PUSHOUT DIAGRAMS OF $H^*$-ALGEBRAS

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In this paper, we obtain some properties of pushout diagrams of $H^*$-algebras. We also prove that, in the commutative diagram of proper $H^*$-algebras and morphisms,

\[
\begin{array}{cccccc}
0 & \longrightarrow & \ker \beta & \overset{\iota}{\longrightarrow} & C & \overset{\beta}{\longrightarrow} & B & \longrightarrow & 0 \\
\downarrow{\alpha|_{\ker \beta}} & & \downarrow{\alpha} & & \downarrow{\gamma} & & & \\
0 & \longrightarrow & \ker \delta & \overset{\iota}{\longrightarrow} & A & \overset{\delta}{\longrightarrow} & X & \longrightarrow & 0
\end{array}
\]

if the right square is pushout, $\alpha$ is surjective and $\ker \alpha \cap \ker \beta^\perp = \{0\}$, then $\gamma$ is injective and $\gamma^* \delta (\alpha(\ker \beta)^\perp) = \beta(\ker \beta^\perp)$. Conversely if $\alpha$ is surjective and $\gamma$ is injective, then the right diagram is pushout. Finally, we deal with the pushout constructions in locally multiplicatively convex $H^*$-algebras.

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1. INTRODUCTION AND PRELIMINARIES

The notion of a pushout diagram was introduced by Pedersen [9] in the category of $C^*$-algebras and some properties of these diagrams were investigated. Some results of pushout diagrams are stable under $H^*$-algebras. In this paper, we use these properties to discover new ones for pushout diagrams of $H^*$-algebras. An $H^*$-algebra, introduced by Ambrose [1] in the associative case, is a Banach algebra $A$ (over $\mathbb{C}$ or $\mathbb{R}$), satisfying the following conditions:

(i) $A$ is itself a Hilbert space under an inner product $\langle ., . \rangle$;
(ii) For each $a$ in $A$ there is an element $a^*$ in $A$, the so-called adjoint of $a$ such that $\langle ab, c \rangle = \langle b, a^* c \rangle$ and $\langle ab, c \rangle = \langle a, cb^* \rangle$ for all $b, c \in A$. In the rest of the paper, we assume that all $H^*$-algebras are complex $H^*$-algebras, unless otherwise specified.

Example 1.1. Any Hilbert space is an $H^*$-algebra, where the product of each pair of elements is zero. Of course, in this case the adjoint $a^*$ of $a$ need not be unique, in fact, every element is an adjoint of every element.

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Recall that \( A_0 = \{ a \in A : aA = \{0\} \} = \{ a \in A : Aa = \{0\} \} \) is called the annihilator ideal of \( A \). A proper \( H^* \)-algebra is an \( H^* \)-algebra with zero annihilator ideal. Ambrose [1] proved that an \( H^* \)-algebra is proper if and only if every element has a unique adjoint. The trace-class \( \tau(A) \) of \( A \) is defined by the set \( \tau(A) = \{ ab : a, b \in A \} \). The trace functional \( \text{tr} \) on \( \tau(A) \) is defined by \( \text{tr}(ab) = \langle ab, b^* \rangle = \langle b, a^* \rangle = \text{tr}(ba) \) for each \( a, b \in A \). In particular, \( \text{tr}(aa^*) = \langle a, a \rangle = \| a \|^2 \) for all \( a \in A \). Many mathematicians worked on \( H^* \)-algebras and developed them in several directions, see ([2, 3, 7, 8]) and references cited therein.

The notion of locally multiplicatively convex \( H^* \)-algebra (l.m.c. \( H^* \)-algebra in short) was introduced in [6] as a natural extension of the \( H^* \)-algebra. A locally multiplicatively convex algebra (l.m.c.a. in short) is a topological algebra \((A, \tau)\), whose topology \( \tau \) is determined by a family \( \{|.|_\lambda \}_{\lambda \in \Lambda} \) of submultiplicative seminorms. If \( A \) is endowed with an involution \( a \mapsto a^* \) such that \( |a|_\lambda = |a^*|_\lambda \), for any \( a \in A \), \( \lambda \in \Lambda \), then \((A, \{|.|_\lambda \}_{\lambda \in \Lambda})\) is called an l.m.c. \(*\)-algebra.

**Definition 1.2.** Suppose that \((A, \{|.|_\lambda \}_{\lambda \in \Lambda})\) is a complete l.m.c. \(*\)-algebra on which a family of positive semi-inner products \( \langle .., . \rangle_\lambda \) is defined such that the following properties hold (i) \( |a|_\lambda^2 = \langle a, a \rangle_\lambda \), (ii) \( \langle ab, c \rangle_\lambda = \langle b, a^* c \rangle_\lambda = \langle c, b^* a \rangle_\lambda \), for all \( x, y, z \in E \) and \( \lambda \in \Lambda \). Then \((A, \{|.|_\lambda \}_{\lambda \in \Lambda})\) is called an l.m.c. \( H^* \)-algebra.

**Example 1.3.** For each \( n \in \mathbb{N} \), put \( A_n = \mathbb{R} \) and \( A = \bigoplus_{n=1}^{\infty} A_n \). Then \( A \) is a real l.m.c. \( H^* \)-algebra, whose topology is determined by the family \( \{|.|_n \}_{n \in \mathbb{N}} \) of submultiplicative seminorms such that \( |.|_{n_0} \) is defined by \( |(x_n)_{n\in \mathbb{N}}|_{n_0} = |x_{n_0}|_{n_0} \) for each \( n_0 \in \mathbb{N} \). On the other hand, \( A \) is not a real \( H^* \)-algebra with usual addition and multiplication. Since the norm \( \|\{n\}\| = \left( \sum_{n=1}^{\infty} n^2 \right)^{1/2} \) is not convergent.

For every \( \lambda \in \Lambda \), \( N_\lambda = \{ a \in A : | a |_\lambda = 0 \} \) is a closed self-adjoint ideal in \( A \) and the quotient space \( A_\lambda = A/N_\lambda \) is an inner product space under \( \langle a_\lambda, b_\lambda \rangle_\lambda = \langle a, b \rangle_\lambda \), where \( a_\lambda = a + N_\lambda \), \( b_\lambda = b + N_\lambda \) are in \( A_\lambda \). The completion \( \hat{A}_\lambda \) of \( A_\lambda \), is a Hilbert space. Moreover, the Banach \(*\)-algebra \((\hat{A}_\lambda, |.|_\lambda)\) is an \( H^* \)-algebra, where \( \| \bar{a} \|_\lambda = | a |_\lambda \), \( (\bar{a} \in \hat{A}_\lambda) \).

If \( A \) and \( B \) are \( H^* \)-algebras (l.m.c. \( H^* \)-algebras), then a continuous \(*\)-homomorphism \( \varphi : A \rightarrow B \) is called a morphism. For more details on l.m.c. \( H^* \)-algebras, see [4, 5]. In the present, work we generalize the concept of pushout diagram in the framework of \( H^* \)-algebras and l.m.c. \( H^* \)-algebras. This paper is organized as follows: in Section 2, we introduce the concept of pushout diagram in the framework of \( H^* \)-algebras and investigate some conditions under which a diagram of \( H^* \)-algebras is pushout. In Section 3, we consider a commutative
diagram of l.m.c. $H^*$-algebras and obtain some conditions under which the corresponding diagrams of $H^*$-algebras are pushout. Throughout this paper, $H^*$-algebras are proper and if $E$ is a subset of an $H^*$-algebra $A$, then $E^\perp$ and $Id(E)$ are denoted the orthogonal complement of $E$ and the smallest closed ideal generated by $E$, respectively.

2. PUSHOUT CONSTRUCTIONS IN $H^*$-ALGEBRAS

In this section, we introduce a pushout diagram of $H^*$-algebras and investigate some of their properties.

**Definition 2.1.** A commutative diagram of $H^*$-algebras and morphisms

\[
\begin{align*}
C & \xrightarrow{\beta} B \\
\downarrow^{\alpha} & \downarrow^{\gamma} \\
A & \xrightarrow{\delta} X
\end{align*}
\]

is pushout if $X$ is generated by $\gamma(B) \cup \delta(A)$ and for every other pair of morphisms $\varphi : A \to Y$ and $\psi : B \to Y$ into an $H^*$-algebra $Y$ satisfying condition $\varphi \circ \alpha = \psi \circ \beta$, there is a unique morphism $\sigma : X \to Y$ such that $\varphi = \sigma \circ \delta$ and $\psi = \sigma \circ \gamma$.

The following theorem is proved in the framework of $C^*$-algebras ([9, Theorem 2.5.]). It is easy to show that it holds in the category of $H^*$-algebras.

**Theorem 2.2.** In a commutative diagram of extensions of $H^*$-algebras and morphisms together with inclusion map $i$

\[
\begin{align*}
0 & \longrightarrow I \xrightarrow{\iota} C \xrightarrow{\beta} B \longrightarrow 0 \\
0 & \longrightarrow J \xrightarrow{\iota} A \xrightarrow{\delta} X \longrightarrow 0
\end{align*}
\]

the right square is pushout if and only if $\alpha(I)$ generates $J$ as an ideal.
Proof. Let $Y$ be an $H^*$-algebra and $\varphi : A \to Y$ and $\psi : B \to Y$ are a pair of morphisms such that $\varphi \circ \alpha = \psi \circ \beta$. Since $I = \ker \beta$, we have $I \subset \ker \psi \circ \beta = \ker \varphi \circ \alpha$. Then $\alpha(I) \subset \ker \varphi$, since $Id(\alpha(I)) = J$ we have $J \subset \ker \varphi$. Suppose that $b \in B$ is arbitrary. Then there exists an element $c \in C$ such that $\beta(c) = b$. Hence
\[\psi(b) = \psi(\beta(c)) = \varphi(\alpha(c)) = \sigma(\delta(\alpha(c))) = \sigma(\gamma(\beta(c))) = \sigma(\gamma(b)),\]
whence $\psi = \sigma \circ \gamma$.

Conversely, let the diagram be pushout and $J_0 = Id(\alpha(I))$. Then $J_0 \subset J$ and consider the pair of morphisms $(\varphi, \psi)$ consisting of the quotient morphism $\varphi : A \to \frac{A}{J_0}$ and the induced morphism $\psi : B \to \frac{A}{J_0}$ defined by $\psi(c + I) = \alpha(c) + J_0$ for each $c \in C$. By the assumption, $\varphi = \sigma \circ \delta$ for some morphism $\sigma : X \to \frac{A}{J_0}$, which insures that
\[J = \ker \sigma \subset \ker \sigma \circ \delta = \ker \varphi = J_0 = Id(\alpha(I)).\]

Remark 2.3. If $I$ is an arbitrary ideal in $H^*$-algebra $A$, then for $a \in I$ and $b \in I^\perp$, we have $ab \in I \cap I^\perp = \{0\}$.

Lemma 2.4. Let $A$ and $B$ be $H^*$-algebras and $\alpha : A \to B$ be a surjective morphism. If $I$ is a closed, self adjoint ideal in $A$ such that $I \cap \ker \alpha = \{0\}$, then $\alpha(I^\perp) = \alpha(I)^\perp$.

Proof. Let $b \in I^\perp$. Then by Remark 2.3, we have $\langle \alpha(b), \alpha(a) \rangle = \text{tr}(\alpha(b)(\alpha(a))^*) = \text{tr}(\alpha(ba^*)) = 0$ for each $a \in I$. So $\alpha(I^\perp) \subseteq \alpha(I)^\perp$. Conversely, let $c \in \alpha(I)^\perp \subseteq B$. By surjectivity of $\alpha$, $c = \alpha(d)$ for some $d \in A$. Also, surjectivity of morphism $\alpha$ implies that $\alpha(I)$ is an ideal in $B$. Applying Remark 2.3, we get $\alpha(di) = c\alpha(i) = 0$ for each $i \in I$. Hence for each $i \in I$, $di \in \ker \alpha \cap I = \{0\}$ and so $\langle d, i \rangle = \text{tr}(di^*) = 0$. It yields that $d \in I^\perp$ and $c = \alpha(d) \in \alpha(I^\perp)$. □

Theorem 2.5. Let
\[
\begin{array}{cccccc}
0 & \rightarrow & \ker \beta & \xrightarrow{\iota} & C & \xrightarrow{\beta} & B & \rightarrow & 0 \\
& & \downarrow{\alpha|_{\ker \beta}} & & \downarrow{\alpha} & & \downarrow{\gamma} & \\
0 & \rightarrow & \ker \delta & \xrightarrow{\iota} & A & \xrightarrow{\delta} & X & \rightarrow & 0
\end{array}
\]
be a commutative diagram of extensions of $H^*$-algebras and morphisms. If the right square is pushout, $\alpha$ is surjective and $\ker \alpha \cap \ker \beta^\perp = \{0\}$, then $\gamma$ is injective and $\gamma^*\delta(\alpha(\ker \beta)^\perp) = \beta(\ker \beta^\perp)$. Conversely if $\alpha$ and $\gamma$ are surjective and injective, respectively, then the right diagram is pushout.

Proof. By the surjectivity of $\alpha$ and Theorem 2.2, it is easy to verify that $Id(\alpha(\ker \beta)) = \ker \delta$ is a closed ideal in $A$. By applying Lemma 2.4, where
$I = \ker \beta^\perp$ we conclude that $\alpha(\ker \beta) = \alpha(\ker \beta^\perp) = \alpha(\ker \beta^\perp)^\perp$. Then $\alpha(\ker \beta)$ is a closed ideal in $A$ and

$$\tag{2.1} A = \alpha(\ker \beta) \oplus \alpha(\ker \beta)^\perp.$$  

On the other hand $\alpha$ is onto and

$$\tag{2.2} A = \alpha(\ker \beta \oplus \ker \beta^\perp) = \alpha(\ker \beta) \oplus \alpha(\ker \beta^\perp)$$

As in the proof of Lemma 2.4, we get $\alpha(\ker \beta^\perp) \subseteq \alpha(\ker \beta)^\perp$. By (2.1) and (2.2), we shall show that $\gamma$ is injective. Let $b$ be a non zero element in $B$. Since $B = \beta(C) = \beta(\ker \beta \oplus \ker \beta^\perp) = \beta(\ker \beta^\perp)$, there exists a non zero element $c_1$ in $\ker \beta^\perp$ such that $b = \beta(c_1)$. In addition, by the commutativity of the diagram, we get $\gamma(b) = \gamma(\beta(c_1)) = \delta \alpha(c_1)$. By the assumption, $c_1$ is not in $\ker \alpha$ and $\alpha(c_1) \in \alpha(\ker \beta^\perp) = \alpha(\ker \beta)^\perp = \ker \delta^\perp$. Hence $\gamma(b) = \delta \alpha(c_1) \neq 0$.

By the injectivity of $\gamma$ we have $\overline{\gamma^*(X)^\perp} = \ker \gamma = \{0\}$, so $\overline{\gamma^*(X)} = B$. The commutativity of diagram, surjectivity of $\beta$ and $\ker \delta^\perp = \alpha(\ker \beta^\perp)$ ensure that

$$\begin{align*}
\overline{\gamma^* \delta \alpha(\ker \beta^\perp)} &= \overline{\gamma^* \delta (\ker \delta^\perp)} \\
&= \overline{\gamma^* \delta (\ker \delta \oplus \ker \delta^\perp)} \\
&= \overline{\gamma^* \delta (A)} \\
&= \overline{\gamma^* (X)} \\
&= B \\
&= \beta(C) \\
&= \beta(\ker \beta \oplus \ker \beta^\perp) \\
&= \beta(\ker \beta^\perp).
\end{align*}$$

Conversely, suppose that $\gamma$ and $\alpha$ are injective and surjective, respectively. The surjectivity of $\alpha$ implies that $\alpha(\ker \beta)$ is an ideal in $A$. By Theorem 2.2, it is enough to show that $\alpha(\ker \beta) = \ker \delta$. Let $c \in \ker \beta$. The commutativity of diagram implies that $\delta \alpha(c) = \gamma \beta(c) = 0$, so $\alpha(c) \in \ker \delta$. This implies that, $\alpha(\ker \beta) \subseteq \ker \delta$. For the reverse direction, assume that $a \in \ker \delta$. By the surjectivity of $\alpha$, there exists $c \in C$ such that $\alpha(c) = a$. Hence $\gamma \beta(c) = \delta \alpha(c) = \delta(a) = 0$. The injectivity of $\gamma$, yields that $\beta(c) = 0$. So $c \in \ker \beta$ and $a = \alpha(c) \in \alpha(\ker \beta)$.

In the following example, we show that the surjectivity of $\alpha$ in Theorem 2.5 is a necessary condition.

**Example 2.6.** The Hilbert space $\mathbb{R}^n$ is a real $H^*$-algebra, where for each $(a_1, ..., a_n)$ and $(b_1, ..., b_n)$ in $\mathbb{R}^n$, $(a_1, ..., a_n)(b_1, ..., b_n) = (a_1b_1, ..., a_nb_n)$, $\langle (a_1, ..., a_n), (b_1, ..., b_n) \rangle = \sum_{i=1}^n a_i b_i$ and $(a_1, ..., a_n)^* = (a_1, ..., a_n)$. Let $2 < m < $
Let $n$ and $\lambda_0 \in \{2, ..., m-1\}$ be fixed. Consider the following commutative diagram of the $H^*$-algebras and morphisms,

\[
\begin{array}{ccc}
\mathbb{R}^m & \xrightarrow{\beta} & \mathbb{R}^{m-1} \\
\downarrow{\alpha} & & \downarrow{\gamma} \\
\mathbb{R}^n & \xrightarrow{\delta} & \mathbb{R}^{m-1}
\end{array}
\]

where $\alpha, \beta, \delta$ and $\gamma$ are defined as follows.

(i) $\alpha((a_1, ..., a_m)) = (a_1, ..., a_m, 0, ..., 0)$.

(ii) $\beta((a_1, ..., a_m)) = (a_1, ..., a_{\lambda_0-1}, a_{\lambda_0+1}, ..., a_m)$.

(iii) $\delta((a_1, ..., a_n)) = (a_1, ..., a_{\lambda_0-1}, a_{\lambda_0+1}, ..., a_m)$.

(iv) $\gamma$ is the identity operator.

Clearly, $\alpha$ is not surjective and $\ker \beta = \{(a_1, ..., a_m) : a_1 = ... = a_{\lambda_0-1} = a_{\lambda_0+1} = ... = a_m = 0\}$. Hence $\alpha(\ker \beta) = \{(a_1, ..., a_n) : a_i = 0, \text{ for every } i \in \{1, ..., \lambda_0 - 1, \lambda_0 + 1, ..., n\}\}$, which it is an ideal in $\mathbb{R}^n$.

On the other hand, $\ker \delta = \{(a_1, ..., a_n) : a_i = 0 \text{ for every } i \in \{1, ..., \lambda_0 - 1, \lambda_0 + 1, ..., m\}\}$. Then $\alpha(\ker \beta) \neq \ker \delta$ and so, by Theorem 2.2, the above diagram is not pushout, although, $\beta$ and $\delta$ are surjective and $\gamma$ is injective.

### 3. PUSHOUT CONSTRUCTIONS IN l.m.c. $H^*$-ALGEBRAS

In this section, we consider a commutative diagram of l.m.c. $H^*$-algebras and morphisms and obtain some conditions under which the corresponding diagrams of $H^*$-algebras and morphisms are pushout.

**Theorem 3.1.** Let $(A, \{\|\lambda_A\}\lambda_A \in \Lambda_A)$, $(B, \{\|\lambda_B\}\lambda_B \in \Lambda_B)$, $(C, \{\|\lambda_C\}\lambda_C \in \Lambda_C)$ and $(X, \{\|\lambda_X\}\lambda_X \in \Lambda_X)$ be l.m.c. $H^*$-algebras and let

\[
\begin{array}{ccc}
C & \xrightarrow{\beta} & B \\
\downarrow{\alpha} & & \downarrow{\gamma} \\
A & \xrightarrow{\delta} & X
\end{array}
\]

be a commutative diagram of l.m.c. $H^*$-algebras and morphisms. Fix $\lambda_C \in \Lambda_C$. If there exist $\lambda_B \in \Lambda_B$, $\lambda_A \in \Lambda_A$ and $\lambda_X \in \Lambda_X$ such that the following conditions hold

(i) $\alpha(N_{\lambda_C}) \subseteq N_{\lambda_A}$,

(ii) $\beta(N_{\lambda_C}) \subseteq N_{\lambda_B}$,

(iii) $\delta(N_{\lambda_A}) \subseteq N_{\lambda_X}$,

(iv) $\gamma(N_{\lambda_B}) \subseteq N_{\lambda_X}$,
(v) $\gamma^{-1}\{x\} \cap N_{\lambda B} \neq \emptyset$ for each $x \in N_{\lambda X}$,

(vi) $\delta$ and $\beta$ are bijective and surjective morphisms, respectively, then the right square in the following diagram of pre-$H^*$-algebras is pushout, where $I' = \frac{\beta^{-1}(N_{\lambda B})}{N_{\lambda C}}$, $J' = \frac{\delta^{-1}(N_{\lambda X})}{N_{\lambda A}}$ and $\alpha'(c + N_{\lambda C}) = \alpha(c) + N_{\lambda A}$ for each $c \in C$. The maps $\beta'$, $\delta'$ and $\gamma'$ are defined similarly.

$$
\begin{array}{c}
0 \longrightarrow I' \xrightarrow{i} C_{N_{\lambda C}} \xrightarrow{\beta'} B_{N_{\lambda B}} \longrightarrow 0 \\
\downarrow \alpha'|_{I'} \quad \downarrow \alpha' \quad \downarrow \gamma' \\
0 \longrightarrow J' \xrightarrow{i} A_{N_{\lambda A}} \xrightarrow{\delta'} X_{N_{\lambda X}} \longrightarrow 0
\end{array}
$$

Proof. Consider $c + N_{\lambda C} \in \ker(\beta')$ if and only if $c \in \beta^{-1}(N_{\lambda B})$. This ensures that $I' = \ker(\beta')$. A similar argument shows that $J' = \ker(\delta')$. The diagram of extensions on pre-$H^*$-algebras is commutative since the initial diagram is commutative. By Theorem 2.2, the right square of the diagram is pushout if and only if $Id(\alpha'(I')) = J'$. We shall show that this equation holds if and only if $\delta(Id(\alpha(\beta^{-1}(N_{\lambda B})))) = N_{\lambda X}$.

If $Y \subseteq A$, then for each finite subset $F$ of $\mathbb{N}$

$$
Id(\delta(Y)) = \left\{ \sum_{i \in F} \lambda_i \delta(y_i)^{m_i}x_i\delta(z_i)^{n_i}, \lambda_i \in \mathbb{C}, m_i, n_i \in \mathbb{N}, y_i, z_i \in Y, x_i \in X \right\}.
$$

By the surjectivity of $\delta$, for each $i \in F$, there exists $a_i \in A$ such that $x_i = \delta(a_i)$. Hence

$$
Id(\delta(Y)) = \left\{ \delta(\sum_{F} \lambda_i y_i^{m_i} a_i z_i^{n_i}), \lambda_i \in \mathbb{C}, m_i, n_i \in \mathbb{N}, y_i, z_i \in Y, a_i \in A \right\} = \delta(Id(Y)).
$$

Since $\alpha(\beta^{-1}(N_{\lambda B})) \subseteq A$, by the above discussion, we get

(3.1) $\delta(Id(\alpha(\beta^{-1}(N_{\lambda B})))) = Id(\delta(\alpha(\beta^{-1}(N_{\lambda B}))))$.

Conditions (iv), (vi) and commutativity of the diagram imply that

$$
\delta(\alpha(\beta^{-1}(N_{\lambda B}))) = \gamma(\beta\beta^{-1}(N_{\lambda B}) = \gamma(N_{\lambda B}) \subseteq N_{\lambda X}.
$$

Since $N_{\lambda X}$ is an ideal in $X$, we get $Id(\delta(\alpha(\beta^{-1}(N_{\lambda B})))) \subseteq N_{\lambda X}$.

Conversely, suppose that there exists $x \in N_{\lambda X}$ such that $x \notin \delta(Id(\alpha(\beta^{-1}(N_{\lambda B}))))$. Thus $\delta^{-1}(x) \notin Id(\alpha(\beta^{-1}(N_{\lambda B})))$. Condition (v) implies that $x = \gamma(b)$ for some $b \in N_{\lambda B}$. By the surjectivity of $\beta$, there is an element $c \in C$ such that $b = \beta(c)$. Hence $c \in \beta^{-1}(N_{\lambda B})$. On the other hand, it follows from the
commutativity of the diagram and bijectivity of \( \delta \) that \( \delta^{-1}(x) = \delta^{-1}(\gamma(b)) = \delta^{-1}(\gamma\beta(c)) = \delta^{-1}(\delta\alpha(c)) = \alpha(c) \). Therefore, \( \delta^{-1}(x) = \alpha(c) \in \alpha\beta^{-1}(N_{\lambda_B}) \), so \( \delta^{-1}(x) \in \text{Id}(\alpha(\beta^{-1}(N_{\lambda_B}))) \). This is a contradiction. Therefore \( N_{\lambda x} \subseteq \text{Id}(\delta(\alpha(\beta^{-1}(N_{\lambda_B})))) \). Hence \( \text{Id}(\delta(\alpha(\beta^{-1}(N_{\lambda_B})))) = N_{\lambda x} \). By (3.1), we reach \( \delta(\text{Id}(\alpha(\beta^{-1}(N_{\lambda_B})))) = \delta^{-1}(N_{\lambda x}) \). So

\[
J' = \frac{\delta^{-1}(N_{\lambda x})}{N_{\lambda_A}} = \frac{\text{Id}(\alpha(\beta^{-1}(N_{\lambda_B})))}{N_{\lambda_A}} = \text{Id} \left( \frac{\alpha(\beta^{-1}(N_{\lambda_B}))}{N_{\lambda_A}} \right) \\
= \text{Id} \left( \alpha' \left( \frac{\beta^{-1}(N_{\lambda_B})}{N_{\lambda_C}} \right) \right) = \text{Id}(\alpha'(I')).
\]

Note that the condition (v) implies that \( N_{\lambda_A} \subseteq \alpha(\beta^{-1}(N_{\lambda_B})) \). \( \square \)

**Corollary 3.2.** If the assumptions of Theorem 3.1 hold, then the right square in the following diagram of \( H^* \)-algebras is pushout, where \( I' = \frac{\beta^{-1}(N_{\lambda_B})}{N_{\lambda_C}} \) and \( J' = \frac{\delta^{-1}(N_{\lambda x})}{N_{\lambda_A}} \).

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & I' & \longrightarrow & C_{N_{\lambda_C}} & \longrightarrow & B_{N_{\lambda_B}} & \longrightarrow & 0 \\
& & \downarrow \alpha'_{|I'} & & \downarrow \alpha' & & \downarrow \gamma' & & \\
0 & \longrightarrow & J' & \longrightarrow & A_{N_{\lambda_A}} & \longrightarrow & X_{N_{\lambda_X}} & \longrightarrow & 0
\end{array}
\]

**Proof.** We show that \( \alpha', \beta', \delta', \gamma' \) are well-defined. Assume that \( x \in \frac{C_{N_{\lambda_C}}}{N_{\lambda_C}} \). Then \( x = \lim_{n \to \infty} (c_n + N_{\lambda_C}) \) for some sequence \( \{c_n\}_n \) in \( C \). The morphism \( \alpha \) is continuous, so it is bounded. For each \( \lambda_C \in \Lambda_C, \lambda_A \in \Lambda_A \), there exists \( M_{\lambda_C, \lambda_A} \geq 0 \) such that \( \|\alpha(x)\|_{\lambda_A} \leq M_{\lambda_C, \lambda_A} \|x\|_{\lambda_C} \) for each \( x \in C \). Since \( \{c_n + N_{\lambda_C}\}_n \) is Cauchy and \( \alpha \) is bounded, it is easy to show that \( \{\alpha(c_n) + N_{\lambda_A}\}_n \) is a Cauchy sequence in \( \frac{A_{N_{\lambda_A}}}{N_{\lambda_A}} \) and so it is convergent. Set \( \alpha'(x) = \lim_{n \to \infty} \alpha(c_n) + N_{\lambda_A} \).

The proof can be complemented by using the same argument as in the proof of Theorem 3.1, so we omit it. \( \square \)

In the next example, we show that the condition (v) of Theorem 3.1 is necessary.

**Example 3.3.** Let \( \Lambda = \{1, \ldots, n\} \), where \( n \geq 2 \) and \( \{(A_\lambda, \|\cdot\|_\lambda)\}_{\lambda \in \Lambda} \) be a family of \( H^* \)-algebras, in which \( A_{n-1} = A_n \). Then \( \boxplus_{\lambda \in \Lambda} A_\lambda = \{(x_\lambda) ; x_\lambda \in A_\lambda\} \) is an l.m.c. \( H^* \)-algebra, whose topology is determined by a family \( \{\|\cdot\|_\lambda\}_{\lambda \in \Lambda} \) of submultiplicative seminorms such that \( \|\cdot\|_{\lambda_0} \) is defined by \( \|(x_\lambda)_{\lambda \in \Lambda}\|_{\lambda_0} = \|x_{\lambda_0}\|_{\lambda_0} \).
for each $\lambda_0 \in \Lambda$. Further, $\bigoplus_{\lambda \in \Lambda} A_\lambda$ is an l.m.c. $H^*$-algebra, whose topology is determined by the family $\{ ||.||_{\lambda, \mu} \}_{\lambda, \mu \in \Lambda}$ of submultiplicative seminorms such that $||.||_{\lambda_0, \mu_0}$ is defined by $|| (x_\lambda)_{\lambda \in \Lambda} ||_{\lambda_0, \mu_0}^2 = || x_{\lambda_0} ||_{\lambda_0}^2 + || x_{\mu_0} ||_{\mu_0}^2$ for each $\lambda_0, \mu_0 \in \Lambda$. Note that $\bigoplus_{\lambda \in \Lambda} A_\lambda$ and $\bigoplus_{\lambda \in \Lambda} A_\lambda$ are both $H^*$-algebras with usual addition and multiplication. Consider the following commutative diagram of l.m.c. $H^*$-algebras and morphisms.

$$
\begin{array}{ccc}
\bigoplus_{\lambda \in \Lambda} A_\lambda & \xrightarrow{\beta} & \bigoplus_{\lambda \in \Lambda} A_\lambda \\
\downarrow{\alpha} & & \downarrow{\gamma} \\
\lambda \in A_\lambda & \xrightarrow{\delta} & \bigoplus_{\lambda \in \Lambda} A_\lambda
\end{array}
$$

The morphisms $\alpha, \beta, \delta$ and $\gamma$ are defined as follow.

(i) $\alpha((x_\lambda)_{\lambda \in \Lambda}) = (y_\lambda)_{\lambda \in \Lambda}$, where $y_\lambda = x_\lambda$ for $\lambda \neq n$ and $y_n = 0$.

(ii) $\beta((x_\lambda)_{\lambda \in \Lambda}) = (z_\lambda)_{\lambda \in \Lambda}$, where $z_\lambda = x_\lambda$ for $\lambda \in \{1, 2, ..., n-2\}$ and $z_{n-1} = x_n, z_n = x_{n-1}$.

(iii) $\gamma((z_\lambda)_{\lambda \in \Lambda}) = (w_\lambda)_{\lambda \in \Lambda}$ such that $w_\lambda = z_\lambda$ for $\lambda \neq n-1$ and $w_{n-1} = 0$.

(iv) $\delta$ is defined in the same way as $\beta$.

Now consider the diagram of extensions,

$$
\begin{array}{ccc}
0 & \xrightarrow{i} & I' \\
\downarrow{\alpha'} & & \downarrow{\alpha'} \\
0 & \xrightarrow{i} & J'
\end{array}
\quad \begin{array}{ccc}
\bigoplus_{\lambda \in \Lambda} A_{\Lambda} & \xrightarrow{\beta'} & \bigoplus_{\lambda \in \Lambda} A_{\Lambda} \\
\downarrow{\gamma'} & & \downarrow{\gamma'} \\
\bigoplus_{\lambda \in \Lambda} A_{\Lambda} & \xrightarrow{\delta'} & \bigoplus_{\lambda \in \Lambda} A_{\Lambda}
\end{array}
\quad \begin{array}{c}
0
\end{array}
$$

where

$$
N_{n-1} = \{(a_\lambda)_{\lambda \in \Lambda} : |(a_\lambda)_{\lambda \in \Lambda}|_{n-1} = 0 \} = \{(a_\lambda)_{\lambda \in \Lambda} : a_{n-1} = 0 \},
$$

$$
N_n = \{(a_\lambda)_{\lambda \in \Lambda} : |(a_\lambda)_{\lambda \in \Lambda}|_{n} = 0 \} = \{(a_\lambda)_{\lambda \in \Lambda} : a_n = 0 \},
$$

$$
N_{n-1,n} = \{(a_\lambda)_{\lambda \in \Lambda} : |(a_\lambda)_{\lambda \in \Lambda}|_{n-1,n} = 0 \} = \{(a_\lambda)_{\lambda \in \Lambda} : a_{n-1} = a_n = 0 \},
$$

$I' = \ker \beta' = \{(a_1, ..., a_{n-2}, 0, a_n) + N_{n-1,n} : a_i \in A, \quad i \in \{1, 2, ..., n-2, n\} \},$ and

$J' = \ker \delta' = \{(a_1, a_2, ..., a_{n-1}, 0) + N_{n-1,n} : a_i \in A, \quad i \in \{1, 2, ..., n-1\} \}.$

Clearly, conditions (i), (ii), (iii), (iv) and (vi) of the preceding theorem hold. If $a = (a_1, a_2, ..., 0, a_n) \in N_{n-1}$ in which $a_n \neq 0$, then $\gamma^{-1}(a) = (a_1, a_2, ..., a_{n-1}, a_n)$ for arbitrary element $a_{n-1} \in A$ and $\gamma^{-1}(a) \cap N_n = \emptyset$.

We are going to show that the right square in the diagram above, is not pushout. It is enough to show that $Id(\alpha'(I')) \neq J'$. This holds, since $\alpha'(I') = \{(a_1, a_2, ..., a_{n-2}, 0, 0) + N_{n-1,n} : a_i \in A$ for $i \in \{1, 2, ..., n-2\} \}$, which is an ideal in $\bigoplus_{\lambda \in \Lambda} A_{\Lambda}$ and not equal to $\ker \delta'$.

In the special case of above example, where $n = 2$ we have the following diagram
\[
0 \longrightarrow I' \xrightarrow{i} A \oplus A \xrightarrow{\beta'} \frac{A \Box A}{A \boxplus 0} \longrightarrow 0 \\
\quad \downarrow \alpha' \quad \downarrow \alpha' \quad \downarrow \gamma' \\
0 \longrightarrow J' \xrightarrow{i} A \oplus A \xrightarrow{\delta'} \frac{A \Box A}{0 \boxplus A} \longrightarrow 0
\]

where \( A = A_1 = A_2 \), so that \( \frac{A \oplus A}{N_{1,2}} = A \oplus A \), \( \frac{A \boxplus A}{N_1} = \frac{A \boxplus A}{0 \boxplus A} \), \( \frac{A \boxplus A}{N_2} = \frac{A \boxplus A}{A \boxplus 0} \).

\[
\alpha'((x_1, x_2)) = (x_1, 0), \\
\beta'((x_1, x_2)) = (x_2, x_1) + A \boxplus 0, \\
\gamma'((x_1, x_2) + A_1 0) = (0, x_2) + 0 \boxplus A \\
\delta'(x_1, x_2) = (x_2, x_1) + 0 \boxplus A
\]

Also \( I' = \ker \beta' = 0 \oplus A \) and \( J' = \ker \delta' = A \oplus 0 \). It is easy to see that all conditions of the preceding theorem hold except \((v)\). In addition, \( \alpha'(I') = \alpha'(0 \oplus A) = 0 \oplus 0 \) and \( J' = A \oplus 0 \) are valid. The right square in the diagram above, however is not pushout, since \( \text{Id}(\alpha'(I')) \neq J' \).

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**REFERENCES**


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