Dedicated to Michael Vaughan-Lee on his 73rd birthday

CHARACTERIZATION OF FINITE p-GROUPS BY THE ORDER OF THEIR SCHUR MULTIPLIERS ($t = 6$)

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Let G be a finite p-group of order p^n. In 1956, Green proved that the order of $\mathcal{M}(G)$, the Schur multiplier of G, is equal to $p^{\frac{1}{2}n(n-1)-t}$ for some integer $t \geq 0$. The p-groups which satisfy $0 \leq t \leq 5$ are determined up to now. In this paper, we classify all finite p-groups with $t = 6$.

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1. INTRODUCTION

The study of the Schur multipliers of groups dates back to 1904 [20]. This study has an impact on other aspects of group theory. There are several bounds on the order of the Schur multiplier of a group. For instance when G is a finite p-group of order p^n, Green [9] has shown that the order of $\mathcal{M}(G)$, the Schur multiplier of G, is at most $p^{\frac{1}{2}n(n-1)}$. Another concept which is closely related to the Schur multiplier and has been studied independently from 1987, is the non-abelian tensor square of groups and it was introduced by R. Brown and J.-L. Loday [4]. When G is a finite p-group of order p^n with commutator subgroup of order p^c, Rocco [19] proved that the order of $G \otimes G$, the non-abelian tensor square of G, cannot exceed $p^{n(n-c)}$. One of the interesting subjects is to classify p-groups when the order of their Schur multipliers and their non-abelian tensor squares is determined. If $|\mathcal{M}(G)| = p^{\frac{1}{2}n(n-1)-t}$ and $|G \otimes G| = p^{n(n-c)-l}$ for some integers $t, l \geq 0$, the characterization of p-groups when $0 \leq t \leq 5$ is given (see [2, 7, 16] and [21]), likewise for $0 \leq l \leq 10$ and $c = 1$, the classification may be found in [11]. In this paper, we improve these classifications when $t = 6$ and $l = 11, 12, 13$.

Notations:

D_n, Q_n, QD_n = Dihedral, Quaternion and QuasiDihedral group of order n;

$E^1_{p^3}, E^2_{p^3}$ = Extra special group of order p^3 with exponent p or p^2, $p > 2$;

$E^d_{2^{2m+1}}, E^1_{p^{2m+1}}$ = Central product of m copies of D_8's or $E^1_{p^3}$'s;

$E^q_{2^{2m+1}}$ = Central product of $(m-1)$ copies of D_8's and a single Q_8;

$E^2_{p^{2m+1}}$ = Central product of $(m-1)$ copies of $E^1_{p^3}$'s and a single $E^2_{p^3}$;

$GE^r_{p^{2+r}}$ = Generalized extra special group of order p^{2+r} with exponent p^r and presentation $\langle x, y, z | x^p = y^p = z^{p^r} = 1, [x, y] = z^{p^{r-1}}, [y, z] = [x, z] = 1\rangle$;

$GE^{r+1}_{p^{2+r}}$ = Generalized extra special group of order p^{2+r} with exponent p^{r+1} and presentation $\langle x, y | x^{p^{r+1}} = y^p = 1, x^y = x^{1+p^r}\rangle$;

$GE^r_{p^{2m+r}}, GE^{r+1}_{p^{2m+r}}$ = Central product of m copies of $GE^r_{p^{2+r}}$'s or $GE^{r+1}_{p^{2+r}}$'s;

T_i, X_i, Y_i, Z_i = Nonabelian p-groups are described in table I.

Theorem A. *Let G be a p-group of order p^n and $|\mathcal{M}(G)| = p^{\frac{1}{2}n(n-1)-t}$. Then*

$t = 6$ if and only if $G \cong C_{p^4},\ C_p \times (C_{p^2})^2,\ (C_p)^5 \times C_{p^2},\ Q_{16},\ QD_{16},\ T_3,\ T_1 \times C_2,$ $T_{10},\ T_{11},\ Q_8 \times (C_2)^3,\ T_4 \times (C_2)^2,\ E^d_{32} \times C_2,\ E^q_{32} \times C_2,\ D_8 \times (C_2)^4,\ X_3,\ E^1_{p^3} \times C_{p^2},$ $X_1 \times C_p,\ X_6 \times C_p,\ Y_4,\ E^1_{p^5} \times C_p,\ E^2_{p^5} \times C_p,\ E^2_{p^3} \times (C_p)^3,\ X_4 \times (C_p)^2,\ GE^2_{p^{4+2}},$ $Y_3 \times C_p$ or $E^1_{p^3} \times (C_p)^5$.

Theorem B. *Let G be a p-group of order p^n with derived subgroup of order p and $|G \otimes G| = p^{n(n-1)-l}$. Then*

(i) $l = 11$ if and only if $G \cong T_{13},\ T_1 \times (C_2)^2,\ T_2 \times (C_2)^2,\ Q_8 \times C_2 \times C_4,\ D_8 \times (C_2)^{10},$
$Q_8 \times (C_2)^{11},\ E^1_{p^3} \times C_{p^2} \times C_p,\ X_1 \times (C_p)^2,\ E^2_{p^3} \times (C_p)^9,\ E^1_{p^3} \times (C_p)^{11},$
$E^i_{p^{2m+1}} \times (C_p)^{11-2m}$ for any integer $2 \le m \le 5$ and $i = 1, 2$, or
$GE^2_{p^{2m+2}} \times (C_p)^{11-(2m+1)}$ for any integer $1 \le m \le 5$.

(ii) $l = 12$ if and only if $G \cong T_{14},\ T_{15},\ T_3 \times (C_2)^2,\ T_7 \times C_2,\ D_8 \times C_2 \times C_4,\ E^d_{2^{13}},\ E^q_{2^{13}},$
$D_8 \times (C_2)^{11},\ Q_8 \times (C_2)^{12},\ Y_5,\ Y_6,\ X_2 \times (C_p)^2,\ X_4 \times C_{p^2},\ E^2_{p^3} \times (C_p)^{10},\ E^1_{p^{13}},\ E^2_{p^{13}},$ $E^1_{p^3} \times (C_p)^{12},\ E^i_{p^{2m+1}} \times (C_p)^{12-2m}$ for any integer $2 \le m \le 5$ and $i = 1, 2$, or
$GE^2_{p^{2m+2}} \times (C_p)^{12-(2m+1)}$ for any integer $1 \le m \le 5$.

(iii) $l = 13$ if and only if $G \cong T_{16},\ T_4 \times C_4,\ T_9 \times C_2,\ T_{18},\ E^d_{2^{13}} \times C_2,\ E^q_{2^{13}} \times C_2,$ $D_8 \times (C_2)^{12},\ Q_8 \times (C_2)^{13},\ Y_7,\ E^2_{p^3} \times C_{p^2} \times C_p,\ X_3 \times (C_p)^2,\ GE^3_{p^{4+2}},\ Z_1,\ Z_2,\ Z_3,$ $E^2_{p^3} \times (C_p)^{11},\ E^1_{p^3} \times (C_p)^{13},\ E^i_{p^{2m+1}} \times (C_p)^{13-2m}$ for any integer $2 \le m \le 6$ and $i = 1, 2$, or $GE^2_{p^{2m+2}} \times (C_p)^{13-(2m+1)}$ for any integer $1 \le m \le 6$.

2. PRELIMINARIES

In this section, we provide the necessary preliminary results. Let G and H be two groups equipped with an action $(g, h) \mapsto {}^g h$ of G on H and an action $(h, g) \mapsto {}^h g$ of H on G. The actions should be compatible, see [4]. The non-abelian tensor product $G \otimes H$ is the group generated by symbols $g \otimes h$ for $g \in G$ and $h \in H$, subject to the relations

$$gg' \otimes h = ({}^g g' \otimes {}^g h)(g \otimes h), \quad g \otimes hh' = (g \otimes h)\ ({}^h g \otimes {}^h h')$$

for all $g, g' \in G$ and $h, h' \in H$. By using the conjugation action of a group on itself we may always define the non-abelian tensor square $G \otimes G$.

Our method which is based on direct computation of non-abelian tensor square of groups depends on the following result.

Lemma 2.1 ([3], Proposition 9). *If Z is a central subgroup of a group G, then the following sequence is exact:*

$$(*) \qquad (G \otimes Z) \times (Z \otimes G) \xrightarrow{\iota} G \otimes G \longrightarrow G/Z \otimes G/Z \longrightarrow 1,$$

where $\iota(g \otimes z, z' \otimes g') = (g \otimes z)(z' \otimes g')$ for all $z, z' \in Z$ and $g, g' \in G$. In particular if $Z \subseteq G'$, then the sequence

$$(**) \qquad Z \otimes G \longrightarrow G \otimes G \longrightarrow G/Z \otimes G/Z \longrightarrow 1$$

is exact.

The following result gives the order of the non-abelian tensor square of a finite group G in terms of the orders of G, $\mathcal{M}(G)$ and $\mathcal{M}(G^{ab})$.

Theorem 2.2 ([11], Lemma 2.3). *Let G be a d-generator finite p-group.*

(i) *If $p > 2$, then $|G \otimes G| = |G||\mathcal{M}(G)||\mathcal{M}(G^{ab})|$.*

(ii) *If $p = 2$ and $G/G' = \prod_{i=1}^{d} C_{2^{e_i}}$ where $1 \leq e_i \leq e_j$, for every $1 \leq i \leq j \leq d$. Then $|G \otimes G| = 2^k|G||\mathcal{M}(G)||\mathcal{M}(G^{ab})|$, where $k \leq d$ is a non-negative integer.*

Now we recall some bounds of order of the Schur multiplier of finite p-groups.

Theorem 2.3 ([15]). *Let G be a d-generator p-group of order p^n. Then*

$$p^{\frac{1}{2}d(d-1)} \leq |G'||\mathcal{M}(G)| \leq p^{\frac{1}{2}n(n-1)}.$$

Theorem 2.4 ([8]). *Let G be a d-generator group of order p^n, the derived factor $\frac{G}{G'}$ of order p^m with exponent p^e and the central factor $\frac{G}{Z(G)}$ be a δ-generator group, then*

$$|\mathcal{M}(G)| \leq p^{d(m-e)/2+(\delta-1)(n-m)-max\{0,\delta-2\}}.$$

3. PROOF OF THE MAIN THEOREMS

Throughout the rest of the paper, we always assume that G is a d-generator p-group of order p^n with

$$|\mathcal{M}(G)| = p^{\frac{1}{2}n(n-1)-t}. \tag{1}$$

We also assume that $\frac{G}{Z(G)}$ is a δ-generator group, $|G'| = p^c$ and the Frattini subgroup $\Phi(G)$ is of order p^a and so $a = n-d$. Ellis [7] established the following inequalities:

$$2(t - c(d+1-\delta)) \geq a^2 - a, \qquad a \geq c \geq 0, d \geq \delta \tag{2}$$

$$2(t-c) \geq a^2 - a \tag{3}$$

Proof of Theorem A. Suppose $t = 6$. Then (3) implies that $(c, a) = (0,0), (0,1), (0,2), (0,3), (0,4), (1,1), (1,2), (1,3), (2,2), (2,3)$ or $(3,3)$.

Suppose $(c, a) = (0,0)$, then $n \geq 1$, $|\mathcal{M}(G)| = p^{\frac{1}{2}n(n-1)}$ and (1) cannot hold. Suppose $(c, a) = (0,1)$, then $n \geq 2$, $|\mathcal{M}(G)| = p^{\frac{1}{2}(n-1)(n-2)}$ and (1) holds if and only if $n = 7$ and $G \cong C_{p^2} \times (C_p)^5$.

Using the same method for cases $(0,2)$ and $(0,3)$, one can observe that (1) holds if and only if $G \cong (C_{p^2})^2 \times C_p$ or $G \cong C_{p^4}$, respectively. Also (1) cannot hold when $(c, a) = (0,4)$.

Suppose $(c, a) = (1,1)$. If $Z(G)$ is cyclic then G is generalized extraspecial and $n = 2m+2$. Thus by [13, Theorem 4.2], $|G| = p^6$ and $G = GE^2_{p^6}$.

If $Z(G)$ is non-cyclic, then $G \cong M \times C_p$ where M is a maximal subgroup of G and $t(M) = t(G) - 1$. So it follows from [16] that (1) holds if and only if G is isomorphic with one of the following groups.

$Q_8 \times (C_2)^3$, $D_8 \times (C_2)^4$, $T_4 \times (C_2)^2$, $E^d_{32} \times C_2$, $E^q_{32} \times C_2$, $E^1_{p^3} \times (C_p)^5$, $E^2_{p^3} \times (C_p)^3$, $E^1_{p^5} \times C_p$, $E^2_{p^5} \times C_p$ or $X_4 \times (C_p)^2$.

Suppose $(c, a) = (1,2)$. Then $G^{ab} \cong C_{p^2} \times (C_p)^{n-3}$ and Theorem 2.4 implies that $2n - 4 \leq 6$, hence $n = 4, 5$. For odd prime p, it follows from [11, Theorem 3.1] that $G = X_3$, $X_1 \times C_p$ or $E^1_{p^3} \times C_{p^2}$. If $p = 2$ then $G = T_3$ or $T_1 \times C_2$ by GAP.

Suppose $(c, a) = (1,3)$. If $G^{ab} \cong C_{p^3} \times (C_p)^{n-4}$ or $(C_{p^2})^2 \times (C_p)^{n-5}$, then by Theorem 2.4, $n \leq 4$ which is a contradiction.

Suppose $(c, a) = (2,2)$. Then $d = \delta$ or $\delta + 1$ by (2). If $d = \delta$ then $n \leq 7$ by Theorem 2.4. If $n = 7$ then $d = 5$ and G has exponent p. By [18] if G may be described as the direct product of its subgroups, the straightforward direct computation of the non-abelian tensor square together with [13, Theorem 2.3] and Theorem 2.2 imply that (1) cannot hold. In other cases if G is a descendant of algebra 5.1 discussed in [18], *i.e.*

(7.27)
$$\langle a, b, c, d, e \mid [d, a]=[e, a]=[c, b]=[d, b]=[e, b]=[d, c]=[e, c]=0, [e, d]=[b, a], \; class\, 2\rangle,$$

(7.28)
$$\langle a, b, c, d, e \mid [d,a]=[e,a]=[c,b]=[e,b]=[d,c]=[e,c]=0, [d,b]=[c,a], [e,d]=[b,a], \; class\, 2\rangle,$$

then the equalities $[b, a] \otimes e = 1$ and $[c, a] \otimes d = 1$, respectively, together with the exact sequence $(**)$, [13, Theorem 2.3] and Theorem 2.2 imply that (1) cannot hold. If G is a descendant of algebra 6.3 discussed in [18], *i.e.*

$$\text{(7.191)} \quad \langle a, b, c, d, e \mid [b, a, b] = [c, a] = [c, b] = [d, a] = [d, b] = [e, a] = [e, c] = [e, d] = 0, [d, c] = [e, b] = [b, a, a], \; class\ 3\rangle,$$

then similarly by the relation $[b, a, a] \otimes c = 1$ we have $t > 6$.

Throughout the rest of this section, all the notations and group presentations of James's classification for p-groups of order less than or equal to p^6, $p \neq 2$ will be used (see [14]). If $n = 6$ then G belongs to one of the families of Φ_{12}, Φ_{13}, Φ_{15} or Φ_{22}. At the first family put $Z = \langle\gamma_1\rangle$ in sequence $(**)$ and observe that $\gamma_1 \otimes \alpha_2 = \gamma_1 \otimes \beta_2 = 1$. Hence $|\mathrm{Im}(Z \otimes G \to G \otimes G)| \leq p^2$. As $|G/Z \otimes G/Z| \leq p^{18}$ by [13, Theorem 2.3], it follows that $|G \otimes G| \leq p^{20}$ which is a contradiction by Theorem 2.2. If G is a direct product of its subgroups, one can easily check that such groups does not satisfy our condition.

Similarly in families Φ_{13} and Φ_{15}, put $Z = \langle\beta_2\rangle$ and $Z = Z(G)$, respectively, and observe that either $|\mathrm{Im}(Z \otimes G \to G \otimes G)| \leq p^2$ or p^4. Finally, in family Φ_{22}, put $Z = \langle\alpha_3\rangle$. It is clear that $\alpha_3 \otimes \beta_1 = \alpha_3 \otimes \beta_2 = 1$ and the same result holds.

Let $n = 5$. Then G may be in the families Φ_4 or Φ_7. If put $Z = Z(G)$ in the first family, one concludes that $|\mathrm{Im}((Z \otimes G) \times (G \otimes Z) \to G \otimes G)| \leq p^2$. For instance, in group $\Phi_4(221)a$ we have $\beta_1 \otimes \alpha_1 = 1$, $\beta_1 \otimes \alpha_2 = \beta_2 \otimes \alpha_1$ and $\beta_1 \otimes \alpha = \alpha_1^p \otimes \alpha = \alpha_1 \otimes \alpha^p = \alpha_1 \otimes \beta_2$, because G has nilpotency class 2 (see [1, Lemma 2.6]). Also $\beta_2 \otimes \alpha = 1$. So the desired assertion holds and $|G \otimes G| \leq p^{11}$, which is a contradiction by the Theorem 2.2. For the group $\Phi_4(221)b$ we have $\beta_1 \otimes \alpha_1 = \alpha_2^p \otimes \alpha_1 = \alpha_2 \otimes \alpha_1^p = 1$, $\beta_1 \otimes \alpha_2 = 1$, $\beta_1 \otimes \alpha = \alpha_2^p \otimes \alpha = (\alpha_2 \otimes \alpha)^p (\alpha_2 \otimes \beta_2)^{\frac{1}{2}p(p-1)} = \alpha_2 \otimes (\alpha)^p = \alpha_2 \otimes \beta_2$ and $\beta_2 \otimes \alpha_1 = \beta_2 \otimes \alpha = 1$. Hence $|G \otimes G| \leq p^{10}$. In other groups of this family, the same result holds and exceptionally the order of non-abelian tensor square of $\Phi_4(1^5)$ is equal to p^{14} by [13, Theorem 3.1].

In family Φ_7, put $Z = Z(G) = \langle\alpha_3\rangle$. We can see that $Im(Z \otimes G \to G \otimes G) = 1$ for all groups G of this family except $\Phi_7(1^5)$. So by the same method discussed above, the desired result holds. In particular for the group $\Phi_7(2111)b_r$, $(\alpha_3 \otimes \alpha)^r = \alpha_1^p \otimes \alpha = (\alpha_1 \otimes \alpha)^p = (\alpha_1 \otimes \alpha^p)(\alpha_3 \otimes \alpha)^{-\frac{1}{6}p(p-1)(p-2)}$ and when $p > 3$, $(\alpha_3 \otimes \alpha)^{-\frac{1}{6}p(p-1)(p-2)} = 1$. So $\alpha_3 \otimes \alpha = 1$. If $p = 3$ then GAP calculation shows that the order of the Schur multiplier of this group is at most p^3.

When $G = \Phi_7(1^5) = Y_4$, we first use the method of [5] to determine a presentation of its Lazard correspondence Lie ring L_p, which has the same order and nilpotency class for $p \geq 5$, *i.e.*

$$L_p = \langle a, a_1, a_2, a_3, b \mid [a_1, a] = a_2 - \tfrac{1}{2}a_3,\ [a_2, a] = a_3,\ [a_1, b] = a_3 \rangle.$$

Since this group has exponent p, the Lie ring L_p may be regarded as a Lie algebra over the field $\mathbb{Z}_p$ and hence it is isomorphic to the nilpotent Lie algebra L(4, 5, 1, 6) of dimension 5 given in [10] which has the Schur multiplier of dimension 4. In addition, the Schur multipliers of L_p and G are isomorphic by [6, Theorem 1]. Therefore $|\mathcal{M}(G)| = p^4$, as desired. Also, GAP shows that the group G does satisfy our condition when $p = 3$.

If $n = 4$ then G belongs to the family Φ_3 and there is no group which satisfies (1).

Now suppose $d = \delta + 1$, then by Theorem 2.4, it follows that $n \leq 6$. Thus our group G must be in one of the families of Φ_3, Φ_4 or Φ_7. If $d = 4$ and $n = 6$, then by Theorem 2.4, G may belong to Φ_4 or Φ_7. In the first family for groups $\Phi_4(2211)g$, $\Phi_4(2211)h$ and $\Phi_4(2211)i$ take $Z = \langle \beta_2 \rangle$. So $|Im(Z \otimes G \to G \otimes G)| \leq p^2$ and $|G \otimes G|$ cannot equal to p^{21}. For group $\Phi_4(21^4)d$ it is enough to consider $Z = \langle \beta_1 \rangle$. If $G \cong H \times C_p$ then the order of $H \otimes H$ should be p^{14} and by [13, Theorem 3.1], we have $H = \Phi_4(1^5) = Y_3$ whence $G \cong Y_3 \times C_p$. If G is in the family Φ_7, just the group $\Phi_7(21^4)d$ has four generators and by putting $Z = \langle \alpha_3 \rangle$ in sequence $(**)$ the desired result holds.

If $d = 3$ and $n = 5$, then the group G should belong to family Φ_3. When G is a direct product of its subgroups, only the group $\Phi_3(1^4) \times C_p = X_6 \times C_p$ satisfies our condition. In other groups, only the group $\Phi_3(2111)c$ has three generators. If take $Z = \langle \alpha_3 \rangle$ then sequence $(**)$ implies that $|G \otimes G| \leq p^{11}$, which is again a contradiction.

For the case $(c, a) = (2, 2)$ if p is even, then $G = Q_{16},\ QD_{16},\ T_{10}$ or T_{11} by GAP and in all groups we have $d = \delta$.

Suppose $(c, a) = (2, 3)$. As $d = n - 3$ it follows that $n \geq 5$. On the other hand $n \leq 4$ by Theorem 2.4, so there is not any group in this case.

Suppose $(c, a) = (3, 3)$. As $d = \delta = n - 3$, so $n = 5$ by Theorem 2.4 and $d = 2$. In this case the order of non-abelian tensor square of our group must be p^{10} by Theorem 2.2. But by [12] the order of a non-abelian p-group which attains the upper bound of tensor given by Rocco in [19], cannot exceed p^3. Therefore the proof is complete. □

Proof of Theorem B. Let $a = 1$. If G is an extraspecial p-group of order p^{2m+1}, then by [11, Corollary 2.4 and Proposition 2.6] we have

$$(2m+1)2m - l = 4m^2.$$

This equality holds if $m = 6$ and $l = 12$. Therefore $|G| = p^{13}$ and $G = E^1_{p^{13}}$, $E^2_{p^{13}}$, $E^d_{2^{13}}$ or $E^q_{2^{13}}$.

If G is not extraspecial and $Z(G)$ is cyclic, then by [13, Theorem 4.2], $|Z(G)| = p^2$ and $n = 2m+2$. Thus $l = 11$ if and only if $G = GE^2_{p^{12}}$ and $l = 13$ if and only if $G = GE^2_{p^{14}}$. Note that the case $l = 12$ does not hold here.

If $Z(G)$ is non-cyclic then $G \cong M \times C_p$ where M is a maximal subgroup of G and $l(M) = l(G) - 1$. So $l = 11$ if and only if G is isomorphic to $E^1_{p^3} \times (C_p)^{11}$, $Q_8 \times (C_2)^{11}$, $E^2_{p^3} \times (C_p)^9$, $D_8 \times (C_2)^{10}$, $E^i_{p^{2m+1}} \times (C_p)^{11-2m}$ for any integer $2 \le m \le 5$ and $i = 1, 2$, or $GE^2_{p^{2m+2}} \times (C_p)^{11-(2m+1)}$ for any integer $1 \le m \le 4$. For $l = 12, 13$ the method is similar.

Let $a > 1$. By [13, Theorem 2.3] we have $2 + a \le n \le \frac{1}{a}(l + a + 2)$. If $11 \le l \le 13$ then $a = 2$ or 3 and consequently $4 \le n \le 8$.

Let $a = 2$ and $p \ne 2$. Then $G^{ab} \cong C_{p^2} \times (C_p)^{n-3}$. By Theorems 2.2 and 2.4 we should have $2n - 4 \le t = l - n + 3$. Thus $n = 4$, 5 or 6. But as in Theorem 2.3, n must be 6. Now by James's classification of p-groups if G is a direct product of its subgroups, one can easily observe by [11, Theorems 3.1 and 3.2] that

$l = 11$ if and only if $G \cong \Phi_2(211)c \times (C_p)^2$ or $\Phi_2(111) \times C_{p^2} \times C_p$;
$l = 12$ if and only if $G \cong \Phi_2(22) \times (C_p)^2$ or $\Phi_2(211)b \times C_{p^2}$;
$l = 13$ if and only if $G \cong \Phi_2(31) \times (C_p)^2$ or $\Phi_2(21) \times C_{p^2} \times C_p$.

For groups $G = \Phi_5(2211)a = Z_1$, $\Phi_5(2211)b = Z_2$, $\Phi_5(21^4)c = Z_3$ and $\Phi_5(311) = GE^3_{p^{4+2}}$, put $Z = G'$ in sequence $(**)$. Therefore $|G \otimes G| = |G^{ab} \otimes G^{ab}| = p^{17}$ and we must have $l = 13$.

If $a = 3$ and $p \ne 2$. Then by the same argument we should have $n = 5$. Hence $l = 12$ if and only if $G \cong \Phi_2(32)a_1 = Y_5$ or $\Phi_2(311)c = Y_6$ and $l = 13$ if and only if $G = \Phi_2(32)a_2 = Y_7$. Note that the case $p = 2$ may be verified by GAP.

Remark. When $l = 10$ and $(c, a) = (1, 3)$ it follows by [17] that $G = G_p(2,2,1,1,1) = \langle a, b | a^{p^2} = b^{p^2} = 1, [a,b]^p = [a,b,a] = [a,b,b] = 1 \rangle$. This was missed in [11].

Table 1

Name	Relations	NumberOfSmallGroup	(c,a)
T_1	$a^4=b^2=c^2=1,[a,c]=b,\ [a,b]=[b,c]=1$	$(16,3)$	$(1,2)$
T_2	$a^4=b^4=1,\ [a,b]=a^2$	$(16,4)$	$(1,2)$
T_3	$a^8=b^2=1,\ [a,b]=a^4$	$(16,6)$	$(1,2)$
T_4	$a^4=b^2=c^2=1,\ [b,c]=a^2,[a,b]=[a,c]=1$	$(16,13)$	$(1,1)$
T_7	$a^4=b^2=c^4=1,\ [a,b]=c^2$	$(32,24)$	$(1,2)$
T_9	$b^2=c^2=1,\ [c,b]=a^4[a,b]=[c,a]=1$	$(32,38)$	$(1,2)$
T_{10}	$a^2=b^2=c^2=1,\ [a,b]=1,$ $[a,c,b]=1,\ [b,c,a]=1,\ [b,c,b]=1$	$(32,27)$	$(1,2)$
T_{11}	$a^4=b^4=c^2=1,$ $[a,b]=1,[a,c]=a^2,\ [b,c]=b^2$	$(32,34)$	$(1,2)$
T_{13}	$a^4=b^8=1,\ [a,b]b^4=1$	$(32,4)$	$(1,3)$
T_{14}	$a^2=b^8=1,\ [a,b,a]=[a,b,b]=1$	$(32,5)$	$(1,3)$
T_{15}	$a^4=b^8=1,\ [a,b]a^2=1$	$(32,12)$	$(1,3)$
T_{16}	$a^2=b^{16},\ [a,b]a^8=1$	$(32,17)$	$(1,3)$
T_{18}	$[a_1,a_2]=[a_1,a_4]=[a_2,a_3]=a_2^2,$ $a_1^4=a_3^4=a_4^2=1,a_2^2=a_3^2$	$(64,200)$	$(1,2)$
X_1	$a^{p^2}=b^p=c^p=1,$ $[a,c]=b,\ [a,b]=[b,c]=1,p>2$	—	$(1,2)$
X_2	$a^{p^2}=b^{p^2}=1,\ [a,b]=a^p,p>2$	—	$(1,2)$
X_3	$a^{p^3}=b^p=1,\ [a,b]=a^{p^2},p>2$	—	$(1,2)$
X_4	$a^{p^2}=b^p=c^p=1,$ $[b,c]=a^p,[a,b]=[a,c]=1,p>2$	—	$(1,1)$
X_6	$[a_i,a]=a_{i+1},$ $a^p=a_i^{(p)}=a_3^p=1,\ (i=1,2),p>2$	—	$(2,2)$
Y_3	$[a_i,a]=b_i,\ a^p=a_i^p=b_i^p=1,\ (i=1,2),p>2$	—	$(2,2)$
Y_4	$[a_i,a]=a_{i+1},[a_1,b]=a_3,$ $a^p=a_1^{(p)}=a_{i+1}^p=b^p=1,\ (i=1,2),p>2$	—	$(2,2)$
Y_5	$[a_1,a]=a^{p^2}=a_2,\ a_1^{p^2}=a_2^p=1,p>2$	—	$(1,3)$
Y_6	$[a_1,a]=a_2,\ a^{p^3}=a_1^p=a_2^p=1,p>2$	—	$(1,3)$
Y_7	$[a_1,a]=a_1^p=a_2,\ a^{p^3}=a_2^p=1,p>2$	—	$(1,3)$
Z_1	$[a_1,a_2]=[a_3,a_4]=a_2^p=b,$ $a_1^{p^2}=a_3^p=a_4^p=b^p=1,p\geq 2$	—	$(1,2)$
Z_2	$[a_1,a_2]=[a_3,a_4]=a_3^p=b,$ $a_1^{p^2}=a_2^p=a_4^p=b^p=1,p\geq 2$	—	$(1,2)$
Z_3	$[a_1,a_2]=[a_3,a_4]=b,$ $a_1^{p^2}=a_2^p=a_3^p=a_4^p=b^p=1,p>2$	—	$(1,2)$

Here $a_{i+1}^{(p)}$ will denote the word $a_{i+1}^p a_{i+2}^{\binom{p}{2}} \ldots a_{i+k}^{\binom{p}{k}} \ldots a_{i+p}$ discussed in [14].

REFERENCES

[1] M.R. Bacon, *On the non-abelian tensor square of a nilpotent group of class* 2. Glasg. Math. J. **36** (1994), 291–296.

[2] Ya.G. Berkovich, *On the order of the commutator subgroups and the Schur multiplier of a finite p-group.* J. Algebra **144** (1991), 269–272.

[3] R. Brown, D.L. Johnson and E.F. Robertson, *Some computations of non-abelian tensor products of groups.* J. Algebra **111** (1987), 177–202.

[4] R. Brown and J.-L. Loday, *Van Kampen theorems for diagrams of spaces.* Topology **26** (1987), 311–335.

[5] S. Cicalò, W.A. De Graaf and M. Vaughan-Lee, *An effective version of the Lazard correspondence.* J. Algebra **352** (2012), 430–450.

[6] B. Eick, M. Horn and S. Zandi, *Schur multipliers and the Lazard correspondence.* Arch. Math. **99** (2012), 217–226.

[7] G. Ellis, *On the Schur multipliers of p-groups.* Comm. Algebra **27(9)** (1999), 4173-4177.

[8] G. Ellis and J. Wiegold, *A bound on the Schur multiplier of a prime-power group.* Bull. Aust. Math. Soc. **60(2)**(1999),191–196.

[9] J.A. Green, *On the number of automorphisms of a finite group.* Proc. Royal Soc. A **237** (1956), 574–581.

[10] P. Hardy, *On characterizing nilpotent Lie algebras by their multipliers III.* Comm. Algebra **33** (2005), 4205–4210.

[11] S.H. Jafari, F. Saeedi and E. Khamseh, *Characterization of finite p-groups by their non-abelian tensor square.* Comm. Algebra **41(5)** (2013), 1954–1963.

[12] S.H. Jafari, *A bound on the order of non-abelian tensor square of a prime-power group.* Comm. Algebra **40(2)** (2012), 528–530.

[13] S.H. Jafari, *Categorizing finite p-groups by the order of their non-abelian tensor squares.* J. Algebra Appl. **15(5)** (2016), 1650095-1–1650095-13.

[14] R. James, *The groups of order p^6 (p an odd prime).* Math. Comp. **34** (1980), 613–637.

[15] M.R. Jones, *Some inequalities for the multiplicator of a finite group.* Proc. Amer. Math. Soc. **39** (1973), 450–456.

[16] E. Khamseh, M.R.R. Moghaddam anf F. Saeedi, *Characterization of finite p-groups by their Schur multiplier.* J. Algebra Appl. **12(5)** (2013), 1250035-1–1250035-9.

[17] A. Magidin and R.F. Morse, *Certain homological functors of 2-generator p-groups of class 2*, Computational group theory and the theory of groups **II**, 127–166, Contemp. Math. **511**, Amer. Math. Soc., Providence, RI, 2010.

[18] E.A. O'Brien and M.R. Vaughan-Lee, *The groups with order p^7 for odd prime p.* J. Algebra **292(1)** (2005), 243–258.

[19] N.R. Rocco, *On a construction related to the nonabelian tensor square of a group.* Bol. Soc. Brasil. Mat. (N. S.) **22** (1991), 63–79.

[20] I. Schur, *Über die Darstellungen der endlichen Gruppen durch gebrochene lineare Substitutionen* J. Reine Angew. Math. **127** (1904), 20–50.

[21] X. Zhou, *On the order of Schur multipliers of finite p-groups.* Comm. Algebra **22(1)** (1994), 1–8.

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