RADIO LABELING OF SOME LADDER-RELATED GRAPHS*

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Let $d(u,v)$ denote the distance between two distinct vertices of a connected graph G, and $diam(G)$ be the diameter of G. A radio labeling c of G is an assignment of positive integers to the vertices of G satisfying $d(u,v)+|c(u)-c(v)| \geq diam(G)+1$ for every two distinct vertices u, v. The maximum integer in the range of the labeling is its span. The radio number of G, $rn(G)$, is the minimum possible span of any radio labeling for G. In this paper, the radio numbers of Mongolian tent graph, diamond graph, fan and double fan are determined.

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Key words: radio labeling, multi-level distance labeling, radio number, Mongolian tent, diamond graph, fan, double fan, radio graceful.

1. INTRODUCTION

Radio labeling (cf. [2,3]) is motivated by the channel assignment problem introduced by Hale [7]. Suppose we are given a set of stations or transmitters; the task is to assign to each station (or transmitter) a channel (non-negative integer) such that the interference is avoided. The interference is closely related to the geographical locations of the stations – the closer are the stations the stronger the interference that might occur. To avoid interference, the separation of the channels assigned to nearby stations must be large enough. To model this problem, we construct a graph so that each station is represented by a vertex, and two vertices are adjacent when their corresponding stations are close. The ultimate goal is to find a *valid* labeling such that the span (range) of the channels used is minimized.

Let G be a connected graph. For any two vertices u and v, the distance between u and v, denoted by $d_G(u,v)$ (or $d(u,v)$ when G is understood in the context), is the length of a shortest $(u,v)-$path in G. A *distance-two labeling* (or λ-*labeling*) with span k is a function $c : V(G) \to \{1, 2, \ldots, k\}$ having the

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maximum value k such that the following relations are satisfied for any two distinct vertices u and v:

$$|c(u) - c(v)| \geq \begin{cases} 2, & \text{if } d(u,v) = 1 \\ 1, & \text{if } d(u,v) = 2. \end{cases}$$

The λ-*number* of G is the smallest k such that G admits a distance-two labeling with span k. Since introduced by Griggs and Yeh [6] in 1992, distance-two labeling has been studied extensively (see [1,14]).

Radio labeling extends the number of interference levels considered in distance-two labeling from two to the largest possible - the diameter of G. The diameter of G, denoted by $diam(G)$, is the maximum distance among all pairs of vertices in G.

A *radio labeling* or *multi-level distance labeling* [11,12] with span k for a graph G is a function $c : V(G) \to \{1, 2, \ldots, k\}$ having the maximum value k such that the following condition holds for any two distinct vertices u and v:

$$d(u,v) + |c(u) - c(v)| \geq 1 + diam(G). \tag{1}$$

This condition is referred to as *radio condition.*

We denote by $S(G,c)$ the set of consecutive integers $\{m, m+1, ..., M\}$, where $m = \min_{u \in V(G)} c(u)$ and $M = \max_{u \in V(G)} c(u)$ is the span of c, denoted $span(c)$.

The *radio number* of G, denoted by $rn(G)$, is the minimum span of a radio labeling for G. A radio labeling c of G with $span(c) = rn(G)$ will be called optimal radio labeling for G.

Note that if $diam(G) = 2$, then radio labeling and λ−labeling become identical.

A graph G with n vertices is called *radio graceful* if $rn(G) = n$.

Besides its motivation by the channel assignment, radio labeling itself is an interesting graph labeling problem and has been studied by several authors. It is computationally complex to calculate the radio number on general graph. The problem is known to be NP-hard for graphs with diameter 2, but the complexity in general is not known [9]. Therefore, research in this area has focused on special classes of graphs, the problem proving to be difficult even for basic families of graphs [12]. The radio numbers for paths and cycles were investigated in [2,3,15], and were completely solved by Liu and Zhu [12]. Sooryanarayana and Raghunath [13] determined the radio number of the cube of C_n for all $n \leq 20$ and for $n \equiv 0$ or 2 or 4 $\pmod 6$. They also determine the values of n for which this graph is radio graceful.

For many types of labelings, studying the problem on cartesian product of graphs and graphs is related to that present interest. For example, in [5], the λ-labeling for product of complete graph is considered. Also, the radio number for grid graphs, which is the cartesian product $P_m \times P_n$, was completely determined in [8], after only upper and lower bounds were previously found. Also, some families of graphs obtained from grids by adding extra vertices (such as Mongolian tent and diamond graph) proved to be interesting for different types of labelings. For example, Lee determined in [10] that some of these graphs are graceful.

In this article, we consider graphs related to the ladder graph, which is the grid $P_2 \times P_n$. We completely determine the radio number for Mongolian tent graph, diamond graph, fan and double fan.

We will use only the positive integers as labels.

The ladder graph, denoted by L_n, is the graph with vertex set

$$V(L_n) = \{u_i, v_i : 1 \leq i \leq n\}$$

and edge set

$$E(L_n) = \{u_i u_{i+1}, v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i v_i : 1 \leq i \leq n\}.$$

L_n is isomorphic to the grid $P_2 \times P_n$.

Mongolian tent, denoted by Mt_n, is the graph obtained from the ladder graph L_n by adding a new vertex z and joining each vertex v_i, $1 \leq i \leq n$ with z.

Diamond graph, denoted by d_n, is the graph obtained from the Mongolian tent graph Mt_n by adding a new vertex z_1 and joining each vertex u_i, $1 \leq i \leq n$ with z_1.

Fan graph, denoted by f_n, is the graph obtained from the path with n vertices P_n, where $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(P_n) = \{v_i v_{i+1} : 1 \leq i \leq n-1\}$ by adding a new vertex z and joining each v_i, $1 \leq i \leq n$ with z.

Double fan graph, denoted by df_n, is the graph obtained from the fan f_n by adding a new vertex z_1 joining each v_i, $1 \leq i \leq n$ with z_1.

The following remark will be useful in our proofs.

Remark 1.1. Let c be an optimum radio labeling of graph G. We can associate to c an ordering of the vertices of G, increasing by their labels. Denote by $a_1, \ldots, a_n$ the vertices of G in this order:

$$c(a_1) < c(a_2) < \ldots < c(a_n).$$

We have

- $c(a_1) = 1$
- $rn(G) = span(c) = 1 + \sum_{i=1}^{n-1} (c(a_{i+1}) - c(a_i))$
- If $c(a_{i+1}) - c(a_i) = 1$, then we must have $d(a_i, a_{i+1}) = diam(G)$.

In order to find a lower bound for $rn(G)$, for graphs with small diameter is sometimes useful to determine how many pairs (a_i, a_{i+1}) with $c(a_{i+1}) - c(a_i) = 1$ we can have. If there can be at most x such pairs, then we have:

$$rn(G) \geq 1 + x + 2(n - 1 - x).$$

Next, we introduce the notion of forbidden values associated to a vertex v for a radio labeling c. Let c be a radio labeling of graph G. Since vertex v has label $c(v)$ then, by radio condition, some values from $S(G, c)$ that are close to $c(v)$ cannot be labels for other vertices. We will call these values forbidden values associated to vertex v.

2. RADIO NUMBER FOR MONGOLIAN TENT GRAPH

THEOREM 2.1. *a) Mongolian tent Mt_2 is radio graceful.*
b) The radio number of Mongolian tent Mt_3 is 11.
c) The radio number of Mongolian tent Mt_4 is 12.

Proof. In order to prove that the values stated in the Theorem are lower bounds for the radio number, we will use the idea from Remark 1.1. Consider c an optimal radio labeling and denote by $a_1, a_2, \ldots, a_m$ the vertices of the graph in increasing order of their labels. We investigate the maximum number of pairs (a_i, a_{i+1}) with $c(a_{i+1}) - c(a_i) = 1$. By radio condition, these pairs must have the property that $d(a_i, a_{i+1}) = diam(G)$.

For proving that the claimed values are upper bounds for the radio numbers of considered graphs, we will provide radio labelings having spans equal to these values.

a) The Mongolian tent Mt_2, is a planar graph with 5 vertices, 6 edges and diameter 2. We have $rn(Mt_2) \geq |V(Mt_2)| = 5$.

The radio labeling c of Mt_2 represented in Fig. 1 (a), shows that $rn(Mt_2) \leq 5$. It implies that $rn(Mt_2) = 5$. Therefore Mt_2 is radio graceful.

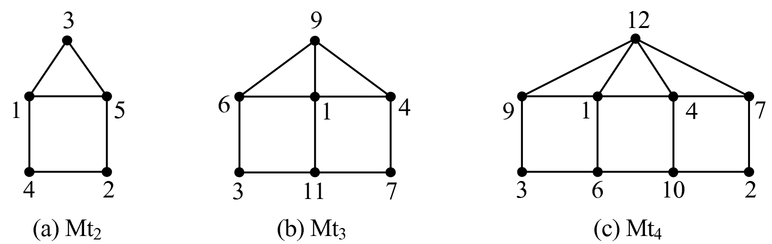

Fig. 1 – Radio labelings for Mongolian tent graphs.

b) Mt_3 has $m = 7$ vertices and $diam(Mt_3) = 3$. There are only two pairs of vertices at distance 3 in Mt_3, hence we have

$$rn(Mt_3) \geq 1 + 2 \cdot 1 + (m - 1 - 2) \cdot 2 = 1 + 2 + 8 = 11.$$

The radio labeling of Mt_3 illustrated in Fig. 1 (b) shows that $rn(Mt_3) \leq 11$. We conclude that $rn(Mt_3) = 11$.

c) Mongolian tent Mt_4 has $m = 9$ vertices and $diam(Mt_4) = 3$. There are 7 pairs of vertices at distance 3 in Mt_4. In order to easily observe these pairs, consider the distance-3 graph associated to Mt_4, that is the graph having the same vertices as Mt_4 and the edge set consisting of the pairs of vertices that are at distance 3 in Mt_4, shown in Fig. 2.

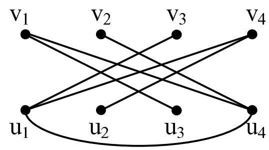

Fig. 2 – Distance-3 graph of Mt_4.

In order to obtain a more precise estimation of the maximum number of pairs (a_i, a_{i+1}) with $c(a_{i+1}) - c(a_i) = 1$, we study how many triplets (a_i, a_{i+1}, a_{i+2}) may have consecutive labels: $c(a_i)$, $c(a_i)+1$, $c(a_i)+2$. By radio condition we must have $d(a_i, a_{i+2}) \geq 2$. Such a triplet corresponds to a path of length 2 in the distance-3 graph associated to Mt_4, whose extremities are at distance at least 2 in Mt_4. It is easy to see that there are only two such path: $[v_2, u_4, u_1]$ and $[u_4, u_1, v_3]$, which have 2 vertices in common.

It follows that we can have at most 5 pairs (a_i, a_{i+1}) with consecutive labels (otherwise more triplets with consecutive labels will occur), hence

$$rn(Mt_4) \geq 1 + 5 \cdot 1 + (m - 1 - 5) \cdot 2 = 1 + 5 + 6 = 12.$$

The radio labeling of Mt_4 represented in Fig. 1 (c) shows that $rn(Mt_4) \leq 12$, hence $rn(Mt_4) = 12$. $\square$

THEOREM 2.2. *For $n \geq 5$, $rn(Mt_n) \geq 4n + 2$.*

Proof. Assume $n \geq 5$. Then $diam(Mt_n) = 4$, so any radio labeling c of Mt_n must satisfy the radio condition

$$d(u, v) + |c(u) - c(v)| \geq 5$$

for all distinct vertices $u, v \in V(Mt_n)$.

Let c be an optimal radio labeling for Mt_n. We count the number of values needed for labels and add the minimum number of forbidden values for c.

Thus, since $d(z,r) \leq 2$ for all vertices $r \neq z$, the values $\{c(z)-2, c(z)-1, c(z)+1, c(z)+2\} \cap S(Mt_n, c)$ are forbidden. Similarly, as $d(v_i, r) \leq 3$ for all v_i and for any $r \neq v_i$, the values $\{c(v_i)-1, c(v_i)+1\} \cap S(Mt_n, c)$ are forbidden, for every $i \in \{1, 2, \ldots, n\}$. However, as $d(u_i, r) = 4$ for some vertex r, it is possible to use consecutive labels on u_i and r. (i.e. there are no forbidden values associated with the vertices $\{u_1, u_2, \ldots, u_n\}$.)

Remark that the number of forbidden values associated to z is $|\{c(z)-2, c(z)-1, c(z)+1, c(z)+2\} \cap S(Mt_n, c)| \geq 2$, with equality only if $c(z) \in \{1, span(c)\}$. Also, $|\{c(v_i)-1, c(v_i)+1\} \cap S(Mt_n, c)| \geq 1$, with equality only if $c(v_i) \in \{1, span(c)\}$. Moreover, these forbidden values are distinct, since by radio condition we must have $|c(z) - c(v_i)| \geq 3$ and $|c(v_i) - c(v_j)| \geq 2$ for every $i \neq j$. The minimum number of forbidden values for c is then obtained in two situations (when there exists i such that $\{c(v_i), c(z)\} = \{1, span(c)\}$ or there exists $i \neq j$ such that $\{c(v_i), c(v_j)\} = \{1, span(c)\}$) and this number is $3 + 2n - 2 = 2n + 1$. Adding in the $2n+1$ values needed to label the $2n+1$ vertices provides a total of $4n+2$ labels, hence $rn(Mt_n) \geq 4n+2$, for $n \geq 5$. □

THEOREM 2.3. *For* $n \geq 5$, $rn(Mt_n) \leq 4n + 2$.

Proof. We shall propose a radio labeling of Mt_n with span $4n+2$, which implies $rn(Mt_n) \leq 4n+2$. Let $n \geq 5$. The radio labeling $c : V(Mt_n) \to \mathbb{Z}^+$ is defined as follows:

$c(z) = 4n + 2$

$$c(u_i) = \begin{cases} 4i, & \text{if } 1 \leq i \leq n-1 \\ 3, & \text{if } i = n \end{cases}$$

Case A - n is odd:

$$c(v_i) = \begin{cases} 2n + 4i, & \text{if } 1 \leq i \leq \frac{n+1}{2} - 1 \\ 1, & \text{if } i = \frac{n+1}{2} \\ 2(2i - n), & \text{if } \frac{n+1}{2} + 1 \leq i \leq n \end{cases}$$

Case B - n is even:

$$c(v_i) = \begin{cases} 2(n + 2i + 1), & \text{if } 1 \leq i \leq \frac{n}{2} - 1 \\ 1, & \text{if } i = \frac{n}{2} \\ 2(2i + 1 - n), & \text{if } \frac{n}{2} + 1 \leq i \leq n \end{cases}$$

In both cases the span of c is equal to $4n + 2$ and it is reached for $c(z)$.

Claim: The labeling c is a valid radio labeling.

We must show that the radio condition

$$d(u,v) + |c(u) - c(v)| \geq diam(Mt_n) + 1 = 5$$

holds for all pairs of vertices (u, v) (where $u \neq v$).

1: Consider the pair (z, r) (for any vertex $r \neq z$). As $d(z, r) \geq 1$ and $c(r) \leq 4n - 2$, we have $d(z,r) + |c(z) - c(r)| \geq 1 + |4n + 2 - (4n - 2)| \geq 5$ for any $r \neq z$. The radio condition is satisfied.

2: Consider the pairs (v_i, v_j) (with $i \neq j$). Note that $d(v_i, v_j) \geq 1$ for $i \neq j$. $|c(v_i) - c(v_j)| \geq 4$ for all $v_i \neq v_j$. Hence, again, the radio condition is satisfied.

3: Consider the pairs (u_i, u_j) (with $i \neq j$). We have $d(u_1, u_n) = 4$, and the labels difference for this pair is $|c(u_1) - c(u_n)| = 1$; so the radio condition for (u_1, u_n) is satisfied. Note that $d(u_i, u_j) \geq 1$ for $i \neq j$ and the label difference for each pair is $|c(u_i) - c(u_j)| \geq 4$, except the pair (u_1, u_n). The radio condition is then satisfied for all distinct u_i.

4: Finally, consider the pairs (u, v), where $u \in \{u_1, u_2, \ldots, u_n\}$ and $v \in \{v_1, v_2, \ldots, v_n\}$. We have $c(u) \in \{3, 4, 8, 12, \ldots, 4(n-1)\}$. If $d(u, v) = 1$, then by the way c was defined, $|c(u) - c(v)| \geq 2n - 3 \geq 7$ for $n \geq 5$. If $d(u, v) = 2$, then $|c(u) - c(v)| \geq 2n - 7 \geq 3$ for $n \geq 5$. When $d(u, v) = 3$, $|c(u) - c(v)| \geq 2$. It follows that the radio condition is satisfied for these pairs.

These four cases establish the claim that c is a radio labeling of Mt_n.

Thus $rn(Mt_n) \leq span(c) \leq 4n + 2$. □

THEOREM 2.4. *The radio number of Mongolian tent Mt_n is $4n + 2$ when $n \geq 5$.*

Proof. Theorem 2.2 shows $rn(Mt_n) \geq 4n + 2$ for $n \geq 5$, and Theorem 2.3 shows $rn(Mt_n) \leq 4n + 2$ for $n \geq 5$. Therefore $rn(Mt_n) = 4n + 2$. □

3. RADIO NUMBER FOR DIAMOND GRAPH

THEOREM 3.1. *For diamond graphs the following relations hold:*

a) $rn(d_2) = 10$

b) $rn(d_3) = 12$

c) $rn(d_4) = 14$

d) $rn(d_5) = 15$.

Proof. In Fig. 3 are shown radio labelings having spans equal to the values stated in the Theorem, hence these values are upper bounds for the radio numbers of considered graphs.

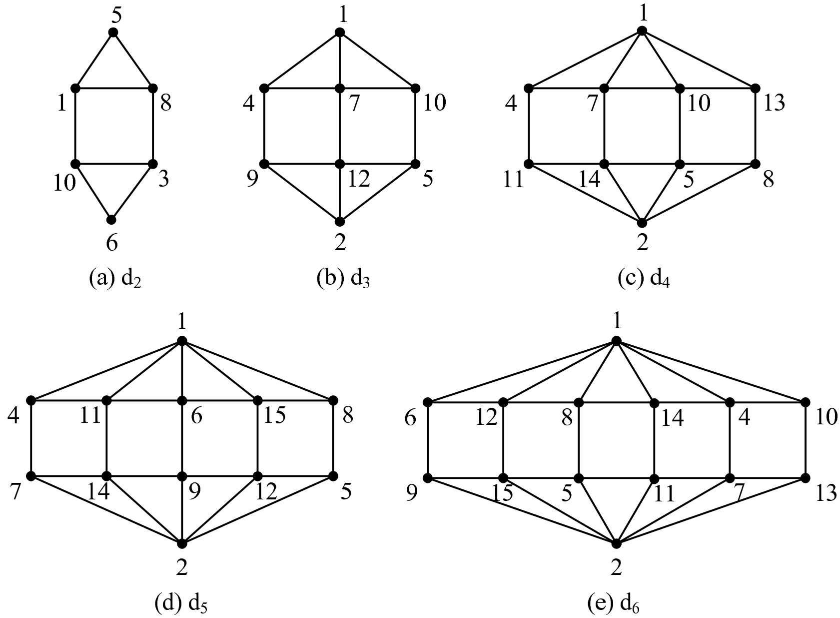

Fig. 3 – Radio labelings for diamond graphs.

In order to prove that they are also lower bounds, we will use the same arguments as in Theorem 2.1, based on Remark 1.1.

Consider c an optimal radio labeling and denote by $a_1, a_2, \ldots, a_m$ the vertices of the graph in increasing order of their labels.

a) We have $m = |V(d_2)| = 6$ and $diam(d_2) = 3$. There is only one pair of vertices at distance 3 in d_2 (that is (z, z_1)), hence we have

$$rn(d_2) \geq 1 + 1 \cdot 1 + (m - 1 - 1) \cdot 2 = 1 + 1 + 8 = 10.$$

b) We have $m = |V(d_3)| = 8$ and $diam(d_3) = 3$. There are three pairs of vertices at distance 3 in d_2: (z, z_1), (v_1, u_3) and (v_3, u_1), hence we have

$$rn(d_3) \geq 1 + 3 \cdot 1 + (m - 1 - 3) \cdot 2 = 1 + 3 + 8 = 12.$$

c) d_4 has $m = 10$ vertices and $diam(d_4) = 3$. Consider the distance-3 graph associated to d_4, shown in Fig. 4 (a). As in proof of Theorem 2.1, we observe that there is no path of length 2 in the distance-3 graph associated to d_4 whose extremities are at distance at least 2 in d_4, hence there are no triplets (a_i, a_{i+1}, a_{i+2}) having consecutive labels.

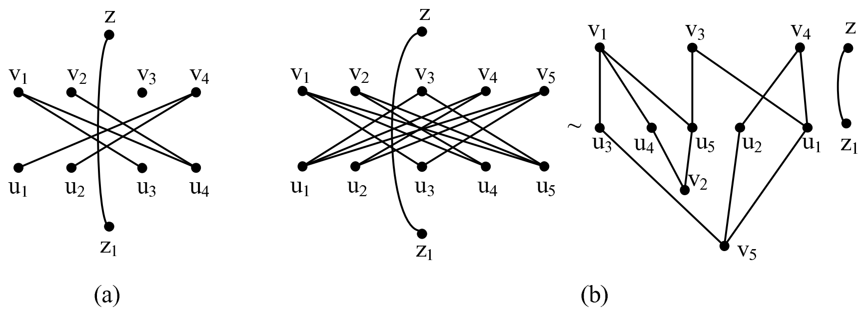

Fig. 4 – Distance-3 graph for d_4 and d_5.

It follows that we can have at most $m/2 = 5$ pairs of vertices (a_i, a_{i+1}) with consecutive labels, hence

$$rn(d_4) \geq 1 + 5 \cdot 1 + (m - 1 - 5) \cdot 2 = 1 + 5 + 8 = 14.$$

d) We have $|V(d_5)| = 12$ and $diam(d_5) = 3$. We consider again paths of length 2 in the distance-3 graph associated to d_5, shown in Fig. 4 (b). There are 3 paths of length 2 in the distance 3 graph associated to d_4 joining vertices at distance at least 2 in d_4: $[u_5, v_1, u_3]$, $[u_1, v_5, u_3]$, $[u_5, v_3, u_1]$. These paths contain 6 of the vertices of the graph, so there are no triplets of vertices with consecutive labels containing some of the other 6 vertices. It follows that there are at most $(6 - 1) + 6/2 = 8$ pairs of vertices with consecutive labels, hence

$$rn(d_5) \geq 1 + 8 \cdot 1 + (m - 1 - 8) \cdot 2 = 1 + 8 + 6 = 15. \quad \square$$

THEOREM 3.2. *For $n \geq 6$, the radio number of diamond graph d_n is $2n + 3$.*

Proof. Recall the vertex set and edge set of diamond graph as follows:

$$V(d_n) = \{v_i, u_i : 1 \leq i \leq n\} \cup \{z, z_1\}$$

$$E(d_n) = \{u_i u_{i+1}, v_i v_{i+1} : 1 \leq i \leq n - 1\} \cup \{u_i v_i, z v_i, z_1 u_i : 1 \leq i \leq n\}.$$

For $n \geq 6$, $diam(d_n) = 3$. The diamond graph contains $2n + 2$ vertices and $5n - 2$ edges.

First we will prove that $rn(d_n) \geq 2n+3$. For that, let c be a radio labeling for d_n. We will prove that c has at least one forbidden value, associated to one of the vertices z and z_1. By symmetry we can assume $c(z) < c(z_1)$. Denote $a = c(z)$.

As z_1 is the only vertex at distance 3 of z, $a - 1$ and $a + 1$ can be used as label only for z_1.

Assume $c(z_1) = a+1 = b$. As $d(z_1, r) \leq 2$ for all $r \notin \{z, z_1\}$, if $b+1 = a+2$ is assigned to any other vertices, then the condition (1) is not satisfied. It follows that if $c(z_1) = a + 1$ then either $c(z) - 1$ is a forbidden value associated to z (if $c(z) > 1$), or $c(z) + 2$ (if $c(z) = 1$). If $c(z_1)$ is not labeled with $a + 1$ then, since $a = c(z) < c(z_1) \leq span(c)$, value $a + 1$ is forbidden.

Therefore $rn(d_n)$ must be greater or equal to $|V(d_n)| + 1 = 2n + 3$. To prove $rn(d_n) \leq 2n + 3$, we define a labeling $c : V(d_n) \to \{1, 2, \ldots, 2n + 3\}$ as follows such that radio condition is satisfied.

For $n = 6$ such a labeling is shown in Fig. 3 (f).

Let $n \geq 7$.

Case A - n is even

$c(z) = 1, c(z_1) = 2$

$$c(v_i) = \begin{cases} n + 6 - i, & \text{if } i \equiv 0 \pmod 2 \\ 4, & \text{if } i = 1 \\ 2n + 5 - i & \text{if } i \geq 3 \text{ and } i \equiv 1 \pmod 2 \end{cases}$$

$$c(u_i) = \begin{cases} n + 9 - i, & \text{if } i \equiv 0 \pmod 2 \\ 8 - i, & \text{if } i = 1, 3 \\ 2n + 8 - i & \text{if } i \geq 5 \text{ and } i \equiv 1 \pmod 2 \end{cases}$$

Case B - n is odd

We divide this case into two subcases.

B.1: $n \equiv 3 \pmod 4$**.** Then we define

$c(z) = 2n + 3, c(z_1) = 2n + 2$

$$c(u_i) = \begin{cases} \frac{3n+i+1}{2}, & \text{if } i \equiv 0 \pmod 4 \\ \frac{i+1}{2}, & \text{if } i \equiv 1 \pmod 4 \\ \frac{n+1+i}{2}, & \text{if } i \equiv 2 \pmod 4 \\ \frac{2n+1+i}{2} & \text{if } i \equiv 3 \pmod 4 \end{cases}$$

$$c(v_i) = \begin{cases} \frac{n+1+i}{2}, & \text{if } i \equiv 0 \pmod 4 \\ \frac{2n+1+i}{2}, & \text{if } i \equiv 1 \pmod 4 \\ \frac{3n+i+1}{2}, & \text{if } i \equiv 2 \pmod 4 \\ \frac{i+1}{2} & \text{if } i \equiv 3 \pmod 4 \end{cases}$$

B.2: $n \equiv 1 \pmod 4$.

$c(z) = 2n + 2, c(z_1) = 2n + 3$

$$c(u_i) = \begin{cases} n+1+\frac{i}{2}, & \text{if } i \equiv 0 \pmod 4 \\ \frac{n+2+i}{2}, & \text{if } i \equiv 1 \pmod 4 \\ 2n, & \text{if } i = 2 \\ \frac{3n+i-1}{2}, & \text{if } i > 2 \text{ and } i \equiv 2 \pmod 4 \\ \frac{i+1}{2} & \text{if } i \equiv 3 \pmod 4 \end{cases}$$

$$c(v_i) = \begin{cases} \frac{3n+i-1}{2}, & \text{if } i \equiv 0 \pmod 4 \\ \frac{i+1}{2}, & \text{if } i \equiv 1 \pmod 4 \\ \frac{2n+2+i}{2}, & \text{if } i \equiv 2 \pmod 4 \\ \frac{n+2+i}{2} & \text{if } i \equiv 3 \pmod 4 \end{cases}$$

Claim: The labeling c is a valid radio labeling. We must show that the radio condition

$$d(u,v) + |c(u) - c(v)| \geq 1 + diam(d_n) = 4 \quad (2)$$

holds for all pairs of vertices (u, v) (where $u \neq v$).

Case A: Assume n is even. We consider all types of pairs of vertices.

1: Consider the pair (z, r) for any vertex $r \notin \{z, z_1\}$. As $1 \leq d(z, r) \leq 2$, $r \notin \{z, z_1\}$, $c(z) = 1, c(r) \geq 4$, $|c(z) - c(r)| \geq 3$, it follows that $d(z, r) + |c(z) - c(r)| \geq 1 + 3 = 4$.

2: For pair (z, z_1), as $d(z, z_1) = 3$ and $|c(z) - c(z_1)| = 1$, the radio condition is satisfied.

3: Consider the pairs (z_1, r) for any vertex $r \notin \{z, v_1\}$. As $1 \leq d(z_1, r) \leq 2$, $r \notin \{z, v_1\}$, $c(z_1) = 2, c(r) \geq 5$, $|c(z_1) - c(r)| \geq 3$ and $d(z_1, v_1) = 2$, $|c(z_1) - c(v_1)| = 2$, the radio condition (2) is satisfied.

4: Consider the pairs (v_i, v_j) (with $i \neq j$.) If $d(v_i, v_j) = 1$ we have $d(v_i, v_j) + |c(v_i) - c(v_j)| \geq 1 + |n - 2| \geq 6$, otherwise $|c(v_i) - c(v_j)| \geq 2$. Therefore the radio condition is satisfied for such pairs.

5: Consider the pairs (u_i, u_j) (with $i \neq j$.) Similar as Case 3.

6: Consider the pairs (u_i, v_j). We examine the label difference for each pair, when distance between vertices is one, two, three. As $1 \leq d(u_i, v_j) \leq 3$, so

- if $d(u_i, v_j) = 1$ then $i = j$ and $|c(u_i) - c(v_j)| \geq 3$
- if $d(u_i, v_j) = 2$ then $i = j \pm 1$ and $|c(u_i) - c(v_j)| \geq |n - 5| \geq 2$ for $n \geq 7$
- if $d(u_i, v_j) = 3$ then $|c(u_i) - c(v_j)| \geq 1$.

Hence the radio condition (2) is satisfied.

Case B: n is odd.

B.1 If $n \equiv 3 \pmod 4$ we have the following cases:

1: Consider the pair (z, r) for any vertex $r \notin \{z, z_1\}$. As $1 \leq d(z, r) \leq 2$, $r \notin \{z, z_1\}$, $c(z) = 2n + 3, c(r) \leq 2n$, $|c(z) - c(r)| \geq 3$. Hence $d(z, r) + |c(z) - c(r)| \geq 1 + 3 = 4$.

2: As $d(z, z_1) = 3$ and $|c(z) - c(z_1)| = 1$ the radio condition is satisfied.

3: Consider the pairs (z_1, r) for any vertex $r \notin \{z, z_1\}$. As $1 \leq d(z_1, r) \leq 2, r \notin \{z, z_1\}$, when $d(z_1, r) = 1$ we have $|c(z_1) - c(r)| \geq 3$ and when $d(z_1, r) = 2$, then $|c(z_1) - c(r)| \geq 2$. It follows that the radio condition (2) is satisfied.

4: Consider the pairs (v_i, v_j) (with $i \neq j$). As $d(v_i, v_j) \leq 2$ for $i \neq j$,
- if $d(v_i, v_j) = 1$ then $i = j \pm 1$ and $|c(v_i) - c(v_j)| \geq \frac{n+1}{2} \geq 4$ for $n \geq 7$
- if $d(v_i, v_j) = 2$ then $|c(v_i) - c(v_j)| \geq 2$.

Therefore the radio condition is satisfied for such pairs.

5: Consider the pairs (u_i, u_j) (with $i \neq j$). As $d(u_i, u_j) \leq 2$ for $i \neq j$,
- if $d(u_i, u_j) = 1$ then $i = j \pm 1$ and $|c(u_i) - c(u_j)| \geq \frac{n+1}{2} \geq 4$ for $n \geq 7$
- if $d(u_i, u_j) = 2$ then $|c(u_i) - c(u_j)| \geq 2$ $n \geq 7$.

Hence the radio condition is also satisfied for these pairs.

6: Consider the pairs (u_i, v_j). We examine the labels difference for each pair, when distance between vertices is one, two, three. As $1 \leq d(u_i, v_j) \leq 3$, so
- if $d(u_i, v_j) = 1$ then $i = j$ and $|c(u_i) - c(v_j)| \geq 3$
- if $d(u_i, v_j) = 2$ then $i = j \pm 1$ and $|c(u_i) - c(v_j)| \geq |n - 5| \geq 2$
- if $d(u_i, v_j) = 3$ then $|c(u_i) - c(v_j)| \geq 1$.

Hence the radio condition (2) is satisfied.

The situation when $n \equiv 1 \pmod 4$ is similar as $n \equiv 3 \pmod 4$.

For all cases we establish the claim that c is a radio labeling of d_n. Thus $rn(d_n) \leq 2n + 3$. Hence $rn(d_n) = 2n + 3$. □

4. RADIO NUMBER FOR FAN AND DOUBLE FAN GRAPHS

THEOREM 4.1. *For $n \geq 4$, $rn(f_n) = n + 2$.*

Proof. Note that $diam(f_n) = 2$. This together with the fact that the center vertex z is adjacent to every other vertex implies we may not use consecutive integers to label the center and another vertex. Since $|V(f_n)| = n + 1$, we see $rn(f_n) \geq n + 2$. Assigning 1 to the center and consecutive integers beginning with 3 to the other vertices, first to the vertices of even indices, and then to the vertices with odd indices, produces a radio labeling with span $n + 2$, so $rn(f_n) = n + 2$. □

THEOREM 4.2. *For $n \geq 4$, $rn(df_n) = n + 3$*

Proof. Note that $diam(df_n) = 2$. There are two center vertices, these together with the fact that the center vertices are adjacent to every other vertex implies we may not use consecutive integers to label the centers and another vertex. Since $|V(df_n)| = n + 2$, we see $rn(df_n) \geq n + 3$. Assigning labels 1 and 2 to the centers and consecutive integers beginning with 4 to the other vertices, first to the vertices of even indices, and then to the vertices with odd indices, produces a radio labeling with span $n+3$, so $rn(df_n) = n+3$. $\square$

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