JOINT SPECTRUM OF n-TUPLE OF UPPER TRIANGULAR MATRICES WITH ENTRIES IN A UNITALL BANACH ALGEBRA

HAMIDEH MOHAMMADZADEHKAN, ALI EBADIAN
and KAZEM HAGHNEJAD AZAR

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Let $(T_1, \ldots, T_n)$ be n-tuple in $(U_m(A))^n$. We investigate formula for joint (Harte) spectrum of $(T_1, \ldots, T_n)$ with respect to upper triangular matrices algebra $U_m(A)$ and obtain condition such that joint spectrum of the n-tuple in $(U_m(A))^n$ equals with respect to $U_m(A)$ and $M_m(A)$.

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1. INTRODUCTION

Suppose that A is a complex Banach algebra with unit 1, we denote the sets of invertible, left invertible, right invertible elements of A, respectively with $Inv(A)$, $Inv_{lt}(A)$, and $Inv_{rt}(A)$. The *spectrum* of an element $a \in A$ is the set $\sigma(a) = \{\lambda \in \mathbb{C} \mid a - \lambda 1 \notin Inv(A)\}$. For $n-$tuples $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ of elements of A we write

$$a.b = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n,$$

if $a.b = 1$ declare that a is $left\ inverse$ for b in A, and b is $right\ inverse$ for a.

The *joint* $(Harte)$ *spectrum* $\sigma(a)$ for $n-$tuples $a = (a_1, \ldots, a_n)$ will be a set of $n-$tuples $\lambda = (\lambda_1, \ldots, \lambda_n)$ of complex numbers, for which the $n-$tuples $a - \lambda = (a_1 - \lambda_1, \ldots, a_n - \lambda_n)$ has no left or right inverses in A.

Definition 1.1. Let A be a unital Banach algebra.

(i) The $left$ *spectrum* $\sigma^{left}(a) = \sigma_A^{left}(a)$ of $n-$tuples a with respect to A is the set of $\lambda \in \mathbb{C}^n$ for which $a - \lambda$ has no left inverse in A.

(ii) The *right spectrum* $\sigma^{right}(a) = \sigma_A^{right}(a)$ of $n-$tuples a with respect to A is the set of $\lambda \in \mathbb{C}^n$ for which $a - \lambda$ has no right inverse in A.

(iii) The *joint spectrum* $\sigma(a) = \sigma_A^{joint}(a)$ of an $n-$tuples a in A is the union of the left and right spectra of a.

In terms of ideals, an $n-$tuple $\lambda \in \mathbb{C}^n$ is in the left spectrum of $n-$tuple $a \in A^n$ if, and only if the left ideal $\sum_j A(a_j - \lambda_j)$ generated by the $n-$tuple $a - \lambda$ is proper. It's similar for the right spectrum.

Note that for a single element $a \in A$ joint spectrum coincides with the usual spectrum of a. But there are many different properties, for example, spectrum of an element is always non-empty, but joint spectrum may be void, for more information see [6].

Definition 1.2. If a is an n-tuple of elements of A, a sequence $(u) = (u_k)_{k=1}^{\infty}$ of elements of A which satisfies

$$\inf_k \| u_k \| > 0 = \lim_k \| u_k a_j \| \quad (j = 1, \dots, n) \tag{1}$$

is called an *approximate left annihilator* for a in A, with again a similar definition of right annihilator.

Definition 1.3. (*i*) The *left approximate point spectrum* $\tau_A^{left}(a)$ of an n-tuple $a \in A^n$ is the set of $s \in \mathbb{C}^n$ for which $a - s$ has approximate right annihilators in A.

(*ii*) The *right approximate point spectrum* $\tau_A^{right}(a)$ of an n-tuple $a \in A^n$ is the set of $s \in \mathbb{C}^n$ for which $a - s$ has approximate left annihilators in A.

(*iii*) The *joint approximate point spectrum* $\tau(a) = \tau_A^{joint}(a)$ of an n-tuple $a \in A^n$ is the union of its left and right approximate point spectra.

Dash, Coburn and Schechter used the concept of joint spectrum most for solving interpolation problems [2,3]. Harte studied joint spectrum and obtained many useful results, in an example he showed that $\sigma(a)$ is compact in $\mathbb{C}^n$, but possibly empty. For n-tuples $f = (f_1, \dots, f_n)$ of noncommutative polynomials in m variables on A there is inclusion $f\sigma(a) \subset \sigma f(a)$ [6, Theorem 3.2] and equality holds if the elements a_j $(j = 1, \dots, n)$ commute with one another [6, Theorem 4.3]. There are many works related to this notion that they are considered on algebras such as Hilbert spaces, commutative Banach algebra, noncommutative normal operators and Waelbroech algebras [4, 5, 7, 8, 10–12].

Let A be a Banach algebra and $M_m(A)$ be the algebra of $m \times m$ matrices with entries in A. We denote the subalgebra of $M_m(A)$ which contains all upper triangular matrices by $U_m(A)$; *i.e.* $U_m(A) = \{T = (T_{ij}) | \ T_{ij} = 0, \textit{whenever } i > j\}$ and by $M_{m,k}(A)$ we mean all $m \times k$ matrices with entries in A. The algebra $M_m(A)$ with the following norm is a Banach algebra

$$\|T\| = \sup_{1 \le k \le m} [\sum_{j=1}^{m} \|T_{jk}\|_A].$$

If a Banach algebra A is unital, then $M_m(A)$ is a unital Banach algebra with unit I_m, where it is a $m \times m$ matrix with $a_{ii} = 1$ and $a_{ij} = 0$ for $i \neq j$.

Let A be a unital Banach algebra, we may write U_m or U for $U_m(A)$, $\sigma(T)$ for $\sigma(T;A)$ or $\sigma_A(T)$. Denote the algebra of $m \times m$ diagonal matrices with entries in A by $D_m(A)$. For $\{R_k\}_{1\leq k\leq m} \subseteq A$, let $dg(R_1, R_2, \ldots, R_m)$ be the matrix in $D_m(A)$ with diagonal entries $R_1, R_2, \ldots, R_m$. For $T \in M_m(A)$, let $dg(T) = dg(T_{11}, T_{22}, \ldots, T_{mm})$.

In this paper, we investigate joint spectrums of the above stated matrix algebras and our aim is characterizing these joint spectrums.

2. MAIN RESULTS

In the whole of this section, we assume that A is a unital Banach algebra. If $J = J_{kj} \in U_m(A)$ is strictly upper triangular, that is $J_{kk} = 0$ for all k, then $I_m - J$ is invertible since J is nilpotent (the inverse is $\sum_{k=0}^{m} J^k$).

Remark 2.1. Let $(T_1, \ldots, T_n) \in (U_m)^n$, then $\sigma(T_1, \ldots, T_n; M_m) \subseteq \sigma(T_1, \ldots, T_n; U_m)$ because if $(T_1, \ldots, T_n)$ is invertible in $(U_m)^n$, so that is invertible in $(M_m)^n$ (because $(U_m)^n \subseteq (M_m)^n$ implies $Inv((U_m)^n) \subseteq Inv((M_m)^n)$).

PROPOSITION 2.1 ([1]). *Assume that $T \in U_m(A)$. Then*

$$\sigma(T) = \bigcup_{k=1}^{m} \sigma(T_{kk}). \tag{2}$$

similar equalities hold for $\sigma^{right}(T)$ and $\sigma^{left}(T)$.

PROPOSITION 2.2. *Let $n-$tuple $(T_1, \ldots, T_n)$ in $U_m(A)$. Then*

$$\sigma(T_1, \ldots, T_n) = \bigcup_{k=1}^{m} (\sigma(T_1(kk)) \times \sigma(T_2(kk)) \times \cdots \times \sigma(T_n(kk))). \tag{3}$$

Proof. Let $(\lambda_1, \ldots, \lambda_n) \notin \sigma(T_1, \ldots, T_n)$, then $n-$tuple $(T_1 - \lambda_1 I_m, \ldots, T_n - \lambda_n I_m)$ has inverse like $n-$tuple $(S_1, \ldots, S_n)$ in $U_m(A)$. Thus

$$(S_1, \ldots, S_n)(T_1 - \lambda_1 I_m, \ldots, T_n - \lambda_n I_m) = I_m.$$

Clearly

$$\sum_{i=1}^{n} S_i(T_i - \lambda_i I_m) = I_m,$$

and so for any $1 \leq k \leq m$

$$\sum_{i=1}^{n} S_i(kk)(T_i(kk) - \lambda_i 1_A) = 1_A.$$

Then in terms of ideals, for any $1 \leq k \leq m$, the left ideal $\sum_{i=1}^{n} A(T_i(kk) - \lambda_i 1_A)$ generated by the $n-$tuple $(T_1(kk) - \lambda_1 1_A, \ldots, T_n(kk) - \lambda_n 1_A)$ equals to A and so contains 1_A. This implies that 1_A is in the sum of n ideals generated by $(T_i(kk) - \lambda_i 1_A)$ for $1 \leq i \leq n$'s and so for some j, 1_A is in ideal generated by $(T_j(kk) - \lambda_j 1_A)$. Then $\lambda_j \notin \sigma(T_j(kk))$ and so

$$(\lambda_1, \ldots, \lambda_n) \notin \sigma(T_1(kk)) \times \sigma(T_2(kk)) \times \cdots \times \sigma(T_n(kk)).$$

This argument holds for any k, therefore

$$(\lambda_1, \ldots, \lambda_n) \notin \bigcup_{k=1}^{m} (\sigma(T_1(kk)) \times \sigma(T_2(kk)) \times \cdots \times \sigma(T_n(kk))).$$

Conversely, let $(\lambda_1, \ldots, \lambda_n) \notin \bigcup_{k=1}^{m} (\sigma(T_1(kk)) \times \sigma(T_2(kk)) \times \cdots \times \sigma(T_n(kk)))$. Then for any $1 \leq k \leq m$,

$$(\lambda_1, \ldots, \lambda_n) \notin \sigma(T_1(kk)) \times \sigma(T_2(kk)) \times \cdots \times \sigma(T_n(kk)).$$

Thus, for arbitrary k there is, $1 \leq i_k \leq n$ such that $\lambda_{i_k} \notin \sigma(T_{i_k}(kk))$. Therefore $T_{i_k}(kk) - \lambda_{i_k} 1_A$ is invertible. Hence, there exists $S_{i_k}(kk)$ such that

$$(T_{i_k}(kk) - \lambda_{i_k} 1_A) S_{i_k}(kk) = 1_A.$$

Assume that $S_1, \ldots, S_n$ are $m \times m$ matrices that $S_{i_k}(kk)$ lies in kk entry of one of them such as S_l $(1 \leq l \leq n)$ which the other entries of S_l are zero and the other matrices are zero matrices. Thus,

$$\sum_{i=1}^{n} (T_i(kk) - \lambda_i 1_A) S_i(kk) = (T_{i_k}(kk) - \lambda_{i_k} 1_A) S_{i_k}(kk) = 1_A.$$

Then there is a strictly upper triangular matrix J such that

$$\sum_{i=1}^{n} (T_i - \lambda_i I_m) S_i = I_m + J.$$

Therefore $\sum_{i=1}^{n} (T_i - \lambda_i I_m) S_i = I_m - (-J)$ is invertible in $U_m(A)$. Let $X = \sum_{k=0}^{m} (-J)^k$ be the inverse of $\sum_{i=1}^{n} (T_i - \lambda_i I_m) S_i$, then

$$(\sum_{i=1}^{n} (T_i - \lambda_i I_m) S_i) X = \sum_{i=1}^{n} (T_i - \lambda_i I_m) S_i X = I_m.$$

This makes a right inverse for $n-$tuple $(T_1 - \lambda_1 I_m, \ldots, T_n - \lambda_n I_m)$. Similar argument shows that $n-$tuple $(T_1 - \lambda_1 I_m, \ldots, T_n - \lambda_n I_m)$ has a left inverse. Hence $(\lambda_1, \ldots, \lambda_n) \notin \sigma(T_1, \ldots, T_n)$. $\square$

Examples:

(i) If A is a commutative Banach algebra and $(T_1, \ldots, T_n) \in U_m(A)^n$, then

$$\sigma(T_1, \ldots, T_n) = \bigcup_{k=1}^{m} (\{\phi(T_1(kk)) : \phi \in \Phi\} \times \{\phi(T_2(kk)) : \phi \in \Phi\} \times \cdots \times \{\phi(T_n(kk)) : \phi \in \Phi\}).$$

The proof is trivial by Proposition 2.2 and using Theorem 2.2 [6] in case $n=1$.

(ii) If X be a non-empty compact Hausdorff space and $C(X)$ the algebra of all continuous functions on X with the sup-norm, $A=C(X)$ and $(T_1,\dots,T_n)\in U_m(A)^n$ then by Proposition 2.2 and using Theorem 2.2 [6] in case $n=1$ obtain equation (4) and by [9, page 23] for any $i=1,\dots,n$, we have $\sigma(T_i(kk))=(T_i(kk))(X)$ and so by Proposition 2.2 we have equation (5):

$$\sigma(T_1,\dots,T_n) = \bigcup_{k=1}^{m}(\tau(T_1(kk))\times\tau(T_2(kk))\times\cdots\times\tau(T_n(kk))) \tag{4}$$

$$= \bigcup_{k=1}^{m}((T_1(kk)(X))\times(T_2(kk)(X))\times\cdots\times(T_n(kk)(X)). \tag{5}$$

(iii) Consider the Banach algebra H^∞ of all bounded analytic functions on the open unit disc and $\mathcal{M}(H^\infty)$ the maximal ideal space, by [9, page 24], $\sigma_{H^\infty}(f)=\overline{f(D)}$ for $f\in H^\infty$.If $A=\mathcal{M}(H^\infty)$ and $(T_1,\dots,T_n)\in U_m(A)^n$, then by Proposition 2.2 we have

$$\sigma(T_1,\dots,T_n)=\bigcup_{k=1}^{m}(\overline{(T_1(kk)(D))}\times\overline{(T_2(kk)(D))}\times\cdots\times\overline{(T_n(kk)(D))}).$$

Suppose that $(T_1,\dots,T_n)$ be $n-$tuple of 2×2 block matrices, we use the following notation for every T_i entries:

(i) T_i^{11} be a $k\times k$ matrix in 11 position,

(ii) T_i^{12} be a $k\times(m-k)$ matrix in 12 position,

(iii) T_i^{22} be a $(m-k)\times(m-k)$ matrix in 22 position.

By the above stated notice we have the following:

Corollary 2.3. *Let*

$$T_1=\begin{pmatrix} T_1^{11} & T_1^{12}\\ 0 & T_1^{22}\end{pmatrix},\dots,T_n=\begin{pmatrix} T_n^{11} & T_n^{12}\\ 0 & T_n^{22}\end{pmatrix}$$

where $T_i^{11}\in M_k$, $T_i^{22}\in M_{m-k}$, $T_i^{12}\in M_{k,m-k}$. Then $T_i\in U_2$ for any $1\le i\le n$, and

(1) $\sigma^{left}(T_1^{11},\dots,T_n^{11})\subseteq\sigma^{left}(T_1,\dots,T_n)$;

(2) $\sigma^{right}(T_1^{22},\dots,T_n^{22})\subseteq\sigma^{right}(T_1,\dots,T_n)$.

Proof. (1) Let $\lambda=(\lambda_1,\dots,\lambda_n)\notin\sigma^{left}(T_1,\dots,T_n)$. Suppose that $S=(S_1,\dots,S_n)$ is a left inverse of $(T_1,\dots,T_n)$, so $\sum_{i=1}^{n}S_i(T_i-\lambda_i)=I_2$. Then

$$\begin{pmatrix} S_i^{11} & S_i^{12}\\ S_i^{21} & S_i^{22}\end{pmatrix}\begin{pmatrix} T_i^{11}-\lambda_i & T_i^{12}\\ 0 & T_i^{22}-\lambda_i\end{pmatrix}$$

$$= \begin{pmatrix} S_i^{11}(T_i^{11}-\lambda_i) & S_i^{11}T_i^{12}+S_i^{11}(T_i^{22}-\lambda_i) \\ S_i^{21}(T_i^{11}-\lambda_i) & S_i^{21}T_i^{12}+S_i^{22}(T_i^{22}-\lambda_i) \end{pmatrix} = I_2.$$

It follows that

$$\sum_{i=1}^{n} S_i^{11}(T_i^{11}-\lambda_i) = I_{M_k} = I_k.$$

This implies that

$$\lambda = (\lambda_1, \ldots, \lambda_n) \notin \sigma^{left}(T_1^{11}, \ldots, T_n^{11}).$$

(2) By the similar method in (1), it is clear that (2) holds. □

Similar to [1], assume that $T \in U_m$. Suppose that a sequence of 2×2 block matrices from T as follows:

For $1 \leq k < m$, let P_k be the block matrix

$$P_k = \begin{pmatrix} P_k[1,1] & P_k[1,2] \\ 0 & P_k[2,2] \end{pmatrix},$$

where the entries are defined by:

(i) $P_k[1,1]$ is the matrix in M_k with entries $\{T_{pq} : 1 \leq p, q \leq k\}$;

(ii) $P_k[1,2]$ is the matrix in $M_{k,m-k}$ with entries $\{T_{pq} : 1 \leq p \leq k, k < q \leq m\}$;

(iii) $P_k[2,2]$; is the matrix in M_{m-k} with entries $\{T_{pq} : k < p, q \leq m\}$.

Note that in the following Proposition we use similar definition of block matrices, ${P_i}^k[1,1]$, ${P_i}^k[1,2]$ and ${P_i}^k[2,2]$ for any T_i:

(i) $P_i^k[1,1]$ is the matrix in M_k with entries $\{T_i(pq) : 1 \leq p, q \leq k\}$;

(ii) $P_i^k[1,2]$ is the matrix in $M_{k,m-k}$ with entries $\{T_i(pq) : 1 \leq p \leq k, k < q \leq m\}$;

(iii) $P_i^k[2,2]$; is the matrix in M_{m-k} with entries $\{T_i(pq) : k < p, q \leq m\}$.

Proposition 2.4. *Let $(T_1, \ldots, T_n) \in (U_m)^n$. Then*

$$\sigma(T_1, \ldots, T_n; (M_m)^n) = \bigcup_{j=1}^{m} (\sigma(T_1(jj)) \times \cdots \times \sigma(T_n(jj))) \tag{6}$$

if and only if

$$\begin{aligned} \sigma(T_1, \ldots, T_n; (M_m)^n) &= (\sigma({P_1}^k[1,1]; (M_k)^n) \times \cdots \times \sigma({P_n}^k[1,1]; (M_k)^n)) \\ &\cup (\sigma({P_1}^k[2,2]; (M_{m-k})^n) \times \cdots \times \sigma({P_n}^k[2,2]; (M_{m-k}))^n), \end{aligned} \tag{7}$$

for any $1 \leq k \leq m$.

Proof. Suppose $\sigma(T_1, \ldots, T_n; (M_m)^n) = \bigcup_{j=1}^{m} (\sigma(T_1(jj)) \times \cdots \times \sigma(T_n(jj)))$. Without loss of generality we take $(\lambda_1, \ldots, \lambda_n) = 0$. Let

$$0 \notin (\sigma({P_1}^k[1,1];M_k) \times \cdots \times \sigma({P_n}^k[1,1];M_k) \cup (\sigma({P_1}^k[2,2];M_{m-k}) \times \cdots \times \sigma({P_n}^k[2,2];M_{m-k})),$$

so for any $1 \le k \le m$,

$$0 \notin (\sigma({P_1}^k[1,1];M_k) \times \cdots \times \sigma({P_n}^k[1,1];M_k) \tag{8}$$

and

$$0 \notin (\sigma({P_1}^k[2,2];M_{m-k}) \times \cdots \times \sigma({P_n}^k[2,2];M_{m-k})). \tag{9}$$

Now we use induction on k to show that for any $1 \le k \le m$,

$$0 \notin \bigcup_{j=1}^{k} \sigma(T_1(jj)) \times \cdots \times \sigma(T_n(jj)).$$

For $k=1$, ${P_i}^1[1,1] = T_i(11)$, then by (8), $0 \notin (\sigma(T_1(11)) \times \cdots \times \sigma(T_n(11))$. For $k = p-1$, assume that $0 \notin \bigcup_{j=1}^{p-1} \sigma(T_1(jj)) \times \cdots \times \sigma(T_n(jj))$.

Claim: For $k=p$, $0 \notin \bigcup_{j=1}^{p} \sigma(T_1(jj)) \times \cdots \times \sigma(T_n(jj))$.

It is sufficient to show that $0 \notin \sigma(T_1(pp)) \times \cdots \times \sigma(T_n(pp))$. By (8), $0 \notin (\sigma({P_i}^p[1,1];M_p)$ for some $1 \le i \le n$. Then for some $1 \le i \le n$, $0 \notin (\sigma^{right}({P_i}^p[1,1];M_p)$ and $0 \notin (\sigma^{left}({P_i}^p[1,1];M_p)$. Similarly by (9) for some $1 \le i \le n$, $0 \notin (\sigma^{right}({P_i}^p[2,2];M_{m-p})$ and $0 \notin (\sigma^{left}({P_i}^p[2,2];M_{m-p})$. Now if $0 \in \sigma(T_1(pp)) \times \cdots \times \sigma(T_n(pp))$, then for any $1 \le i \le n$, $0 \in \sigma(T_i(pp))$. Therefore $0 \in \sigma^{right}(T_i(pp))$ or $0 \in \sigma^{left}(T_i(pp))$. Rewrite definition of ${P_i}^p[1,1]$:

$$P_i^p[1,1] = \begin{pmatrix} P_i^{p-1}[1,1] & \begin{pmatrix} T_i(1p) \\ \vdots \\ T_i((p-1)p) \end{pmatrix} \\ 0 & T_i(pp) \end{pmatrix}.$$

If $0 \in \sigma^{right}(T_i(pp))$, by Corollary 2.3 case (1), $0 \in \sigma^{right}(P_i^p[1,1])$ and this is a contradiction. Hence for any $1 \le i \le n$, $0 \in \sigma^{left}(T_i(pp)$. Similarly,

$$P_i^{p-1}[2,2] = \begin{pmatrix} T_i(pp) & \begin{pmatrix} T_i(p(p+1)) & \dots & T_i(pm) \end{pmatrix} \\ 0 & P_i^p[2,2] \end{pmatrix}.$$

Again by Corollary 2.3 case (1), $0 \in \sigma^{left}(P_i^{p-1}[2,2])$, which is a contradiction. Thus

$$0 \notin \sigma(T_1(pp)) \times \cdots \times \sigma(T_n(pp)).$$

This shows that the claim is true, *i.e.* for any $1 \le k \le m$,

$$0 \notin \bigcup_{j=1}^{k} \sigma(T_1(jj)) \times \cdots \times \sigma(T_n(jj)).$$

Now set $k = m$ and obtain $0 \notin \bigcup_{j=1}^{m} \sigma(T_1(jj)) \times \ldots \times \sigma(T_n(jj))$ and by (6), $0 \notin \sigma(T_1, \ldots, T_n; M_m)$.

Let $0 \notin \sigma(T_1, \ldots, T_n; (M_m)^n)$. By (6), $0 \notin \bigcup_{j=1}^{m}(\sigma(T_1(jj)) \times \cdots \times \sigma(T_n(jj)))$, then for any $1 \leq j \leq m$, $0 \notin \sigma(T_1(jj)) \times \cdots \times \sigma(T_n(jj))$. Thus for some $1 \leq i \leq m$, $0 \notin \sigma(T_i(jj))$. By P_i^k's definition we have

$$T_i = \begin{pmatrix} P_i^k[1,1] & P_i^k[1,2] \\ 0 & P_i^k[2,2] \end{pmatrix} \quad \text{and} \quad P_i^k[1,1] = \begin{pmatrix} T_i(11) & \ldots & T_i(1k) \\ & \ddots & \\ 0 & \ldots & T_i(kk) \end{pmatrix} \in U_k.$$

By Proposition 2.1, $\sigma({P_i}^k[1,1]; U_k) = \bigcup_{j=1}^{k} \sigma(T_i(jj))$. Similar to above, by induction on k, we can show that for any $1 \leq k \leq m$, $0 \notin \sigma({P_i}^k[1,1]; U_k)$ and we know that $\sigma({P_i}^k[1,1]; M_k) \subseteq \sigma({P_i}^k[1,1]; U_k)$, so for some $1 \leq i \leq n$ and any $1 \leq k \leq m$, $0 \notin \sigma({P_i}^k[1,1]; M_k)$. Thus $0 \notin (\sigma({P_1}^k[1,1]; M_k) \times \cdots \times \sigma({P_n}^k[1,1]; M_k)$.

Similarly we can show that $0 \notin (\sigma({P_1}^k[2,2]; M_{m-k}) \times \cdots \times \sigma({P_n}^k[2,2]; M_{m-k}))$. Therefore

$$0 \notin (\sigma({P_1}^k[1,1]; M_k) \times \cdots \times \sigma({P_n}^k[1,1]; M_k) \cup (\sigma({P_1}^k[2,2]; M_{m-k}) \times \cdots \times \sigma({P_n}^k[2,2]; M_{m-k})).$$

Conversely, suppose that (7) holds. By remark 2.1, we have

$$\sigma(T_1, \ldots, T_n; M_m) \subseteq \sigma(T_1, \ldots, T_n; U_m) = \bigcup_{j=1}^{m} (\sigma(T_1(jj)) \times \cdots \times \sigma(T_n(jj))).$$

Let $0 \notin \sigma(T_1, \ldots, T_n; M_m)$, so by equation (7) we have

$$0 \notin (\sigma({P_1}^k[1,1]; M_k) \times \cdots \times \sigma({P_n}^k[1,1]; M_k)$$

and

$$0 \notin (\sigma({P_1}^k[2,2]; M_{m-k}) \times \cdots \times \sigma({P_n}^k[2,2]; M_{m-k})).$$

Similar to the above argument, by induction on k, we can show that

$$0 \notin \bigcup_{j=1}^{m} \sigma(T_1(jj)) \times \cdots \times \sigma(T_n(jj)).$$

Therefore $\bigcup_{j=1}^{m} \sigma(T_1(jj)) \times \cdots \times \sigma(T_n(jj)) \subseteq \sigma(T_1, \ldots, T_n; M_m)$. □

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Department of Mathematics,
Payame Noor University,
Tehran, Iran
mohammadzadeh83@gmail.com

Department of Mathematics,
Payame Noor University,
Tehran, Iran
ebadian.ali@gmail.com

Department of Mathematics,
University of Mahaghegh Ardabili,
Ardabil, Iran
haghnejad@uma.ac.ir