5 \in J(R)$. Hence, $\bar{2} \in R/J(R)$ is potent. Thus, we can find a potent $w \in R$ such that $2 - w \in J(R)$. Write $w = \frac{m}{n}$, where $(m, n) = 1, 5 \nmid n$ and $w = w^s(s \geq 2)$. Then $w(1 - w^{s-1}) = 0$, and so $w = 0$ or $w^{s-1} = 1$. If $w = 0$, then $2 \in J(R)$, a contradiction. If $w^{s-1} = 1$, then $\frac{m^{s-1}}{n^{s-1}} = 1$; whence, $m = \pm n$. This implies that $w = \pm 1$; hence, $2 - w = 1, 3 \not\in J(R)$, a contradiction. Therefore $R$ is not J-clean-like, by Lemma 4.7. \qed

**Lemma 4.11.** Let $R$ be J-clean-like. Then $N(R) \subseteq J(R)$.

**Proof.** Let $x \in N(R)$. Then $x^m = 0$ for some $m \geq 2$. Moreover, there exists a potent $p \in R$ such that $w := x - p \in J(R)$. Write $p = p^n$ for some $n \geq 2$. Then $p = p^n = (p^n)^n = p^{n^2} = (p^n)^{n^2} = p^{n^3} = \cdots = p^{n^m}$. Clearly, $n^m = (1 + (n - 1))m \geq m(n - 1) \geq m$, and so $x^m = 0$. As $x^{n^m} - p^{n^m} \in J(R)$, we have $x = p + w = p^{n^m} + w \in J(R)$. Therefore $x \in J(R)$, hence the result. \qed

**Lemma 4.12.** Let $R$ be a ring. Then the following are equivalent:

1. $R$ is a periodic ring in which every nilpotent is contained in $J(R)$.
2. $R$ is J-clean-like and $J(R)$ is nil.

**Proof.** $(1) \Rightarrow (2)$ Suppose $R$ is a periodic ring with $N(R) \subseteq J(R)$. Let $x \in J(R)$. Then we have $m, n \in \mathbb{N}$ such that $x^m = x^n(n > m)$. Hence, $x^m(1 - x^{n-m}) = 0$, and so $x^m = 0$. This shows that $J(R)$ is nil. Let $a \in R$. In
view of Theorem 1.1, there exists a potent \( p \in R \) such that \( a - p \in N(R) \). By hypothesis, \( a - p \in J(R) \). Therefore \( R \) is J-clean-like.

\[(2) \Rightarrow (1) \] For any \( a \in R \), there exists a potent \( p \in R \) such that \( w := a - p \in J(R) \). Hence, \( a = p + w \) and \( p = p^n \) for some \( n \geq 2 \). Thus, \( a^n = p^n + v \) for a \( v \in J(R) \). This implies that \( a - a^n = w - v \in J(R) \subseteq N(R) \). Therefore \( R \) is periodic, by Theorem 1.1. In light of Lemma 4.11, every nilpotent of \( R \) is contained in \( J(R) \), as desired. \( \square \)

**Theorem 4.13.** Let \( R \) be a ring. If for any sequence of elements \( \{a_i\} \subseteq R \) there exists a \( k \in \mathbb{N} \) and \( n_1, \ldots, n_k \geq 2 \) such that \((a_1 - a_1^{n_1}) \cdots (a_k - a_k^{n_k}) = 0\), then \( R \) is J-clean-like.

**Proof.** For any \( a \in R \), we have a \( k \in \mathbb{N} \) and \( n_1, \ldots, n_k \geq 2 \) such that \((a - a^{n_1}) \cdots (a - a^{n_k}) = 0\). This implies that \( a^k = a^{k+1}f(a) \) for some \( f(t) \in \mathbb{Z}[t] \). In view of Theorem 1.1, \( R \) is periodic.

Clearly, \( R/J(R) \) is isomorphic to a subdirect product of some primitive rings \( R_i \). Case 1. There exists a subring \( S_i \) of \( R_i \) which admits an epimorphism \( \phi_i : S_i \to M_2(D_i) \) where \( D_i \) is a division ring. Case 2. \( R_i \cong M_{k_i}(D_i) \) for a division ring \( D_i \). Clearly, the hypothesis is inherited by all subrings, all homomorphic images of \( R \), we claim that, for any sequence of elements \( \{a_i\} \subseteq M_2(D_i) \) there exists \( s \in \mathbb{N} \) and \( m_1, \ldots, m_s \geq 2 \) such that \((a_1 - a_1^{m_1}) \cdots (a_s - a_s^{m_s}) = 0\). Choose \( a_i = e_{12} \) if \( i \) is odd and \( a_i = e_{21} \) if \( i \) is even. Then \((a_1 - a_1^{m_1})(a_2 - a_2^{m_2}) \cdots (a_s - a_s^{m_s}) = a_1a_2 \cdots a_s \neq 0 \), a contradiction. Thus, Case I do not happen. Further, in Case II, \( k_i = 1 \) for all \( i \). This shows that each \( R_i \) is reduced, and then so is \( R/J(R) \). If \( a \in N(R) \), we have some \( n \in \mathbb{N} \) such that \( a^n = 0 \), and thus \( \overline{a}^n = 0 \) is \( R/J(R) \). Hence, \( \overline{a} \in J(R/J(R)) = 0 \). This implies that \( a \in J(R) \), and so \( N(R) \subseteq J(R) \). Therefore \( R \) is J-clean-like, by Lemma 4.12. \( \square \)

Recall that a subset \( I \) of a ring \( R \) is left (resp., right) \( T \)-nilpotent in case for every sequence \( a_1, a_2, \cdots \) in \( I \) there is an \( n \) such that \( a_1 \cdots a_n = 0 \) (resp., \( a_n \cdots a_1 = 0 \)). Every nilpotent ideal is left and right \( T \)-nilpotent. The Jacobson radical \( J(R) \) of a ring \( R \) is left (resp., right) \( T \)-nilpotent if and only if for any nonzero left (resp., right) \( R \)-module \( M \), \( J(R)M \neq M \) (resp., \( MJ(R) \neq M \)).

**Corollary 4.14.** Let \( R \) be a ring. If \( R/J(R) \) is potent and \( J(R) \) is left (resp., right) \( T \)-nilpotent, then \( R \) is J-clean-like.

**Proof.** We may assume \( R/J(R) \) is potent and \( J(R) \) is left \( T \)-nilpotent. For every sequence \( a_1, a_2, \cdots, a_m, \cdots \) in \( R \), there exists some \( n_i \in \mathbb{N} \) such that \( a_i - a_i^{n_i} \in J(R) \) for all \( i \). We choose \( b_1 = a_1 - a_1^{n_1}, b_2 = (1 - b_1)^{-1}(a_2 - a_2^{n_2}), b_3 = (1 - b_2)^{-1}(a_3 - a_3^{n_3}), \ldots, b_m = (1 - b_{m-1})^{-1}(a_m - a_m^{n_m}), \ldots \). By hypothesis, we can find some \( k \in \mathbb{N} \) such that \( b_1(1 - b_1)b_2(1 - b_2) \cdots b_{k-1}(1 - b_{k-1}) = 0 \).
Hence, $b_1(1 - b_1)b_2(1 - b_2)\cdots b_{k-1}(1 - b_{k-1})b_k = 0$. This shows that $(a_1 - a_1^{n_1})\cdots (a_s - a_k^{n_k}) = 0$. Therefore $R$ is J-clean-like, by Theorem 4.13. □

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REFERENCES


Received 3 April 2015

Hangzhou Normal University, 
Department of Mathematics,
Hangzhou 310036, China 
huanyinchen@aliyun.com

Statistics and Computer Science
Semnan University, 
Faculty of Mathematics,
Semnan, Iran 
m.sheibani1@gmail.com