A POWER OF A MEROMORPHIC FUNCTION SHARING ONE VALUE WITH ITS DERIVATIVE

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Let f be a non-constant meromorphic function, n, k be two positive integers and $a(z)(\not\equiv 0,\infty)$ be a meromorphic small function of f. Suppose that $f^n - a$ and $(f^n)^{(k)} - a$ share the value 0 CM. If either (1) $n \geq k+1$ and $\overline{N}(r,\infty;f) = S(r,f)$, or (2) $n > k+1$ and $\overline{N}(r,\infty;f) = \lambda\, T(r,f)(\lambda \in [0,1))$, then $f^n \equiv (f^n)^{(k)}$ and f assume the form $f(z) = ce^{\frac{\lambda}{n}z}$, where c is a nonzero constant and $\lambda^k = 1$. This result shows that Brück conjecture is true for meromorphic function when $F = f^n$ with $\overline{N}(r,\infty;f) = S(r,f)$ and $n \geq 2$.

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1. INTRODUCTION, DEFINITIONS AND RESULTS

In this paper, by a meromorphic function we will always mean meromorphic function in the complex plane $\mathbb{C}$. We adopt the standard notations of the Nevanlinna theory of meromorphic functions as explained in [5]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For any non-constant meromorphic function f, we denote by $S(r,f)$ any quantity satisfying

$$\lim_{r\to\infty} \frac{S(r,f)}{T(r,f)} = 0, \quad r \notin E.$$

Also for any non-constant meromorphic function f, we define the order of growth of f by

$$\rho(f) := \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

A meromorphic function a is said to be a small function of f provided that $T(r,a) = S(r,f)$, *i.e.*, $T(r,a) = o(T(r,f))$ as $r \longrightarrow \infty$, $r \notin E$.

Let k be a positive integer and $a \in \mathbb{C}\cup\{\infty\}$. We use $N_{k)}(r,a;f)$ to denote the counting function of a-points of f with multiplicity $\leq k$, $N_{(k+1}(r,a;f)$ to denote the counting function of a-points of f with multiplicity $> k$. Similarly $\overline{N}_{k)}(r, a; f)$ and $\overline{N}_{(k+1}(r, a; f)$ are their reduced functions respectively.

For $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer p we denote by $N_p(r, a; f)$ the sum

$$\overline{N}_{(1}(r, a; f) + \overline{N}_{(2}(r, a; f) + \ldots + \overline{N}_{(p}(r, a; f).$$

Let f and g be two non-constant meromorphic functions and let a be a complex number. We say that f and g share a CM, provided that $f - a$ and $g - a$ have the same zeros with the same multiplicities.

Rubel-Yang [9] proposed to investigate the uniqueness of an entire function f under the assumption that f and its derivative f' share two complex values. Subsequently, related to one or two value sharing similar considerations have been made with respect to higher derivatives and more general (linear) differential expressions by Brück [1], Gundersen [3], Mues-Steinmetz [8], Yang [11] *et al.*

In this direction, an interesting problem still open is the following conjecture proposed by Brück [1]

Conjecture 1.1 ([1]). *Let F be a non-constant entire function. Suppose*

$$\rho_1(F) := \limsup_{r\to\infty} \frac{\log\log T(r, F)}{\log r}$$

is not a positive integer or infinite. If F and F' share one finite value a CM, then

$$\frac{F' - a}{F - a} = c$$

for some non-zero constant c.

The case that $a = 0$ and that $N(r, 0; f') = S(r, f)$ had been proved by Brück [1] while the case that f is of finite order had been proved by Gundersen-Yang [4]. However, the corresponding conjecture for meromorphic functions fails in general (see [4]).

To the knowledge of the author, perhaps Yang-Zhang [13] (see also [14]) were the first to consider the uniqueness of a power of a meromorphic(entire) function $F = f^n$ and its derivative F' when they share certain value as this type of considerations gives the most specific form of the function.

As a result during the last decade, growing interest has been devoted to this setting of meromorphic functions. Improving all the results obtained in [13], Zhang [14] proved the following theorem.

Theorem A ([14]). *Let f be a non-constant meromorphic function, n, k be positive integers and $a(z)(\not\equiv 0, \infty)$ be a meromorphic small function of f.*

Suppose $f^n - a$ *and* $(f^n)^{(k)} - a$ *share the value* 0 *CM and*

$$(n-k-1)(n-k-4) > 3k+6,$$

then $f^n \equiv (f^n)^{(k)}$, *and* f *assumes the form*

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c *is a nonzero constant and* $\lambda^k = 1$.

In 2009, Zhang and Yang [15] further improved the above result in the following manner.

THEOREM B ([15]). *Let* f *be a non-constant meromorphic function,* n, k *be positive integers and* $a(z)(\not\equiv 0, \infty)$ *be a meromorphic small function of* f. *Suppose* $f^n - a$ *and* $(f^n)^{(k)} - a$ *share the value* 0 *CM and*

$$n > k+1+\sqrt{k+1}.$$

Then the conclusion of Theorem A holds.

THEOREM C ([15]). *Let* f *be a non-constant entire function,* n, k *be positive integers and* $a(z)(\not\equiv 0, \infty)$ *be a meromorphic small function of* f. *Suppose* $f^n - a$ *and* $(f^n)^{(k)} - a$ *share the value* 0 *CM and*

$$n > k+1.$$

Then the conclusion of Theorem A holds.

COROLLARY A (Corollary 1.3, [15]). *Let* f *be a non-constant entire function and* $n \geq 3$ *be an integer. Denote* $F = f^n$. *If* F *and* F' *share* 1 *CM, then* $F \equiv F'$ *and* f *assumes the form*

$$f(z) = ce^{\frac{1}{n}z},$$

where c *is a nonzero constant.*

At the end of the paper, the following open problem was posed by the authors in [15].

Open problem. Can n in *Corollary 1.3* [15] be reduced?

One of our objectives in writing this paper is to solve this open problem. Recently, Sheng and Zongsheng [10] proved the following result.

THEOREM D ([10]). *Let* f *be a non-constant meromorphic function such that* $\overline{N}(r,\infty;f) = S(r,f)$. *Denote* $F = f^n$. *Suppose that* F *and* F' *share* 1 *CM. If* (1) $n \geq 3$, *or* (2) $n = 2$ *and* $N(r,0;f) = O(N_{(3}(r,0;f))$, *then* $F \equiv F'$, *and* f *assumes the form*

$$f(z) = ce^{\frac{1}{n}z},$$

where c *is a nonzero constant.*

In 2014, Li [7] obtained the following result.

THEOREM E ([7]). *Let f be a non-constant meromorphic functions, n, k be two positive integers. Suppose that f^n and $(f^n)^{(k)}$ share the value $a(\neq 0,\infty)$ CM. If either (1) $n>k+2$, or (2) $n>k+1$ and $\overline{N}(r,\infty;f)=\lambda\, T(r,f)(\lambda\in[0,\frac{1}{2}))$, then conclusion of Theorem A holds.*

Now observing the above results the following questions are inevitable.

Question 1: Can one remove the condition "$N(r,0;f)=O(N_{(3}(r,0;f))$" in Theorem D?
Question 2: Can one replace the condition " $\overline{N}(r,\infty;f)=\lambda\, T(r,f)(\lambda\in[0,\frac{1}{2}))$" in Theorem E by a weaker one?
In this paper, taking the possible answer of the above questions into background we obtain the following results.

THEOREM 1.1. *Let f be a non-constant meromorphic function such that $\overline{N}(r,\infty;f)=S(r,f)$, n, k be two positive integers and $a(z)(\not\equiv 0,\infty)$ be a meromorphic small function of f. Suppose f^n-a and $(f^n)^{(k)}-a$ share the value 0 CM and $n\geq k+1$. Then the conclusion of Theorem A holds.*

Remark 1.1. Clearly Theorem 1.1 improves Theorem C.

Remark 1.2. Theorem 1.1 shows that Brück conjecture is true for meromorphic function when $F=f^n$ with $\overline{N}(r,\infty;f)=S(r,f)$ and $n\geq 2$.

COROLLARY 1.1. *Let f be a non-constant meromorphic function such that $\overline{N}(r,\infty;f)=S(r,f)$, n be a positive integer such that $n\geq 2$ and $a(z)(\not\equiv 0,\infty)$ be a meromorphic small function of f. Suppose f^n-a and $(f^n)'-a$ share the value 0 CM. Then the conclusion of Theorem A holds.*

Remark 1.3. Clearly Corollary 1.1 improves Theorem D as well as Corollary A.

THEOREM 1.2. *Let f be a non-constant meromorphic functions, n, k be two positive integers and $a(z)(\not\equiv 0,\infty)$ be a meromorphic small function of f. Suppose that f^n-a and $(f^n)^{(k)}-a$ share the value 0 CM. If $n>k+1$ and $\overline{N}(r,\infty;f)=\lambda\, T(r,f)$, where $\lambda\in[0,1)$. Then the conclusion of Theorem A holds.*

Remark 1.4. It is easy to see that the condition $n\geq k+1$ in *Theorem 1.1* is sharp by the following examples.

Example 1.1. Let
$$f(z)=e^{e^z}+1,$$
where $a(z)=\frac{1}{1-e^{-z}}$. Then f and f' share the value a CM, but $f\not\equiv f'$.

Example 1.2. Let

$$f(z) = e^{c_1 z} + c_2,$$

where $c_1, c_2 \in \mathbb{C} \setminus \{0\}$ such that $c_1 \neq 1$. Then f and f' share the value c_3 CM, where $c_1 c_2 = c_3(c_1 - 1)$, but $f \not\equiv f'$.

Example 1.3. Let

$$f(z) = e^{3z} + \frac{2z}{3} + \frac{2}{9}.$$

Note that $f' - z = 3(f - z)$. Then $f - z$ and $f' - z$ share 0 CM, but $f \not\equiv f'$.

2. LEMMAS

In this section, we present the lemmas which will be needed in the sequel.

The following Hadamard's theorem for entire function of infinite order is well known.

LEMMA 2.1 ([6]). *Let f be a transcendental entire function of infinite order, then f can be represented by*

$$f(z) = U(z)e^{V(z)},$$

U and V are entire functions with

$$\lambda(f) = \lambda(U) = \rho(U), \quad \lambda_1(f) = \lambda_1(U) = \rho_1(U),$$

$$\rho_1(f) = \max\{\rho_1(U), \rho_1(e^V)\},$$

where $\lambda_1(f)$ is given by

$$\lambda_1(f) := \limsup_{r\to\infty} \frac{\log\log N(r, 0; f)}{\log r}.$$

LEMMA 2.2 ([2]). *Suppose that f is a transcendental meromorphic function and that*

$$f^n P(f) = Q(f),$$

where $P(f)$ and $Q(f)$ are differential polynomials in f with functions of small proximity related to f as the coefficients and the degree of $Q(f)$ is at most n. Then

$$m(r, P(f)) = S(r, f).$$

LEMMA 2.3 ([12]). *Let f be a non-constant meromorphic function and let $a_n(z)(\not\equiv 0)$, $a_{n-1}(z)$, ... , $a_0(z)$ be meromorphic functions such that $T(r, a_i(z)) =$*

$S(r,f)$ *for* $i=0,1,2,...,n$. *Then*

$$T(r,a_nf^n+a_{n-1}f^{n-1}+...+a_1f+a_0)=nT(r,f)+S(r,f).$$

3. **PROOFS OF THE THEOREMS**

Proof of Theorem 1.1. Let $F_1=\frac{f^n}{a}$ and $G_1=\frac{(f^n)^{(k)}}{a}$. Clearly F_1 and G_1 share 1 CM except for the zeros and poles of $a(z)$ and so

$$\overline{N}(r,1;F_1)=\overline{N}(r,1;G_1)+S(r,f).$$

Let

$$\Phi=\frac{F_1'(F_1-G_1)}{F_1(F_1-1)}. \tag{3.1}$$

We now consider following two cases.

Case 1. Let $n>k+1$.

First we suppose $\Phi\not\equiv 0$.

From (3.1) it is clear that $T(r,\Phi)=S(r,f)$. Also from (3.1) we get

$$\frac{1}{F_1}=\frac{1}{\Phi}\frac{F_1'}{F_1(F_1-1)}[1-\frac{(f^n)^{(k)}}{f^n}]$$

and so

$$m(r,\frac{1}{F_1})=S(r,f).$$

Hence

$$m(r,\frac{1}{f})=S(r,f). \tag{3.2}$$

Let z_0 be a zero of f of multiplicity p such that $a(z_0)\neq 0,\infty$. Then z_0 will be a zero of F_1 and G_1 of multiplicities np and $np-k$ respectively and so from (3.1) we get

$$N(r,0;f)=S(r,f). \tag{3.3}$$

Now from (3.2) and (3.3) we get $T(r,f)=S(r,f)$, which contradicts the fact that f is non-constant function.

Hence $\Phi\equiv 0$. From (3.1) we get $F_1\equiv G_1$, *i.e.*, $(f^n)^{(k)}\equiv f^n$ and so the conclusion of the Theorem holds.

Case 2. Let $n=k+1$.

First we suppose that $\Phi\not\equiv 0$.

Let

$$F=f^n. \tag{3.4}$$

Since $F-a$ and $F^{(k)}-a$ share 0 CM, we see that

$$\Psi=\frac{F-a}{F^{(k)}-a}$$

has no poles and $\Psi(z)=0$ if and only if $f(z)=\infty$. Note that if z_* is a pole of f of multiplicity p_*, then z_* will be a zero of Ψ of multiplicity k.

Thus by Lemma 2.1 and the standard Hadamard's theorem for entire function, we can write

$$\frac{F-a}{F^{(k)}-a}=\Psi=g_1e^{g_2}, \tag{3.5}$$

where g_1, g_2 are entire functions such that

$$N(r,0;g_1)=k\,\overline{N}(r,\infty;f)=S(r,f).$$

Now from (3.5) we have

$$F^{(k)}-a=He^{\alpha}(F-a), \tag{3.6}$$

where $H=\frac{1}{g_1}$, $\alpha=-g_2$. Also

$$N(r,0;H)=0 \;\; \textit{and} \;\; N(r,\infty;H)=S(r,f). \tag{3.7}$$

First we suppose that He^{α} is a non-constant meromorphic function.
By differentiation from (3.6) we get

$$F^{(k+1)}-a'=(H'+H\alpha')e^{\alpha}(F-a)+He^{\alpha}(F'-a'). \tag{3.8}$$

Now combining (3.6) and (3.8) we get

$$\begin{aligned} &F^{(k+1)}F-(\frac{H'}{H}+\alpha')F^{(k)}F-F^{(k)}F' \\ = \;\; &aF^{(k+1)}-\{(\frac{H'}{H}+\alpha')a+a'\}F^{(k)}+\{a'-(\frac{H'}{H}+\alpha'+1)a\}F \\ &+(\frac{H'}{H}+\alpha')a^2. \end{aligned} \tag{3.9}$$

Note that from (3.6) we get

$$T(r,He^{\alpha})\le(k+2)T(r,f^n)+S(r,f)=n(k+2)T(r,f)+S(r,f).$$

Consequently $T(r,He^{\alpha})=O(T(r,f))$ and so $S(r,He^{\alpha})$ can be replaced by $S(r,f)$.
Let

$$\xi=\frac{H'}{H}+\alpha'.$$

From (3.7) we have

$$N(r,\infty;\xi)=S(r,f).$$

Note that

$$m(r,\xi)=m(r,\frac{H'}{H}+\alpha')=m(r,\frac{(He^{\alpha})'}{He^{\alpha}})=S(r,He^{\alpha}).$$

Consequently $T(r,\xi)=S(r,f)$.
Now from (3.9) we get

$$\begin{aligned}&F^{(k+1)}F-\xi F^{(k)}F-F^{(k)}F' \\ =\;& aF^{(k+1)}-(\xi a+a')F^{(k)}+\{a'-(\xi+1)a\}F+\xi a^2.\end{aligned} \tag{3.10}$$

By induction, we deduce from (3.4) that

$$F'=nf^{n-1}f'$$

$$F''=n(n-1)f^{n-2}(f')^2+nf^{n-1}f''$$

$$F'''=n(n-1)(n-2)f^{n-3}(f')^3+3n(n-1)f^{n-2}f'f''+nf^{n-1}f'''$$

and so on.
Thus in general we have

$$F^{(k)}=\sum_{\lambda}a_{\lambda}f^{l_0^{\lambda}}(f')^{l_1^{\lambda}}\ldots(f^{(k)})^{l_k^{\lambda}}, \tag{3.11}$$

where $l_0^{\lambda},l_1^{\lambda},\ldots,l_k^{\lambda}$ are non-negative integers satisfying $\sum\limits_{j=0}^{k}l_j^{\lambda}=n$, $n-k\leq l_0^{\lambda}\leq n-1$ and a_{λ} are constants. Clearly

$$F^{(k)}=(k+1)!f(f')^k+\frac{k(k-1)}{4}(k+1)!f^2(f')^{k-2}f''+\ldots+(k+1)f^kf^{(k)}.$$

Therefore

$$\frac{f'}{f}F^{(k)}=(k+1)!(f')^{k+1}+\frac{k(k-1)}{4}(k+1)!f(f')^{k-1}f''+\ldots+(k+1)f^{k-1}f'f^{(k)}.$$

Also we set

$$F^{(k+1)}=\sum_{\lambda}b_{\lambda}f^{p_0^{\lambda}}(f')^{p_1^{\lambda}}\ldots(f^{(k+1)})^{p_{k+1}^{\lambda}}, \tag{3.12}$$

where $p_0^{\lambda},p_1^{\lambda},\ldots,p_{k+1}^{\lambda}$ are non-negative integers satisfying $\sum\limits_{j=0}^{k+1}p_j^{\lambda}=n$, $n-k-1\leq p_0^{\lambda}\leq n-1$, *i.e.*, $0\leq p_0^{\lambda}\leq n-1$ and b_{λ} are constants. Clearly

$$F^{(k+1)}=(k+1)!(f')^{k+1}+\frac{k(k+1)}{2}(k+1)!f(f')^{k-1}f''+\ldots+(k+1)f^kf^{(k+1)}.$$

Substituting (3.4), (3.11) and (3.12) into (3.10), we have

$$f^nP(f)=Q(f), \tag{3.13}$$

where $Q(f)$ is a differential polynomial in f of degree n and

$$
\begin{aligned}
(3.14)\ P(f) &= \sum_{\lambda} b_\lambda f^{p_0^\lambda}(f')^{p_1^\lambda}\ldots(f^{(k)})^{p_{k+1}^\lambda} - \xi \sum_{\lambda} a_\lambda f^{l_0^\lambda}(f')^{l_1^\lambda}\ldots(f^{(k)})^{l_k^\lambda} \\
&\quad -nf' \sum_{\lambda} a_\lambda f^{l_0^\lambda - 1}(f')^{l_1^\lambda}\ldots(f^{(k)})^{l_k^\lambda} \\
&= -k(k+1)!(f')^{k+1} - (k+1)!\xi f (f')^k \\
&\quad + \frac{k(k+1)(3-k)(k+1)!}{4} f(f')^{k-1}f'' + \ldots \\
&\quad + (k+1)f^k f^{(k+1)} - (k+1)\xi f^k f^{(k)} - (k+1)^2 f^{k-1} f' f^{(k)} \\
&= A(f')^{k+1} + R_1(f),
\end{aligned}
$$

is a differential polynomial in f of the degree $k+1$, where $A = -k(k+1)!$ and $R_1(f)$ is a differential polynomial in f. In particular, every monomial of R_1 has the form

$$R(\xi) f^{q_0^\lambda}(f')^{q_1^\lambda}\ldots(f^{(k+1)})^{q_{k+1}^\lambda},$$

where $q_0^\lambda, \ldots, q_{k+1}^\lambda$ are non-negative integers satisfying $\sum_{j=0}^{k+1} q_j^\lambda = n$ and $1 \le q_0^\lambda \le n-1$, $R(\xi)$ is a polynomial in ξ with constant coefficients.

First we suppose $P \not\equiv 0$. We assert that f is a transcendental meromorphic function. If not, then f must be a polynomial with a as a nonzero constant. But this is impossible since in that case $F-a$ and $G-a$ cannot have the same zeros with the same multiplicities.

Then by Lemma 2.2 we get $m(r,P) = S(r,f)$ and so

$$(3.15) \qquad T(r,P) = S(r,f), \quad T(r,P') = S(r,f).$$

Let z_1 be a zero of f of multiplicity $p(\ge 2)$ such that $a(z_0) \ne 0, \infty$. Then from (3.1) we get $N_{(2}(r,0;f) = S(r,f)$. Now from (3.2) we have

$$(3.16) \qquad T(r,f) = N_{1)}(r,0;f) + S(r,f).$$

Note that from (3.14) we get

$$(3.17) \qquad P' = A_1(f')^k f'' + B_1 \xi (f')^{k+1} + S_1(f),$$

is a differential polynomial in f, where $A_1 = -\frac{1}{4}k(k+1)^2(k+1)!$, $B_1 = -(k+1)!$ and $S_1(f)$ is a differential polynomial in f. In particular, every monomial of S_1 has the form

$$S(\xi) f^{r_0^\lambda}(f')^{r_1^\lambda}\ldots(f^{(k+1)})^{r_{k+1}^\lambda},$$

where $r_0^{\lambda}, \ldots, r_{k+1}^{\lambda}$ are non-negative integers satisfying $\sum_{j=0}^{k+1} r_j^{\lambda} = n$ and $1 \leq r_0^{\lambda} \leq n-1$, $S(\xi)$ is a polynomial in ξ with constant coefficients.

Let z_2 be a simple zero of f. Then from (3.14) and (3.17) we have

$$P(z_2) = A\{f'(z_2)\}^{k+1}$$

and

$$P'(z_2) = A(k+1)\{f'(z_2)\}^k f''(z_2) + B\xi(z_2)\{f'(z_2)\}^{k+1}.$$

This shows that z_2 is a zero of $Pf'' - [K_1P' - K_2\xi P]f'$, where K_1 and K_2 are suitably constants. Let

$$\Phi_1 = \frac{Pf'' - [K_1P' - K_2\xi P]f'}{f}. \tag{3.18}$$

Clearly

$$T(r, \Phi_1) = S(r, f).$$

From (3.18) we obtain

$$f'' = \alpha_1 f + \beta_1 f', \tag{3.19}$$

where

$$\alpha_1 = \frac{\Phi_1}{P}, \qquad \beta_1 = K_1\frac{P'}{P} - K_2\xi. \tag{3.20}$$

From (3.19) we have

$$f^{(i)} = \alpha_{i-1} f + \beta_{i-1} f', \tag{3.21}$$

where $i \geq 2$ and

$$T(r, \alpha_{i-1}) = S(r, f), \qquad T(r, \beta_{i-1}) = S(r, f).$$

Now from (3.14), (3.17) and (3.21) we have

$$P = A(f')^{k+1} + \sum_{j=1}^{k+1} t_j f^j (f')^{k+1-j}, \tag{3.22}$$

$$P' = (A_1\beta_1 + B_1\xi)(f')^{k+1} + \sum_{j=1}^{k+1} s_j f^j (f')^{k+1-j},$$

where $T(r, t_j) = S(r, f)$ and $T(r, s_j) = S(r, f)$. Also (3.20) yields

$$P' \;=\; (\frac{\beta_1}{K_1} + \frac{K_2}{K_1}\xi)P. \tag{3.23}$$

By (3.22) and (3.23) we get

$$h_1(f')^k + h_2 f(f')^{k-1} + \ldots + h_{k+1} f^k \equiv 0, \tag{3.24}$$

where $h_j = s_j - (\frac{\beta_1}{K_1} + \frac{K_2}{K_1}\xi)t_j$ and $T(r, h_j) = S(r, f)$. Also not all h_j' s are identically zero. Note that $f^{(j)}(z) \not\equiv 0$ for $j = 0, 1$, where we define $f^{(0)}(z) = f(z)$. Hence from (3.24) we get

$$N_{1)}(r, 0; f) = S(r, f). \tag{3.25}$$

Therefore we arrive at a contradiction from (3.16) and (3.25).
Hence $P(f) \equiv 0$ and so we obtain

$$F^{(k+1)}F - \xi F^{(k)}F - F^{(k)}F' \equiv 0,$$

i.e.,

$$F^{(k+1)}F - \left(\frac{H'}{H} + \alpha'\right)F^{(k)}F - F^{(k)}F' \equiv 0,$$

i.e.,

$$\frac{F^{(k+1)}}{F^{(k)}} \equiv \frac{H'}{H} + \alpha' + \frac{F'}{F}. \tag{3.26}$$

By integration we have $F^{(k)} = dHFe^{\alpha}$, where d is a non-zero constant. Substituting this and (3.4) into (3.6) we have

$$(d-1)f^n \equiv a\frac{1 - He^{\alpha}}{He^{\alpha}}. \tag{3.27}$$

First we suppose $d = 1$. Then $He^{\alpha} \equiv 1$, which is a contradiction, since we suppose first that He^{α} is a non-constant meromorphic function.

Next we suppose $d \neq 1$. From (3.27) see that all zeros of $1 - He^{\alpha}$ have the multiplicities at least n. Now using (3.27) and Lemma 2.3 we get

$$\begin{aligned} n\,T(r,f) &= T(r,(d-1)f^n) + O(1) = T(r, a\frac{1-He^{\alpha}}{He^{\alpha}}) + O(1) \\ &\leq T(r,a) + T(r, 1-He^{\alpha}) + T(r, He^{\alpha}) + S(r,f) \\ &\leq 2\,T(r, He^{\alpha}) + S(r,f). \end{aligned}$$

This shows that $T(r,f) = O(T(r,He^{\alpha}))$. Also we have $T(r,He^{\alpha}) = O(T(r,f))$. Consequently $S(r,f) = S(r,He^{\alpha})$.

Noting that $n = k+1 \geq 2$, and using (3.7), we get from the second fundamental theorem that

$$\begin{aligned} T(r,He^{\alpha}) &\leq \overline{N}(r,0;He^{\alpha}) + \overline{N}(r,\infty;He^{\alpha}) + \overline{N}(r,1;He^{\alpha}) + S(r,He^{\alpha}) \\ &\leq \overline{N}(r,0;H) + \overline{N}(r,\infty;H) + \overline{N}(r,1;He^{\alpha}) + S(r,He^{\alpha}) \end{aligned}$$

$$\begin{aligned}
&\leq \overline{N}(r,1;He^{\alpha})+S(r,f)+S(r,He^{\alpha})\\
&\leq \frac{1}{n}N(r,1;He^{\alpha})+S(r,He^{\alpha})\\
&\leq \frac{1}{n}T(r,He^{\alpha})+S(r,He^{\alpha}),
\end{aligned}$$

which is a contradiction since we suppose first that He^{α} is a non-constant meromorphic function.

Next we suppose that He^{α} is a non-zero constant, say D. Then from (3.6) we have

$$F^{(k)}-DF\equiv a(1-D). \tag{3.28}$$

Since $n=k+1$, it follows from (3.28) that $N(r,0;f)=S(r,f)$. Now by (3.2) we get $T(r,f)=S(r,f)$, which contradicts the fact that f is non-constant function.

Hence $\Phi\equiv 0$ and so from (3.1) we get $F_1\equiv G_1$, *i.e.*, $(f^n)^{(k)}\equiv f^n$ and so conclusion of Theorem holds.

This completes the proof. □

Proof of Theorem 1.2. Let $F=\frac{f^n}{a}$ and $G=\frac{(f^n)^{(k)}}{a}$. Clearly F and G share 1 CM except for the zeros and poles of $a(z)$. Let

$$\begin{aligned}
\Phi_2 &= \frac{1}{F}\left(\frac{G'}{G-1}-\frac{F'}{F-1}\right)\\
&= \frac{G}{F}\left(\frac{G'}{G-1}-\frac{G'}{G}\right)-\left(\frac{F'}{F-1}-\frac{F'}{F}\right). \qquad (3.29)
\end{aligned}$$

We now consider the following two cases.

Case 1: Let $\Phi_2\equiv 0$. On integration we get

$$F-1\equiv c(G-1), \tag{3.30}$$

where c is a nonzero constant.

This implies that $\overline{N}(r,\infty;f)=S(r,f)$. Let $c\neq 1$. Then from (3.30) we get

$$\frac{1}{F}\equiv\frac{1}{c-1}\left(c\frac{G}{F}-1\right). \tag{3.31}$$

Now using (3.31) and Lemma 2.3 we get

$$\begin{aligned}
n\,T(r,f)&=T(r,F)+O(1)\leq T(r,\frac{G}{F})+S(r,f)=N(r,\infty;\frac{(f^n)^{(k)}}{f^n})+S(r,f)\\
&\leq N_k(r,0;f^n)+k\overline{N}(r,\infty;f)+S(r,f)\leq k\overline{N}(r,0;f)+S(r,f),
\end{aligned}$$

which is impossible since $n>k+1$.

Hence $c = 1$. From (3.30) we get $F \equiv G$, *i.e.*, $f^n \equiv (f^n)^{(k)}$ and so the conclusion of the Theorem holds.

Case 2: Let $\Phi_2 \not\equiv 0$. Clearly $F \not\equiv G$. From (3.29) we get $m(r, \Phi_2) = S(r, f)$ and

$$m(r, F) \leq m(r, \frac{1}{\Phi_2}) + S(r, f). \tag{3.32}$$

Then from (3.29) we get

$$\begin{aligned} N(r, \infty; F) - \overline{N}(r, \infty; F) &\leq N(r, 0; \Phi_2) + S(r, f) \\ &\leq T(r, \Phi_2) - m(r, \frac{1}{\Phi_2}) + S(r, f) \\ &= N(r, \infty; \Phi_2) + m(r, \Phi_2) - m(r, \frac{1}{\Phi_2}) + S(r, f) \\ &\leq N_{k+1}(r, 0; F) - m(r, \frac{1}{\Phi_2}) + S(r, f) \\ &\leq (k+1)\, \overline{N}(r, 0; f) - m(r, \frac{1}{\Phi_2}) + S(r, f). \end{aligned} \tag{3.33}$$

Now using (3.32), (3.33) and Lemma 2.3 we get

$$n\, T(r, f) = T(r, F) + O(1) \leq (k+1)\overline{N}(r, 0; f) + \overline{N}(r, \infty; f) + S(r, f). \tag{3.34}$$

Let

$$\Phi_3 = \frac{F'}{F-1} - \frac{G'}{G-1}. \tag{3.35}$$

Clearly $\Phi_3 \not\equiv 0$. Since $n > k+1$, from (3.35) we get

$$\begin{aligned} (n-k-1)\, \overline{N}(r, 0; f) &\leq N(r, 0; \Phi_3) + S(r, f) \\ &\leq T(r, \Phi_3) + S(r, f) \\ &\leq N(r, \infty; \Phi_3) + m(r, \Phi_3) + S(r, f) \leq \overline{N}(r, \infty; f) + S(r, f). \end{aligned} \tag{3.36}$$

Then using (3.34), (3.36) we get

$$T(r, f) \leq \frac{1}{n-k-1}\, \overline{N}(r, \infty; f) + S(r, f) \leq \frac{\lambda}{n-k-1} T(r, f) + S(r, f),$$

which is a contradiction.

This completes the proof. □

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