PUSHOUT DIAGRAMS OF H^*-ALGEBRAS

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In this paper, we obtain some properties of pushout diagrams of H^*-algebras. We also prove that, in the commutative diagram of proper H^*-algebras and morphisms,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker\beta & \xrightarrow{\iota} & C & \xrightarrow{\beta} & B & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \alpha|_{\ker\beta}} & & \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & \ker\delta & \xrightarrow{\iota} & A & \xrightarrow{\delta} & X & \longrightarrow & 0 \end{array}$$

if the right square is pushout, α is surjective and $\ker\alpha \cap \ker\beta^\perp = \{0\}$, then γ is injective and $\gamma^*\delta(\alpha(\ker\beta)^\perp) = \beta(\ker\beta^\perp)$. Conversely if α is surjective and γ is injective, then the right diagram is pushout. Finally, we deal with the pushout constructions in locally multiplicatively convex H^*-algebras.

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1. INTRODUCTION AND PRELIMINARIES

The notion of a pushout diagram was introduced by Pedersen [9] in the category of C^*-algebras and some properties of these diagrams were investigated. Some results of pushout diagrams are stable under H^*-algebras. In this paper, we use these properties to discover new ones for pushout diagrams of H^*-algebras. An H^*-algebra, introduced by Ambrose [1] in the associative case, is a Banach algebra A (over $\mathbb{C}$ or $\mathbb{R}$), satisfying the following conditions:
(i) A is itself a Hilbert space under an inner product $\langle .,. \rangle$;
(ii) For each a in A there is an element a^* in A, the so-called adjoint of a such that $\langle ab, c\rangle = \langle b, a^*c\rangle$ and $\langle ab, c\rangle = \langle a, cb^*\rangle$ for all $b, c \in A$. In the rest of the paper, we assume that all H^*-algebras are complex H^*-algebras, unless otherwise specified.

Example 1.1. Any Hilbert space is an H^*-algebra, where the product of each pair of elements is zero. Of course, in this case the adjoint a^* of a need not be unique, in fact, every element is an adjoint of every element.

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Recall that $A_0 = \{a \in A : \ aA = \{0\}\} = \{a \in A : \ Aa = \{0\}\}$ is called the annihilator ideal of A. A proper H^*-algebra is an H^*-algebra with zero annihilator ideal. Ambrose [1] proved that an H^*-algebra is proper if and only if every element has a unique adjoint. The trace-class $\tau(A)$ of A is defined by the set $\tau(A) = \{ab : \ a, b \in A\}$. The trace functional tr on $\tau(A)$ is defined by $\mathrm{tr(ab)} = \langle \mathrm{a, b^*} \rangle = \langle \mathrm{b, a^*} \rangle = \mathrm{tr(ba)}$ for each $a, b \in A$. In particular, $\mathrm{tr(aa^*)} = \langle \mathrm{a, a} \rangle = \|\mathrm{a}\|^2$ for all $a \in A$. Many mathematicians worked on H^*-algebras and developed them in several directions, see ([2, 3, 7, 8]) and references cited therein.

The notion of locally multiplicatively convex H^*-algebra (l.m.c. H^*-algebra in short) was introduced in [6] as a natural extension of the H^*-algebra. A locally multiplicatively convex algebra (l.m.c.a. in short) is a topological algebra (A, τ), whose topology τ is determined by a family $\{|.|_\lambda\}_{\lambda \in \Lambda}$ of submultiplicative seminorms. If A is endowed with an involution $a \mapsto a^*$ such that $|a|_\lambda = |a^*|_\lambda$, for any $a \in A, \ \lambda \in \Lambda$, then $(A, \{|.|_\lambda\}_{\lambda \in \Lambda})$ is called an l.m.c. $*$-algebra.

Definition 1.2. Suppose that $(A, \{|.|_\lambda\}_{\lambda \in \Lambda})$ is a complete l.m.c. $*$-algebra on which a family of positive semi-inner products $(\langle ., . \rangle_\lambda)_{\lambda \in \Lambda}$ is defined such that the following properties hold (i) $|a|_\lambda^2 = \langle a, a \rangle_\lambda$,
(ii) $\langle ab, c \rangle_\lambda = \langle b, a^* c \rangle_\lambda = \langle a, cb^* \rangle_\lambda$, for all $x, y, z \in E$ and $\lambda \in \Lambda$.
Then $(A, \{|.|_\lambda\}_{\lambda \in \Lambda})$ is called an l.m.c. H^*-algebra.

Example 1.3. For each $n \in \mathbb{N}$, put $A_n = \mathbb{R}$ and $A = \oplus_{n=1}^{\infty} A_n$. Then A is a real l.m.c. H^*-algebra, whose topology is determined by the family $\{|.|_n\}_{n \in \mathbb{N}}$ of submultiplicative seminorms such that $|.|_{n_0}$ is defined by $|(x_n)_{n \in \mathbb{N}}|_{n_0} = |x_{n_0}|_{n_0}$ for each $n_0 \in \mathbb{N}$. On the other hand, A is not a real H^*-algebra with usual addition and multiplication. Since the norm $\|\{n\}\| = (\sum_{n=1}^{\infty} n^2)^{\frac{1}{2}}$ is not convergent.

For every $\lambda \in \Lambda$, $\quad N_\lambda = \{a \in A : \ | \ a \ |_\lambda = 0\}$ is a closed self-adjoint ideal in A and the quotient space $A_\lambda = A/N_\lambda$ is an inner product space under $\langle a_\lambda, b_\lambda \rangle_\lambda = \langle a, b \rangle_\lambda$, where $a_\lambda = a + N_\lambda, \ b_\lambda = b + N_\lambda$ are in A_λ. The completion $\widehat{A}_\lambda$ of A_λ, , is a Hilbert space. Moreover, the Banach $*$-algebra $(\widehat{A}_\lambda, \|.\|_\lambda)$ is an H^*-algebra, where $\|\overline{a}\|_\lambda = |a|_\lambda, \quad (\overline{a} \in \widehat{A}_\lambda)$.

If A and B are H^*-algebras (l.m.c. H^*-algebras), then a continuous $*$-homomorphism $\varphi : A \to B$ is called a morphism. For more details on l.m.c. H^*-algebras, see [4, 5]. In the present, work we generalize the concept of pushout diagram in the framework of H^*-algebras and l.m.c. H^*-algebras. This paper is organized as follows: in Section 2, we introduce the concept of pushout diagram in the framework of H^*-algebras and investigate some conditions under which a diagram of H^*-algebras is pushout. In Section 3, we consider a commutative diagram of l.m.c. H^*-algebras and obtain some conditions under which the corresponding diagrams of H^*-algebras are pushout. Throughout this paper, H^*-algebras are proper and if E is a subset of an H^*-algebra A, then $E^{\perp}$ and $Id(E)$ are denoted the orthogonal complement of E and the smallest closed ideal generated by E, respectively.

2. PUSHOUT CONSTRUCTIONS IN H^*-ALGEBRAS

In this section, we introduce a pushout diagram of H^*-algebras and investigate some of their properties.

Definition 2.1. A commutative diagram of H^*-algebras and morphisms

$$\begin{tikzcd}
C \arrow[r, "\beta"] \arrow[d, "\alpha"] & B \arrow[d, "\gamma"] \\
A \arrow[r, "\delta"] & X
\end{tikzcd}$$

is pushout if X is generated by $\gamma(B) \cup \delta(A)$ and for every other pair of morphisms $\varphi : A \to Y$ and $\psi : B \to Y$ into an H^*-algebra Y satisfying condition $\varphi o \alpha = \psi \circ \beta$, there is a unique morphism $\sigma : X \to Y$ such that $\varphi = \sigma \circ \delta$ and $\psi = \sigma \circ \gamma$.

$$\begin{tikzcd}
Y & & \\
& X \arrow[ul, dotted, "\sigma"'] & B \arrow[l, "\gamma"] \arrow[ull, bend right, "\psi"'] \\
& A \arrow[u, "\delta"] \arrow[uul, bend left, "\varphi"] & C \arrow[l, "\alpha"] \arrow[u, "\beta"']
\end{tikzcd}$$

The following theorem is proved in the framework of C^*-algebras ([9, Theorem 2.5.]). It is easy to show that it holds in the category of H^*-algebras.

THEOREM 2.2. *In a commutative diagram of extensions of H^*-algebras and morphisms together with inclusion map i*

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r, "\iota"] \arrow[d, "\alpha|_I"] & C \arrow[r, "\beta"] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & J \arrow[r, "\iota"] & A \arrow[r, "\delta"] & X \arrow[r] & 0
\end{tikzcd}$$

the right square is pushout if and only if $\alpha(I)$ generates J as an ideal.

Proof. Let Y be an H^*-algebra and $\varphi : A \to Y$ and $\psi : B \to Y$ are a pair of morphisms such that $\varphi \circ \alpha = \psi \circ \beta$. Since $I = \ker\beta$, we have $I \subset \ker\psi \circ \beta = \ker\varphi \circ \alpha$. Then $\alpha(I) \subset \ker\varphi$, since $Id(\alpha(I)) = J$ we have $J \subset \ker\varphi$. Suppose that $b \in B$ is arbitrary. Then there exists an element $c \in C$ such that $\beta(c) = b$. Hence

$$\psi(b) = \psi(\beta(c)) = \varphi(\alpha(c)) = \sigma(\delta(\alpha(c))) = \sigma(\gamma(\beta(c))) = \sigma(\gamma(b))),$$

whence $\psi = \sigma \circ \gamma$.

Conversely, let the diagram be pushout and $J_0 = Id(\alpha(I))$. Then $J_o \subset J$ and consider the pair of morphisms (φ, ψ) consisting of the quotient morphism $\varphi : A \to \frac{A}{J_0}$ and the induced morphism $\psi : B \to \frac{A}{J_0}$ defined by $\psi(c + I) = \alpha(c) + J_0$ for each $c \in C$. By the assumption, $\varphi = \sigma \circ \delta$ for some morphism $\sigma : X \to \frac{A}{J_0}$, which insures that

$$J = \ker\sigma \subset \ker\sigma \circ \delta = \ker\varphi = J_0 = Id(\alpha(I). \quad \square$$

Remark 2.3. If I is an arbitrary ideal in H^*-algebra A, then for $a \in I$ and $b \in I^{\perp}$, we have $ab \in I \cap I^{\perp} = \{0\}$.

Lemma 2.4. *Let A and B be H^*-algebras and $\alpha : A \to B$ be a surjective morphism. If I is a closed, self adjoint ideal in A such that $I \cap \ker\alpha = \{0\}$, then $\alpha(I^{\perp}) = \alpha(I)^{\perp}$.*

Proof. Let $b \in I^{\perp}$. Then by Remark 2.3, we have $\langle\alpha(b), \alpha(a)\rangle = \mathrm{tr}(\alpha(b)(\alpha(a))^*) = \mathrm{tr}(\alpha(ba^*)) = 0$ for each $a \in I$. So $\alpha(I^{\perp}) \subseteq \alpha(I)^{\perp}$. Conversely, let $c \in \alpha(I)^{\perp} \subseteq B$. By surjectivity of α, $c = \alpha(d)$ for some $d \in A$. Also, surjectivity of morphism α implies that $\alpha(I)$ is an ideal in B. Applying Remark 2.3, we get $\alpha(di) = c\alpha(i) = 0$ for each $i \in I$. Hence for each $i \in I$, $di \in \ker\alpha \cap I = \{0\}$ and so $\langle d, i\rangle = \mathrm{tr}(di^*) = 0$. It yields that $d \in I^{\perp}$ and $c = \alpha(d) \in \alpha(I^{\perp})$. $\square$

Theorem 2.5. *Let*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker\beta & \xrightarrow{\iota} & C & \xrightarrow{\beta} & B & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \alpha|_{\ker\beta}} & & \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & \ker\delta & \xrightarrow{\iota} & A & \xrightarrow{\delta} & X & \longrightarrow & 0
\end{array}$$

be a commutative diagram of extensions of H^*-algebras and morphisms. If the right square is pushout, α is surjective and $\ker\alpha \cap \ker\beta^{\perp} = \{0\}$, then γ is injective and $\gamma^*\overline{\delta(\alpha(\ker\beta)^{\perp})} = \beta(\ker\beta^{\perp})$. Conversely if α and γ are surjective and injective, respectively, then the right diagram is pushout.

Proof. By the surjectivity of α and Theorem 2.2, it is easy to verify that $Id(\alpha(\ker\beta)) = \ker\delta$ is a closed ideal in A. By applying Lemma 2.4, where

$I = \ker\beta^\perp$ we conclude that $\alpha(\ker\beta) = \alpha(\ker\beta^{\perp\perp}) = \alpha(\ker\beta^\perp)^\perp$. Then $\alpha(\ker\beta)$ is a closed ideal in A and

$$A = \alpha(\ker\beta) \oplus \alpha(\ker\beta)^\perp. \tag{2.1}$$

On the other hand α is onto and

$$A = \alpha(\ker\beta \oplus \ker\beta^\perp) = \alpha(\ker\beta) \oplus \alpha(\ker\beta^\perp) \tag{2.2}$$

As in the proof of Lemma 2.4, we get $\alpha(\ker\beta^\perp) \subseteq \alpha(\ker\beta)^\perp$. By (2.1) and (2.2), we shall show that γ is injective. Let b be a non zero element in B. Since $B = \beta(C) = \beta(\ker\beta \oplus \ker\beta^\perp) = \beta(\ker\beta^\perp)$, there exists a non zero element c_1 in $\ker\beta^\perp$ such that $b = \beta(c_1)$. In addition, by the commutativity of the diagram, we get $\gamma(b) = \gamma(\beta(c_1)) = \delta\alpha(c_1)$. By the assumption, c_1 is not in $\ker\alpha$ and $\alpha(c_1) \in \alpha(\ker\beta^\perp) = \alpha(\ker\beta)^\perp = \ker\delta^\perp$. Hence $\gamma(b) = \delta\alpha(c_1) \neq 0$. By the injectivity of γ we have $\overline{\gamma^*(X)}^\perp = \ker\gamma = \{0\}$, so $\overline{\gamma^*(X)} = B$. The commutativity of diagram, surjectivity of β and $\ker\delta^\perp = \alpha(\ker\beta^\perp)$ ensure that

$$\begin{aligned}
\overline{\gamma^*\delta\alpha(\ker\beta^\perp)} &= \overline{\gamma^*\delta(\ker\delta^\perp)} \\
&= \overline{\gamma^*\delta(\ker\delta \oplus \ker\delta^\perp)} \\
&= \overline{\gamma^*\delta(A)} \\
&= \overline{\gamma^*(X)} \\
&= B \\
&= \beta(C) \\
&= \beta(\ker\beta \oplus \ker\beta^\perp) \\
&= \beta(\ker\beta^\perp).
\end{aligned}$$

Conversely, suppose that γ and α are injective and surjective, respectively. The surjectivity of α implies that $\alpha(\ker\beta)$ is an ideal in A. By Theorem 2.2, it is enough to show that $\alpha(\ker\beta) = \ker\delta$. Let $c \in \ker\beta$. The commutativity of diagram implies that $\delta\alpha(c) = \gamma\beta(c) = 0$, so $\alpha(c) \in \ker\delta$. This implies that, $\alpha(\ker\beta) \subseteq \ker\delta$. For the reverse direction, assume that $a \in \ker\delta$. By the surjectivity of α, there exists $c \in C$ such that $\alpha(c) = a$. Hence $\gamma\beta(c) = \delta\alpha(c) = \delta(a) = 0$. The injectivity of γ, yields that $\beta(c) = 0$. So $c \in \ker\beta$ and $a = \alpha(c) \in \alpha(\ker\beta)$. $\square$

In the following example, we show that the surjectivity of α in Theorem 2.5 is a necessary condition.

Example 2.6. The Hilbert space $\mathbb{R}^n$ is a real H^*-algebra, where for each $(a_1, ..., a_n)$ and $(b_1, ..., b_n)$ in $\mathbb{R}^n$, $(a_1, ..., a_n)(b_1, ..., b_n) = (a_1b_1, ..., a_nb_n)$, $\langle (a_1, ..., a_n), (b_1, ..., b_n)\rangle = \sum_{i=1}^n a_ib_i$ and $(a_1, ..., a_n)^* = (a_1, ..., a_n)$. Let $2 < m <$

n and $\lambda_0 \in \{2, ..., m-1\}$ be fixed. Consider the following commutative diagram of the H^*-algebras and morphisms,

$$\begin{array}{ccc} \mathbb{R}^m & \xrightarrow{\beta} & \mathbb{R}^{m-1} \\ \Big\downarrow \alpha & & \Big\downarrow \gamma \\ \mathbb{R}^n & \xrightarrow{\delta} & \mathbb{R}^{m-1} \end{array}$$

where α, β, δ and γ are defined as follows.

(i) $\alpha((a_1, ..., a_m)) = (a_1, ..., a_m, 0, ..., 0)$.

(ii) $\beta((a_1, ..., a_m)) = (a_1, ..., a_{\lambda_0 - 1}, a_{\lambda_0 + 1}, ..., a_m)$.

(iii) $\delta((a_1, ..., a_n)) = (a_1, ..., a_{\lambda_0 - 1}, a_{\lambda_0 + 1}, ..., a_m)$.

(iv) γ is the identity operator.

Clearly, α is not surjective and $\ker \beta = \{(a_1, ..., a_m) : \ a_1 = ... = a_{\lambda_0 - 1} = a_{\lambda_0 + 1} = ... = a_m = 0\}$. Hence $\alpha(\ker \beta) = \{(a_1, ..., a_n) : \ a_i = 0, \text{for every} \ \ i \in \{1, ..., \lambda_0 - 1, \lambda_0 + 1, ..., n\}\}$, which it is an ideal in $\mathbb{R}^n$.

On the other hand, $\ker \delta = \{(a_1, ..., a_n) : \ a_i = 0$ for every $i \in \{1, ..., \lambda_0 - 1, \lambda_0 + 1, ..., m\}$. Then $\alpha(\ker \beta) \neq \ker \delta$ and so, by Theorem 2.2, the above diagram is not pushout, although, β and δ are surjective and γ is injective.

3. PUSHOUT CONSTRUCTIONS IN l.m.c. H^*-ALGEBRAS

In this section, we consider a commutative diagram of l.m.c. H^*-algebras and morphisms and obtain some conditions under which the corresponding diagrams of H^*-algebras and morphisms are pushout.

THEOREM 3.1. *Let* $(A,\{|.|_{\lambda_A}\}_{\lambda_A \in \Lambda_A})$, $(B,\{|.|_{\lambda_B}\}_{\lambda_B \in \Lambda_B})$, $(C,\{|.|_{\lambda_C}\}_{\lambda_C \in \Lambda_C})$ *and* $(X, \{|.|_{\lambda_X}\}_{\lambda_X \in \Lambda_X})$ *be l.m.c.* H^**-algebras and let*

$$\begin{array}{ccc} C & \xrightarrow{\beta} & B \\ \Big\downarrow \alpha & & \Big\downarrow \gamma \\ A & \xrightarrow{\delta} & X \end{array}$$

be a commutative diagram of l.m.c. H^**-algebras and morphisms. Fix* $\lambda_C \in \Lambda_C$*. If there exist* $\lambda_B \in \Lambda_B$*,* $\lambda_A \in \Lambda_A$ *and* $\lambda_X \in \Lambda_X$ *such that the following conditions hold*

(i) $\alpha(N_{\lambda_C}) \subseteq N_{\lambda_A}$,

(ii) $\beta(N_{\lambda_C}) \subseteq N_{\lambda_B}$,

(iii) $\delta(N_{\lambda_A}) \subseteq N_{\lambda_X}$,

(iv) $\gamma(N_{\lambda_B}) \subseteq N_{\lambda_X}$,

(v) $\gamma^{-1}(\{x\}) \cap N_{\lambda_B} \neq \phi$ *for each* $x \in N_{\lambda_X}$,

(vi) δ *and* β *are bijective and surjective morphisms, respectively,*

then the right square in the following diagram of pre-H^**-algebras is pushout, where* $I' = \frac{\beta^{-1}(N_{\lambda_B})}{N_{\lambda_C}}$, $J' = \frac{\delta^{-1}(N_{\lambda_X})}{N_{\lambda_A}}$ *and* $\alpha'(c + N_{\lambda_C}) = \alpha(c) + N_{\lambda_A}$ *for each* $c \in C$*. The maps* β'*,* δ' *and* γ' *are defined similarly.*

$$\begin{tikzcd}
0 \arrow[r] & I' \arrow[r, "i"] \arrow[d, "\alpha'|_{I'}"] & \frac{C}{N_{\lambda_C}} \arrow[r, "\beta'"] \arrow[d, "\alpha'"] & \frac{B}{N_{\lambda_B}} \arrow[r] \arrow[d, "\gamma'"] & 0_{\frac{B}{N_{\lambda_B}}} \\
0 \arrow[r] & J' \arrow[r, "i"] & \frac{A}{N_{\lambda_A}} \arrow[r, "\delta'"] & \frac{X}{N_{\lambda_X}} \arrow[r] & 0_{\frac{X}{N_{\lambda_X}}}
\end{tikzcd}$$

Proof. Consider $c + N_{\lambda_C} \in \ker(\beta')$ if and only if $c \in \beta^{-1}(N_{\lambda_B})$. This ensures that $I' = \ker(\beta')$. A similar argument shows that $J' = \ker(\delta')$. The diagram of extensions on pre-H^*-algebras is commutative since the initial diagram is commutative. By Theorem 2.2, the right square of the diagram is pushout if and only if $Id(\alpha'(I')) = J'$. We shall show that this equation holds if and only if $\delta(Id(\alpha(\beta^{-1}(N_{\lambda_B})))) = N_{\lambda_X}$.

If $Y \subseteq A$, then for each finite subset F of $\mathbb{N}$

$$Id(\delta(Y)) = \left\{ \sum_{i\in F} \lambda_i \delta(y_i)^{m_i} x_i \delta(z_i)^{n_i},\ \lambda_i \in \mathbb{C},\ m_i, n_i \in \mathbb{N},\ y_i, z_i \in Y,\ x_i \in X \right\}.$$

By the surjectivity of δ, for each $i \in F$, there exists $a_i \in A$ such that $x_i = \delta(a_i)$. Hence

$$Id(\delta(Y)) = \left\{ \delta(\sum_F \lambda_i y_i^{m_i} a_i z_i^{n_i}),\ \lambda_i \in \mathbb{C},\ m_i, n_i \in \mathbb{N},\ y_i, z_i \in Y,\ a_i \in A \right\}$$
$$= \delta(Id(Y)).$$

Since $\alpha(\beta^{-1}(N_{\lambda_B})) \subseteq A$, by the above discussion, we get

$$\delta(Id(\alpha(\beta^{-1}(N_{\lambda_B})))) = Id(\delta(\alpha(\beta^{-1}(N_{\lambda_B})))). \tag{3.1}$$

Conditions (iv), (vi) and commutativity of the diagram imply that

$$\delta(\alpha(\beta^{-1}(N_{\lambda_B}))) = \gamma(\beta\beta^{-1}(N_{\lambda_B}) = \gamma(N_{\lambda_B}) \subseteq N_{\lambda_X}.$$

Since N_{λ_X} is an ideal in X, we get $Id(\delta(\alpha(\beta^{-1}(N_{\lambda_B})))) \subseteq N_{\lambda_X}$.

Conversely, suppose that there exists $x \in N_{\lambda_X}$ such that $x \notin \delta(Id(\alpha(\beta^{-1} (N_{\lambda_B}))))$. Thus $\delta^{-1}(x) \notin Id(\alpha(\beta^{-1}(N_{\lambda_B})))$. Condition (v) implies that $x = \gamma(b)$ for some $b \in N_{\lambda_B}$. By the surjectivity of β, there is an element $c \in C$ such that $b = \beta(c)$. Hence $c \in \beta^{-1}(N_{\lambda_B})$. On the other hand, it follows from the commutativity of the diagram and bijectivity of δ that $\delta^{-1}(x) = \delta^{-1}(\gamma(b)) = \delta^{-1}(\gamma\beta(c)) = \delta^{-1}(\delta\alpha(c)) = \alpha(c)$. Therefore, $\delta^{-1}(x) = \alpha(c) \in \alpha\beta^{-1}(N_{\lambda_B})$, so $\delta^{-1}(x) \in Id(\alpha(\beta^{-1}(N_{\lambda_B})))$. This is a contradiction. Therefore $N_{\lambda_X} \subseteq Id(\delta(\alpha(\beta^{-1}(N_{\lambda_B}))))$. Hence $Id(\delta(\alpha(\beta^{-1}(N_{\lambda_B})))) = N_{\lambda_X}$. By (3.1), we reach $\delta(Id(\alpha(\beta^{-1}(N_{\lambda_B})))) = N_{\lambda_X}$. Since δ is one to one, we have $Id(\alpha(\beta^{-1}(N_{\lambda_B}))) = \delta^{-1}(N_{\lambda_X})$. So

$$\begin{aligned} J' &= \frac{\delta^{-1}(N_{\lambda_X})}{N_{\lambda_A}} = \frac{Id(\alpha(\beta^{-1}(N_{\lambda_B})))}{N_{\lambda_A}} = Id\left(\frac{\alpha(\beta^{-1}(N_{\lambda_B}))}{N_{\lambda_A}}\right) \\ &= Id\left(\alpha'\left(\frac{\beta^{-1}(N_{\lambda_B})}{N_{\lambda_C}}\right)\right) = Id(\alpha'(I')). \end{aligned}$$

Note that the condition (v) implies that $N_{\lambda_A} \subseteq \alpha(\beta^{-1}(N_{\lambda_B}))$. $\square$

COROLLARY 3.2. *If the assumptions of Theorem 3.1 hold, then the right square in the following diagram of H^*-algebras is pushout, where* $I' = \frac{\widehat{\beta^{-1}(N_{\lambda_B})}}{N_{\lambda_C}}$ *and* $J' = \frac{\widehat{\delta^{-1}(N_{\lambda_X})}}{N_{\lambda_A}}$.

$$\begin{tikzcd} 0 \arrow[r] & I' \arrow[r, "\iota"] \arrow[d, "\alpha'|_{I'}"] & \widehat{\frac{C}{N_{\lambda_C}}} \arrow[r, "\beta'"] \arrow[d, "\alpha'"] & \widehat{\frac{B}{N_{\lambda_B}}} \arrow[r] \arrow[d, "\gamma'"] & 0_{\frac{B}{N_{\lambda_B}}} \\ 0 \arrow[r] & J' \arrow[r, "\iota"] & \widehat{\frac{A}{N_{\lambda_A}}} \arrow[r, "\delta'"] & \widehat{\frac{X}{N_{\lambda_X}}} \arrow[r] & 0_{\frac{X}{N_{\lambda_X}}} \end{tikzcd}$$

Proof. We show that $\alpha', \beta', \delta', \gamma'$ are well-defined. Assume that $x \in \widehat{\frac{C}{N_{\lambda_C}}}$. Then $x = \lim\limits_{n\to\infty}(c_n + N_{\lambda_C})$ for some sequence $\{c_n\}_n$ in C. The morphism α is continuous, so it is bounded. For each $\lambda_C \in \Lambda_C, \lambda_A \in \Lambda_A$, there exists $M_{\lambda_C,\lambda_A} \geq 0$ such that $\|\alpha(x)\|_{\lambda_A} \leq M_{\lambda_C,\lambda_A}\|x\|_{\lambda_C}$ for each $x \in C$. Since $\{c_n + N_{\lambda_C}\}_n$ is Cauchy and α is bounded, it is easy to show that $\{\alpha(c_n) + N_{\lambda_A}\}_n$ is a Cauchy sequence in $\widehat{\frac{A}{N_{\lambda_A}}}$ and so it is convergent. Set $\alpha'(x) = \lim\limits_{n\to\infty}\alpha(c_n) + N_{\lambda_A}$. The proof can be complemented by using the same argument as in the proof of Theorem 3.1, so we omit it. $\square$

In the next example, we show that the condition (v) of Theorem 3.1 is necessary.

Example 3.3. Let $\Lambda = \{1, \ldots, n\}$, where $n \geq 2$ and $\{(A_\lambda, \|.\|_\lambda)\}_{\lambda\in\Lambda}$ be a family of H^*-algebras, in which $A_{n-1} = A_n$. Then $\boxplus_{\lambda\in\Lambda}A_\lambda = \{(x_\lambda);\ \ x_\lambda \in A_\lambda\}$ is an l.m.c. H^*-algebra, whose topology is determined by a family $\{|.|_\lambda\}_{\lambda\in\Lambda}$ of submultiplicative seminorms such that $|.|_{\lambda_0}$ is defined by $|(x_\lambda)_{\lambda\in\Lambda}|_{\lambda_0} = \|x_{\lambda_0}\|_{\lambda_0}$

for each $\lambda_0 \in \Lambda$. Further, $\oplus_{\lambda\in\Lambda}A_\lambda$ is an l.m.c. H^*-algebra, whose topology is determined by the family $\{|.|_{\lambda,\mu}\}_{\lambda,\mu\in\Lambda}$ of submultiplicative seminorms such that $|.|_{\lambda_0,\mu_0}$ is defined by $|(x_\lambda)_{\lambda\in\Lambda}|^2_{\lambda_0,\mu_0} = \|x_{\lambda_0}\|^2_{\lambda_0}+\|x_{\mu_0}\|^2_{\mu_0}$ for each $\lambda_0, \mu_0 \in \Lambda$. Note that $\boxplus_{\lambda\in\Lambda}A_\lambda$ and $\oplus_{\lambda\in\Lambda}A_\lambda$ are both H^*-algebras with usual addition and multiplication. Consider the following commutative diagram of l.m.c. H^*-algebras and morphisms.

$$\begin{tikzcd}
\oplus_{\lambda\in\Lambda}A_\lambda \arrow[r, "\beta"] \arrow[d, "\alpha"] & \boxplus_{\lambda\in\Lambda}A_\lambda \arrow[d, "\gamma"] \\
{}_{\lambda\in\Lambda}A_\lambda \arrow[r, "\delta"] & \boxplus_{\lambda\in\Lambda}A_\lambda
\end{tikzcd}$$

The morphisms α, β, δ and γ are defined as follow.

(i) $\alpha((x_\lambda)_{\lambda\in\Lambda}) = (y_\lambda)_{\lambda\in\Lambda}$, where $y_\lambda = x_\lambda$ for $\lambda \neq n$ and $y_n = 0$.

(ii) $\beta((x_\lambda)_{\lambda\in\Lambda}) = (z_\lambda)_{\lambda\in\Lambda}$, where $z_\lambda = x_\lambda$ for $\lambda \in \{1, 2, ..., n-2\}$ and $z_{n-1} = x_n$, $z_n = x_{n-1}$.

(iii) $\gamma((z_\lambda)_{\lambda\in\Lambda}) = (w_\lambda)_{\lambda\in\Lambda}$ such that $w_\lambda = z_\lambda$ for $\lambda \neq n-1$ and $w_{n-1} = 0$.

(iv) δ is defined in the same way as β.

Now consider the diagram of extensions,

$$\begin{tikzcd}
0 \arrow[r] & I' \arrow[r, "i"] \arrow[d, "\alpha'"] & \frac{\oplus_{\lambda\in\Lambda}A_\lambda}{N_{n-1,n}} \arrow[r, "\beta'"] \arrow[d, "\alpha'"] & \frac{\boxplus_{\lambda\in\Lambda}A_\lambda}{N_n} \arrow[r] \arrow[d, "\gamma'"] & 0 \\
0 \arrow[r] & J' \arrow[r, "i"] & \frac{\oplus_{\lambda\in\Lambda}A_\lambda}{N_{n-1,n}} \arrow[r, "\delta'"] & \frac{\boxplus_{\lambda\in\Lambda}A_\lambda}{N_{n-1}} \arrow[r] & 0
\end{tikzcd}$$

where

$$N_{n-1} = \{(a_\lambda)_{\lambda\in\Lambda} : |(a_\lambda)_{\lambda\in\Lambda}|_{n-1} = 0\} = \{(a_\lambda)_{\lambda\in\Lambda} : \ a_{n-1} = 0\},$$

$$N_n = \{(a_\lambda)_{\lambda\in\Lambda} : \ \ |(a_\lambda)_{\lambda\in\Lambda}|_n = 0\} = \{(a_\lambda)_{\lambda\in\Lambda} : \ a_n = 0\},$$

$$N_{n-1,n} = \{(a_\lambda)_{\lambda\in\Lambda} : |(a_\lambda)_{\lambda\in\Lambda}|_{n-1,n} = 0\} = \{(a_\lambda)_{\lambda\in\Lambda} : \ a_{n-1} = a_n = 0\},$$

$I' = \ker\beta' = \{(a_1, ..., a_{n-2}, 0, a_n) + N_{n-1,n} : a_i \in A, \quad i \in \{1, 2, ..., n-2, n\}\}$, and $J' = \ker\delta' = \{(a_1, a_2, ..., a_{n-1}, 0) + N_{n-1,n} : a_i \in A, \quad i \in \{1, 2, ..., n-1\}\}$.

Clearly, conditions (i), (ii), (iii), (iv) and (vi) of the preceding theorem hold. If $a = (a_1, a_2, ..., 0, a_n) \in N_{n-1}$ in which $a_n \neq 0$, then $\gamma^{-1}(a) = (a_1, a_2, ..., a_{n-1}, a_n)$ for arbitrary element $a_{n-1} \in A$ and $\gamma^{-1}(a) \cap N_n = \emptyset$.

We are going to show that the right square in the diagram above, is not pushout. It is enough to show that $Id(\alpha'(I')) \neq J'$. This holds, since $\alpha'(I') = \{(a_1, a_2, ..., a_{n-2}, 0, 0) + N_{n-1,n} : a_i \in A$ for $i \in \{1, 2, ..., n-2\}\}$, which is an ideal in $\frac{\oplus_{\lambda\in\Lambda}A_\lambda}{N_{n-1,n}}$ and not equal to $\ker\delta'$.

In the special case of above example, where $n = 2$ we have the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I' & \xrightarrow{\ i\ } & A \oplus A & \xrightarrow{\ \beta'\ } & \frac{A\boxplus A}{A\boxplus 0} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \alpha'} & & \Big\downarrow{\scriptstyle \alpha'} & & \Big\downarrow{\scriptstyle \gamma'} & & \\
0 & \longrightarrow & J' & \xrightarrow{\ i\ } & A \oplus A & \xrightarrow{\ \delta'\ } & \frac{A\boxplus A}{0\boxplus A} & \longrightarrow & 0
\end{array}$$

where $A = A_1 = A_2$, so that $\frac{A\oplus A}{N_{1,2}} = A \oplus A$, $\frac{A\boxplus A}{N_1} = \frac{A\boxplus A}{0\boxplus A}$, $\frac{A\boxplus A}{N_2} = \frac{A\boxplus A}{A\boxplus 0}$,

$$\begin{array}{rcl}
\alpha'((x_1, x_2)) & = & (x_1, 0), \\
\beta'((x_1, x_2)) & = & (x_2, x_1) + A \boxplus 0, \\
\gamma'((x_1, x_2) + A_1 0) & = & (0, x_2) + 0 \boxplus A \\
\delta'(x_1, x_2) & = & (x_2, x_1) + 0 \boxplus A
\end{array}$$

Also $I' = \ker \beta' = 0 \oplus A$ and $J' = \ker \delta' = A \oplus 0$. It is easy to see that all conditions of the preceding theorem hold except (v). In addition, $\alpha'(I') = \alpha'(0 \oplus A) = 0 \oplus 0$ and $J' = A \oplus 0$ are valid. The right square in the diagram above, however is not pushout, since $Id(\alpha'(I')) \neq J'$.

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