RATIONALITY IN QUALITATIVE GAMES

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We extend the study of the iterated elimination of strictly dominated strategies (IESDS) from Nash strategic games to a class of qualitative games. In this case, the IESDS process also leads us to a kind of "rationalizable" result. We define a dominance relation and a game reduction and we establish conditions under which a unique and non-empty maximal reduction exists.

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1. INTRODUCTION

Bernheim [4] and Pearce [18] studied the rationalizable strategic behavior in the framework of non-cooperative strategic games introduced by Nash [16]. The rational behavior of the players is a fundamental assumption in game theory. It implies that each strategic game can be characterized by a process of iterated elimination of strictly dominated strategies (IESDS). The result of this process is known as the maximal reduction of the game.

The iterated elimination of strictly dominated strategies has several different definitions. We must refer to the approaches of Gilboa, Kalai and Zemel [11,12], Milgrom and Roberts [15], Marx and Swinkels [14], Ritzberger [19], Dufwenberg and Stegeman [10], Chen, Long and Luo [6], or Apt [1–3] as some important ones in literature. Osborne and Rubinstein [17] and Rubinstein [20] also developed some topics concerning rationality.

The main problems concerning the IESDS procedure are related to the non-emptiness and to the uniqueness of the limit game. In the case of infinite games, the order of reductions is important, and the maximal reduction may not be unique if different paths are considered. Dufwenberg and Stegeman [10] proved the uniqueness and the non-emptiness of the maximal reduction for a strategic game with compact strategy sets and continuous payoff functions. Apt [2] treated the various definitions of IESDS in a unitary way, specifying the games where the definitions coincide. His approach is based on complete lattice and the study of operators.

In order to develop the ideas concerning the rationality, we consider a model which generalizes the strategic game. We consider the qualitative games which have a strategy set and a preference correspondence constructed by using the utility functions, for every player. Nash's equilibrium point is seen in this framework as a maximal element. We also consider different types of majorized correspondences which generalize the well-known semicontinuous ones. Subsequently, we work with U-majorized correspondences defined by Yuan and Tarafdar [23], $Q_\theta-$majorized correspondences introduced by Liu and Cai [13] and L_S-majorized correspondences due to G.X. Yuan [22]. We use theorems which prove the existence of maximal elements for qualitative games which have these types of considered correspondences. These results are due to Ding [9], Liu and Cai [13], and Chang [5].

Our new approach wants to emphasize that the IESDS process leads to a kind of ´rationalizable´ result in the extended games. In the new context, the idea of rationality, obtained in an iterated process of elimination the unfitted strategies, is underlined. We want to highlight the concept, rather than the context where it was initially defined. We introduce a dominance relation and a game reduction and we establish conditions under which a unique and non-empty maximal reduction exists. An open problem is to rigorously formalize concepts concerning rationality in different classes of games, and to rediscuss the problems in a unified framework which implies economic settings.

The paper is organized in the following way: Section 2 presents preliminaries concerning qualitative games. Section 3 contains the introduction of definitions and the problem of game reduction and order independence. The main results which concern the existence and the uniqueness of maximal reductions are settled in Section 4. In Section 5 other conditions for uniqueness of the maximal reduction are provided. The last section is dedicated to proving that the set of maximal elements is preserved in any game by the process of iterated elimination of strictly dominated strategies. Concluding remarks are stated at the end. A list with the main notations used in the paper is added in Appendix.

2. QUALITATIVE GAMES

Let I be a non-empty and countable set (the set of agents). For each $i \in I$, let G_i be a non-empty topological vector space representing the set of actions $\overline{G} = \prod_{i\in I} G_i$, and $P_i : \overline{G} \to 2^{G_i}$ be the preference correspondence.

The family $G = (G_i, P_i)_{i\in I}$ is said to be a *qualitative game.* A *maximal element* for G is defined as a point $s^* \in \overline{G}$ such that for each $i \in I$, $P_i(s^*) = \emptyset$.

Remark 1. A list with the main notations used in the paper is added in Appendix, in order to make the reading easier.

Example 1. Let $I = \{1, 2\}$, $G_1 = G_2 = [0, 1]$, and the symmetric functions $u_i : [0, 1] \times [0, 1] \to \mathbb{R}$, $i \in I$, be defined by $u_1(x, y) = x$ and $u_2(x, y) = y$ for each $(x, y) \in [0, 1] \times [0, 1]$.

The qualitative game corresponding to $\Gamma = (G_i, u_i)_{i\in I}$ is $G = (G_i, P_i)_{i\in I}$, where

$P_1(x, y) = \{z \in [0, 1] : u_1(z, y) > u_1(x, y)\} = (x, 1]$ if $(x, y) \in [0, 1) \times [0, 1]$ and

$P_1(x, y) = \emptyset$ if $(x, y) \in \{1\} \times [0, 1]$;

$P_2(x, y) = \{z \in [0, 1] : u_2(x, z) > u_2(x, y)\} = (y, 1]$ if $(x, y) \in [0, 1] \times [0, 1)$ and

$P_2(x, y) = \emptyset$ if $(x, y) \in [0, 1] \times \{1\}$;

$(1, 1)$ is a maximal element: $P_1(1, 1) = P_2(1, 1) = \emptyset$.

Now, we define a transitivity type of correspondences.

Definition 1. Let I be a non-empty and countable set, let $\overline{G} = \prod_{i\in I} G_i$ be a product space and, for some $i \in I$, let $P : \overline{G} \to 2^{G_i}$ be a correspondence. We say that P *has the property* T if $z \in P(s)$ and $s \in \overline{G}$ imply $\mathrm{cl}P(z, s_{-i}) \subset P(s)$.

Example 2. P_1 and P_2 from Example 1 have the property T :

if $z \in P_1(x, y)$ and $(x, y) \in [0, 1] \times [0, 1]$, then $z \in (x, 1]$ and $\mathrm{cl}P_1(z, y) = [z, 1] \subset (x, 1]$

Definition 2. Let I be a non-empty and countable set, let $\overline{G} = \prod_{i\in I} G_i$ be a product space and, for some $i \in I$, let $P, Q : \overline{G} \to 2^{G_i}$ be correspondences. We say that the pair (P, Q) *has the property* T if for each $s \in \overline{G}$, $P(s) \subset Q(s)$ and $z \in P(s)$ imply $Q(z, s_{-i}) \subset P(s)$.

Example 3. If for each $i \in \{1, 2\}$, we take $Q_i = \mathrm{cl}P_i$ in Example 1, we obtain that the pair (P_i, Q_i) has the property T.

Example 4. Let $P, Q : [0, 2] \times [0, 2] \to 2^{[0,2]}$ be defined by

$$P(x, y) = \begin{cases} (1, y], & \text{if } x \in [0, 1] \times (1, 2]; \\ \emptyset, & \text{otherwise} \end{cases} \quad \text{and}$$

$$Q(x, y) = \begin{cases} [x, y], & \text{if } 0 \le x < y \le 2; \\ [y, x], & \text{if } 0 \le y < x \le 2; \\ \{x\}, & \text{if } 0 \le x = y \le 2. \end{cases}$$

We have that $P(x, y) \subset Q(x, y)$ for each $(x, y) \in [0, 2] \times [0, 2]$, and if $z \in P(x, y)$, it follows that $x \in [0, 1]$, $y \in (1, 2]$ and $z \in (1, y]$, which imply $Q(z, y) = [z, y] \subset (1, y] = P(x, y)$.

We note that $x \in Q(x,y)$ for each $x \in [0,2] \times [0,2]$ and Q has convex closed values, so that Q verifies the assumptions stated in the hypothesis of the theorems from Section 4.

3. GAME REDUCTION

Firstly, this section gives preliminary definitions on restrictions, strict dominance and game reduction for qualitative games, which generalize the ones that exist in the literature and which are due to Dufwenberg and Stegeman [10]. The main theorems of the paper will be stated in Section 4, where we will consider mainly the game reduction defined in Section 3. The new classes of games will be introduced in the following sections with the main tools we will use in the proofs, that is, the maximal elements theorems. Several simple examples will illustrate our assertions and will underline the importance of the extension proposed to the reader.

We focus on the concepts of rationalizability and iterated elimination of strictly dominated strategies (IESDS). Both terms refer to the reasoning used by rational players. In the definition of the first concept, the deletion of a strategy depends on the type of strict dominance considered by the players.

Firstly, we introduce a relation of strict dominance for the qualitative games, with respect to a restriction.

Definition 3. A restriction of $G = (G_i, P_i)_{i \in I}$ is $H = (H_i, P_{i|\prod_{k\in I} H_k})_{i \in I}$, where $H_i \subseteq G_i$ for each $i \in I$.

We note that a restriction of a qualitative game is not a qualitative game, since for each $i \in I$, the images of $P_{i|\prod_{k\in I} H_k}$ are not contained in H_i. We use the term "restriction" as in [2], rather than "paring" as in [10].

In game theory, the players are supposed to be rational, that is, none of them could possibly use any strictly dominated strategy. In this paper, we will consider the following notion of strict dominance.

Definition 4. Given a restriction H of G and $i \in I$, the strict dominance relation $\overset{H}{\succ}_i$ on G_i, is defined as follows:

for $x, y \in G_i$, $y \overset{H}{\succ}_i x$ if $H_{-i} \neq \emptyset$ and $y \in \cap_{s_{-i} \in H_{-i}} P_i(x, s_{-i})$.

We will remove from G all the strategies that are strictly dominated with respect to H by some strategy in G.

If $H = G$, we obtain the following definition, which will be used in order to give Definition 5

for $x, y \in G_i$, $y \overset{G}{\succ}_i x$ if $G_{-i} \neq \emptyset$ and $y \in \cap_{s_{-i} \in G_{-i}} P_i(x, s_{-i})$.

Let us consider the restrictions K and H of G with the property that $H_i \subseteq K_i$ for each $i \in I$. We generalize the types of game reduction used by Dufwenberg and Stegeman [10] in the following way.

Definition 5. i) We define the reduction $K \to H$ if, for each $i \in I$ and $x \in K_i \backslash H_i$, there exists $y \in K_i$ such that $y \overset{K}{\succ_i} x$, that is, $K_{-i} \neq \emptyset$ and $\cap_{s_{-i} \in K_{-i}} P_i(x, s_{-i}) \cap K_i \neq \emptyset$.

ii) The reduction $K \to H$ is called fast if for each $i \in I$, $K_{-i} \neq \emptyset$ and $\cap_{s_{-i} \in K_{-i}} P_i(x, s_{-i}) \cap K_i \neq \emptyset$ for some $x \in K_i$ implies $x \notin H_i$.

We note that the notion of reduction captures the idea of removing the dominated strategies.

The definition we present below takes care of the possibility that the elimination process has a finite or countable infinite number of iteration steps.

Definition 6. The reduction $K \to^* H$ is defined by the existence of a (finite or countable infinite) sequence of restrictions R^t of H, $t = 0, 1, 2...$, such that $R^0 = K$, $R^t \to R^{t+1}$ fast for each $t \geq 0$ and $H_i = \cap_t R^t_i$ for each $i \in I$.

We note that at each step $t \geq 0$, in the reduction $R^t \to R^{t+1}$, for each $i \in I$, the images of the preference correspondences $P_{i|\prod_{k \in I} R^{t+1}_k}$ are intersected with $(R^t)_i$.

The outcome of the elimination process is provided by the 'maximal reduction'.

Definition 7. H is said to be a maximal $(\to^*)$-reduction of K if $K \to^* H$ and $H \to H'$ only for $H = H'$.

We are interested in finding conditions which can ensure the existence of a non-empty and unique maximal reduction of a game.

4. THE EXISTENCE AND UNIQUENESS OF MAXIMAL REDUCTIONS

In this section, we state the results concerning the iterated elimination of strictly dominated strategies for several classes of qualitative games. The existence and the uniqueness of maximal reductions are proven.

Theorem 2 gives conditions for the existence of a non-dominated element with respect to a dominance relation generated by a restriction. It considers qualitative games with L_S-majorized correspondences. A proof of the uniqueness of the maximal reduction of G is given in this case. This theorem will be used in order to obtain the main result of this paper, that is, Theorem 3.

4.1. $L_{\theta,S}$-MAJORIZED CORRESPONDENCES

Firstly, we recall the notion of $L_{\theta,S}$-majorized correspondence, which generalizes the classical correspondences with open lower sections. It was used by Chang in [5] in order to provide new results on maximal elements and equilibrium. Since the key of the proof of our main theorems is the application of a corollary of Chang's Theorem, which concerns the existence of maximal elements for the $L_{\theta,S}$-majorized correspondences, we begin our research by introducing these necessary preliminaries, before stating our results on maximal reductions of qualitative games.

A non-empty subset D of a topological space X is said to be *compactly open* (Ding [8]), if for every non-empty and compact subset C of X, $D \cap C$ is open in C.

Let $T : X \to 2^Y$ be a correspondence. The *lower sections* of T are defined by $T^{-1}(y) := \{x \in X : y \in T(x)\}$, for each $y \in Y$.

Let X be a topological space, Y be a non-empty subset of a vector space E, $\theta : X \to E$ be a function and let $P : X \to 2^Y$ be a correspondence.

Definition 8 (Yuan, [5]). *1)* P is said *to have compactly open lower sections in* X if for each $y \in Y$, the set $P^{-1}(y) = \{x \in X : y \in P(x)\}$ is compactly open in X.

2) P is said to be *of class* $L_{\theta,S}$, if $\theta(x) \notin \text{co}P(x)$ for each $x \in X$, and P has compactly open lower sections in X.

3) P is said to be $L_{\theta,S}$*-majorized,* if for each $x \in X$, there exists an open neighborhood $N(x)$ of x in X and a correspondence $P_x : X \to 2^Y$ such that:

i) for each $z \in N(x)$, $P(z) \subset P_x(z)$;

ii) for each $z \in N(x)$, $\theta(z) \notin \text{co}P_x(z)$ and

iii) for each $y \in Y$, $P_x^{-1}(y)$ is compactly open in X.

In this paper, we deal mainly with the case $X = Y$, which is a non-empty and convex subset of a topological vector space E and $\theta = I_X$, the identity map on X. In this case, we write L_S in place of $L_{\theta,S}$.

We also have the following theorem due to Chang [5] on non-compact spaces.

THEOREM 1 (Chang, [5]). *Let X be a convex subset of a Hausdorff topological vector space E and let $P : X \to 2^X$ be a L_S-majorized correspondence. Suppose that there exists a compact set D in X such that, for each finite subset S of X, there exists a convex and compact set K, which contains S and which satisfies $K \setminus \cup_{x\in K} P^{-1}(x) \subset D$. Then, there exists $x^* \in D$ such that $P(x^*) = \emptyset$.*

Further, we will use the following corollary of Chang's Theorem, where X is compact.

Corollary 1. *Let X be a compact and convex subset of a Hausdorff topological vector space E and let $P : X \to 2^X$ be a L_S-majorized correspondence. Then, there exists $x^* \in X$ such that $P(x^*) = \emptyset$.*

4.2. THE MAIN RESULTS

We are now stating the following key result, which will be used to prove Theorem 3. The demonstration of Theorem 2 is based on Corollary 1.

Theorem 2. *Let I be a non-empty and countable set of players. Let $G = (G_i, P_i)_{i \in I}$ be a qualitative game, $\overline{G} = \prod_{i \in I} G_i$, where for each $i \in I$, G_i is a non-empty, compact subset of a Hausdorff topological vector space, and let us suppose that there exists $Q_i : \overline{G} \to 2^{G_i}$ such that the pair (P_i, Q_i) satisfies the property T and the following assumptions are fulfilled:*

i) $s_i \in Q_i(s)$ for each $s \in G$;

ii) Q_i has convex and closed values;

iii) $P_i(., s_{-i})$ is L_S-majorized on G_i for each $s_{-i} \in G_{-i}$.

Then,

a) If $G \to^* H$ is a game reduction and if there exists $y \overset{H}{\succ}_i x$, for some $x, y \in G_i$ and $i \in I$, there exists $x^* \in H_i$ such that $z \overset{H}{\not\succ}_i x^* \overset{H}{\succ}_i x$ for each $z \in G_i$;

b) a non-empty maximal $\to^*$reduction of G is the unique maximal reduction of G.

Proof. a) Let R^t be the sequence of restrictions of G, $t = 0, 1, 2...$, such that $R^0 = G$, $R^t \to R^{t+1}$ for each $t \geq 0$ and $H_i = \cap_t R_i^t$ for each $i \in I$.

Let $i \in I$ be arbitrarily fixed. Assume that there exists $x, y \in G_i$ such that $y \overset{H}{\succ}_i x$. Let $Z_i = \cap_{s_{-i} \in H_{-i}} Q_i(y, s_{-i})$. According to i), we have that $Z_i \neq \emptyset$. The set Z_i is convex. It is also closed and included in G_i, which is compact, so Z_i is compact. Since $y \overset{H}{\succ}_i x$, we have that $H_{-i} \neq \emptyset$. Let $s^*_{-i} \in H_{-i}$ be fixed and $F_i : Z_i \to 2^{Z_i}$, $F_i = (P_i \cap Z_i)_{|Z_i \times \{s^*_{-i}\}}$. We note that, since $P_i(\cdot, s^*_{-i})$ is L_s-majorized on G_i and Z_i is a compact set, then, F_i is L_s-majorized on Z_i. In addition, we have already mentioned that Z_i is convex and compact. Hence, all conditions of Corollary 1 are fulfilled. According to this corollary, there exists $x^* \in Z_i$ such that $F_i(x^*) = \emptyset$ and thus, $P_i(x^*, s^*_{-i}) \cap Z_i = \emptyset$.

We have that $x^* \in Q_i(y, s_{-i})$ for each $s_{-i} \in H_{-i}$. The relation $y \overset{H}{\succ}_i x$ implies that $y \in P_i(x, s_{-i})$ for each $s_{-i} \in H_{-i}$ and since the pair (P_i, Q_i) has the property T on $\prod_{i \in I} H_i$, it follows that $x^* \in P_i(x, s_{-i})$ for each $s_{-i} \in H_{-i}$.

If there exists $z \in G_i$ such that $z \overset{H}{\succ}_i x^*$, that is, $z \in P_i(x^*, s_{-i}) \subset Q_i(x^*, s_{-i})$ for each $s_{-i} \in H_{-i}$, then $z \in Z_i$ and $z \in P_i(x^*, s^*_{-i})$, which is a contradiction. It remains that $z \overset{H}{\not\succ}_i x^* \overset{H}{\succ}_i x$ for each $z \in G_i$.

Now, we claim that $z \in H_i$. Indeed, since for each $t \in T$, $H_{-i} \subseteq R^t_{-i}$ and $\cap_{s_{-i} \in H_{-i}} P(x^*, s_{-i}) = \emptyset$, then, $\cap_{s_{-i} \in R^t_{-i}} P(x^*, s_{-i}) \cap R^t_i = \emptyset$. We conclude that $x^* \in R^t_i$ for each $t \geq 0$ and this implies $x^* \in H_i$. The claim is shown.

b) Let M and M' be maximal $(\to^*)-$ reductions of G, M being non-empty. Let us consider $G \to^* M'$ and R'^t, $t = 0, 1, 2...$, be the implied finite or infinite sequence of restrictions. If $M_i \nsubseteq M'_i$ for some i, it follows that $M_i \nsubseteq R'^t_i$, for each $t > T$ and for the largest T such that R'^{T+1}_i is well-defined and $M_i \subseteq R'^T_i$ for each $i \in I$. Let us take $x \in M_i \backslash R'^{T+1}_i$ for a fixed i. We have that $x \in R'^T_i \backslash R'^{T+1}_i$, so that there exists $y \in R'^T_i$ such that $y \in \cap_{s_{-i} \in R'^T_{-i}} P_i(x, s_{-i})$. Since, in addition, $\emptyset \neq M_i \subseteq R'^T_i$ for each $i \in I$, it follows that $y \in \cap_{s_{-i} \in M_{-i}} P_i(x, s_{-i})$. According to a) there exists $z^* \in M_i$ such that $z^* \in \cap_{s_{-i} \in M_{-i}} P_i(x, s_{-i})$, which contradicts the fact that M is a maximal $(\to^*)-$reduction. It remains that $M_i \subseteq M'_i$ for each $i \in I$ and therefore M' is non-empty. We also can prove that $M'_i \subseteq M_i$ for each $i \in I$, implying that $M = M'$. $\square$

Remark 2. We mention that the relevance of Theorem 2 comes from the fact that it is an extension of known results concerning maximal reduction of strategic games to qualitative games. This extension considers correspondences with weak continuities. A further research can be extended to other models of games, for instance, to abstract economy. This model is also a generalization of the exchange economy, introduced by Debreu [7]. So, the possible economic applications of our results refer to the situations which can be seen as strategic games, and also to the market scenarios which can be interpreted as exchange economies (this case can be analysed in the future). We recall that strategic games can be used to model the Cournot oligopoly, Bertrand competition, production with discontinuities, coordination situations, managerial applications, business and so on.

We establish the following corollary.

Corollary 2. *Let I be a non-empty, countable set of players. Let $G = (G_i, P_i)_{i \in I}$ be a qualitative game, $\overline{G} = \prod_{i \in I} G_i$, where for each $i \in I$, G_i is a non-empty compact subset of a Hausdorff topological vector space, $u_i : \overline{G} \to \mathbb{R}$ and the preference correspondence $P_i : \overline{G} \to 2^{G_i}$ is defined by $P_i(s) = \{x \in G_i : u_i(x, s_{-i}) > u_i(s)\}$ for each $s \in \overline{G}$. Suppose that the following assumptions are fulfilled:*

i) P_i has convex values;

ii) $P_i(.,s_{-i})$ is L_S-majorized on G_i for each $s_{-i}\in G_{-i}$.

Then,

a) If $G\to^* H$ is a game reduction and if there exists $y \overset{H}{\succ}_i x$, for some $x,y\in G_i$ and $i\in I$, there exists $x^*\in H_i$ such that $z \overset{H}{\not\succ}_i x^* \overset{H}{\succ}_i x$ for each $z\in G_i$;

b) a non-empty maximal $\to^*$reduction of G is the unique maximal reduction of G.

Proof. We note that if, for each $i\in I$, we define the correspondence $Q_i:\overline{G}\to 2^{G_i}$ by $Q_i(s)=\{x\in G_i: u_i(x,s_{-i})\geq u_i(s)\}$ for each $s\in\overline{G}$, then, the pair (P_i,Q_i) satisfies the property T and $x\in Q_i(x,s_{-i})$ for each $x\in G_i$ and $s_{-i}\in G_{-i}$. Then, we are under the hypotheses of Theorem 2. □

Since a correspondence of class L_S is L_S-majorized, we obtain Corollary 3.

Notation. We will use the following notation, for each $i\in I$ and $s_{-i}\in G_{-i}$:

$P_i^{s_{-i}}:G_i\to 2^{G_i}$ is the correspondences defined by $P_i^{s_{-i}}(x)=P_i(x,s_{-i})$ for each $x\in G_i$ and its lower sections are defined by $(P_i^{s_{-i}})^{-1}(y)=\{x\in G_i: y\in P_i(x,s_{-i})\}$ for each $y\in G_i$.

Corollary 3. *Let I be a non-empty, countable set of players. Let $G=(G_i,P_i)_{i\in I}$ be a qualitative game, $\overline{G}=\prod_{i\in I}G_i$, where for each $i\in I$, G_i is a non-empty compact subset of a Hausdorff topological vector space, and let us suppose that there exists $Q_i:\overline{G}\to 2^{G_i}$ such that the pair (P_i,Q_i) satisfies the property T and the following assumptions are fulfilled:*

i) $s_i\in Q_i(s)$ and $s_i\notin P_i(s)$ for each $s\in G$;

ii) Q_i has convex closed values and P_i has convex values;

iii) $(P_i^{s_{-i}})^{-1}(y)$ is compactly open in G_i for each $y\in G_i$ and $s_{-i}\in G_{-i}$, where $(P_i^{s_{-i}})(x)=P_i(x,s_{-i})$ for each $x\in G_i$.

Then,

a) If $G\to^* H$ is a a game reduction and if there exists $y \overset{H}{\succ}_i x$, for some $x,y\in G_i$ and $i\in I$, there exists $x^*\in H_i$ such that $z \overset{H}{\not\succ}_i x^* \overset{H}{\succ}_i x$ for each $z\in G_i$;

b) a non-empty maximal $\to^*$reduction of G is the unique maximal reduction of G.

Remark 3. We make here a short discussion concerning the conditions of Theorem 2. If, for each $i\in I$, $u_i:\overline{G}\to\mathbb{R}$ and $P_i,Q_i:\overline{G}\to 2^{G_i}$ are defined by $P_i(s)=\{x\in G_i: u_i(x,s_{-i})>u_i(s)\}$, respectively, $Q_i(s)=\{x\in G_i: u_i(x,s_{-i})\geq u_i(s)\}$ for each $s\in\overline{G}$, the condition i) is a very natural one. We note that $x \notin P_i(x, s_{-i})$ for each $x \in G_i$ and $s_{-i} \in G_{-i}$. This last relation is known as "irreflexivity of the preferences". Also in this simple case, the convexity of the images of correspondences P_i and Q_i $(i \in I)$ reflects the quasi-concavity of the functions u_i, which comes naturally from the convexity of the agents' preferences in their economic behavior. The closedness of the images of correspondences Q_i $(i \in I)$ is asked from mathematical reasons.

Corollary 2 expresses the particular case presented above. The condition iii) of Theorem 2 refers to a property of preference correspondences, which are supposed to be L_S-majorized in the ith argument. This property generalizes the notion of correspondences with open lower sections, which is very common in game theory. A reference paper on this topic is the one written by Yannelis and Prabhakar [21], where the authors studied the existence of equilibrium points for the model of abstract economy with correspondences having open lower sections. This last model extends the qualitative game, since it has not only preference correspondences, but also constraint ones (the meaning is that the players can make their choices only from certain sets). Corollary 3 illustrates the important case of the qualitative games with preference correspondences having compactly open lower sections.

We recall that a function $f : X \to \mathbb{R} \cup \{-\infty, \infty\}$ is *upper semicontinuous* if and only if $\{x \in X : f(x) < \alpha\}$ is an open set for every $\alpha \in R$. The game $\Gamma = (G_i, u_i)_{i \in I}$ is called *own-upper semicontinuous* [10], if $u_i(\cdot, s_{-i})$ is upper semicontinuous for each $i \in I$ and for each $s_{-i} \in G_{-i}$.

Remark 4. Theorem 2 and its corollary subsume Dufwenberg and Stageman's result concerning the order independence. We will show that the qualitative games with the hypotheses we assumed in the above theorem include the own-upper semicontinuous strategic games.

Let us suppose that the preference correspondences are defined by using the functions $u_i : \overline{G} \to \mathbb{R}$, $i \in I$. Then, $P_i(s) = \{x \in G_i : u_i(x, s_{-i}) > u_i(s)\}$ for each i. If for each $s_{-i} \in G_{-i}$, $u_i(\cdot, s_{-i})$ is upper semicontinuous, then, for each fixed $y \in G_i$, $(P_i^{s_{-i}})^{-1}(y)$ is an open set. Consequently, Corollary 3 can be applied.

Remark 5. As Dufwenberg and Stageman [10] proved, the existence of the undominated strategies does not ensure the non-emptiness of the maximal reductions or the order independence of the IESDS procedure. In order to see this, the reader is referred to Examples 1 and 2 in [10]. Example 5 (Example 1 in [10] revisited) shows that the above observation maintains for our model of game.

The following example shows that our results handle games that violate own-upper semicontinuity.

Example 5. Let $I = \{1,2\}$, $G_1 = G_2 = [0,2]$. For each $i \in \{1,2\}$, let the symmetric functions $u_i : [0,2] \times [0,2] \to \mathbb{R}$, $i \in I$, be defined in the following way:

$$u_1(x,y) = \begin{cases} 2x+1, \text{ if } x \in [0,\frac{1}{2}), y \in [0,2]; \\ 2, \text{ if } x \in [\frac{1}{2},1], \ y \in [0,2]; \text{ and} \\ -x+2, \text{ if } x \in (1,2], \ y \in [0,2] \end{cases}$$

$$u_2(x,y) = \begin{cases} 2y+1, \text{ if } x \in [0,2] \text{ and } y \in [0,\frac{1}{2}); \\ 2, \text{ if } x \in [0,2] \text{ and } y \in [\frac{1}{2},1]; \\ -y+2, \text{ if } x \in [0,2] \text{ and } y \in (1,2]. \end{cases}$$

We note that $u_1(x,\cdot)$ and $u_2(\cdot,y)$ are not upper semicontinuous for each $x \in [0,2]$ and respectively for each $y \in [0,2]$.

The correspondences $P_i : [0,2] \times [0,2] \to 2^{[0,2]}$, $i \in I$, are defined by:

$$P_1(x,y) = \{z \in [0,2] : u_1(z,y) > u_1(x,y)\} = \begin{cases} (x,1], \text{ if } (x,y) \in [0,\frac{1}{2}) \times [0,2]; \\ \emptyset, \text{ if } (x,y) \in [\frac{1}{2},1] \times [0,2]; \\ [0,x), \text{ if } (x,y) \in (1,2] \times [0,2]. \end{cases}$$

$$P_2(x,y) = \{z \in [0,2] : u_2(x,z) > u_2(x,y)\} = \begin{cases} (y,1], \text{ if } (x,y) \in [0,2] \times [0,\frac{1}{2}); \\ \emptyset, \text{ if } (x,y) \in [0,2] \times [\frac{1}{2},1]; \\ [0,y), \text{ if } (x,y) \in [0,2] \times (1,2]. \end{cases}$$

Note that P_1 and P_2 have convex values; $x \notin P_1(x,y)$ and $y \notin P_2(x,y)$ for each $x,y \in [0,2]$.

For each $y \in [0,2]$, $P_1^y : [0,2] \to 2^{[0,2]}$ is defined by

$$P_1^y(x) = \begin{cases} (x,1], \text{if } x \in [0,\frac{1}{2}); \\ \emptyset, \text{ if } x \in [\frac{1}{2},1]; \\ [0,x), \text{ if } x \in (1,2]. \end{cases}$$

For each $x \in [0,2]$, $P_1^x : [0,2] \to 2^{[0,2]}$ is defined by

$$P_1^x(y) = \begin{cases} (y,1], \text{if } y \in [0,\frac{1}{2}); \\ \emptyset, \text{ if } y \in [\frac{1}{2},1]; \\ [0,y), \text{ if } y \in (1,2]. \end{cases}$$

P_1^y has open lower sections in the topology of $[0,2]$, for each $y \in [0,2]$.

Indeed, $(P_1^y)^{-1}(0) = (1,2]$.

If $z \in (0,1]$, $(P_1^y)^{-1}(z) = \{x \in [0,2] : z \in P_1^y(x)\} = [0,z) \cup (1,2]$.

If $z \in (1,2)$, $(P_1^y)^{-1}(z) = \{x \in [0,2] : z \in P_1^y(x) = [0,x)\} = (1,2]$.

Similarly, we can prove that P_2^x has open lower sections in the topology of $[0,2]$, for each $x \in [0,1]$.

We set the correspondences $Q_i : [0,2] \times [0,2] \to 2^{[0,2]}$, $i \in I$, as follows:

$$Q_1(x,y) = \{z \in [0,2] : u_1(z,y) \geq u_1(x,y)\} = \begin{cases} [x,1], \text{ if } (x,y) \in [0,\frac{1}{2}) \times [0,2]; \\ [\frac{1}{2},1], \text{ if } (x,y) \in [\frac{1}{2},1] \times [0,2]; \\ [0,x], \text{ if } (x,y) \in (1,2] \times [0,2]; \end{cases}$$

$$Q_2(x,y) = \{z \in [0,2] : u_2(x,z) \geq u_2(x,y)\} = \begin{cases} [y,1], \text{ if } (x,y) \in [0,2] \times [0,\frac{1}{2}); \\ [\frac{1}{2},1], \text{ if } (x,y) \in [0,2] \times [\frac{1}{2},1]; \\ [0,y], \text{ if } (x,y) \in [0,2] \times (1,2]. \end{cases}$$

Note that Q_1 and Q_2 have convex and closed values; $x \in Q_1(x,y)$ and $y \in Q_2(x,y)$ for each $x, y \in [0,2]$.

Now, we prove that, for each $i \in \{1,2\}$, the pair (P_i, Q_i) has the property T.

Let $i = 1$ and $(x,y) \in [0,2] \times [0,2]$.

If $x \in [0,\frac{1}{2})$, $P_1(x,y) \subset Q_1(x,y)$ and if $z \in P_1(x,y) = (x,1]$, then, $Q_1(z,y) = [z,1] \subseteq (x,1] = P_1(x,y)$.

If $x \in (1,2]$, $P_1(x,y) \subset Q_1(x,y)$ and if $z \in P_1(x,y) = [0,x)$, then, $Q_1(z,y) = [0,z] \subseteq [0,x) = P_1(x,y)$.

Therefore, for each $(x,y) \in [0,2] \times [0,2]$, $P_1(x,y) \subset Q_1(x,y)$ and $z \in P_1(x,y)$ imply $Q_1(z,y) \subset P_1(x,y)$, and then, the pair (P_1, Q_1) has the property T.

We can show, similarly, that the pair (P_2, Q_2) has the property T.

All the assumptions of Corollary 3 are satisfied. There exists H, the unique non-empty maximal $\to^*$reduction of G, where $H_1 = H_2 = [\frac{1}{2}, 1]$.

Note that there exists $x, y \in G_i$, $i = 1, 2$ such that $y \overset{H}{\succ}_i x$. There also exists $x^* = 2 \in H_1 \cap H_2$ such that $z \overset{H}{\not\succ}_i x^* \overset{H}{\succ}_i x$ for each $z \in G_i$ and $i \in \{1,2\}$.

Remark 6. For $H = G$, we obtain that under the conditions of Corollary 3, we have the following:

If there exists $y \overset{G}{\succ}_i x$, for some $x, y \in G_i$ and $i \in I$, there exists $x^* \in G_i$ such that $z \overset{G}{\not\succ}_i x^* \overset{G}{\succ}_i x$ for each $z \in G_i$.

The next theorem is the main result of our paper and it concerns the class of the qualitative games.

Theorem 3. *Let I be a non-empty, countable set of players. Let $G = (G_i, P_i)_{i \in I}$ be a qualitative game, $\overline{G} = \prod_{i \in I} G_i$, where for each $i \in I$, G_i is a non-empty and compact subset of a Hausdorff topological vector space, and let us suppose that there exists $Q_i : \overline{G} \to 2^{G_i}$ such that the pair (P_i, Q_i) satisfies the property T and the following assumptions are fulfilled:*

i) $s_i \notin P_i(s)$ and $s_i \in Q_i(s)$ for each $s \in G$;

ii) Q_i and P_i have convex and closed values;

iii) $P_i^{s_{-i}}$ has compactly open lower sections in G_i for each $s_{-i} \in G_{-i}$.

Then, G has a unique maximal $\rightarrow^*$reduction M. Further, for each $i \in I$, *M_i is non-empty, compact and $P_{i|M}$ has compactly open lower sections.*

Proof. Further, we will prove that G has a non-empty maximal reduction. The proof of its uniqueness is a consequence of Corollary 3.

1) We first establish that if $G \rightarrow H$ is fast and G_i is compact for each $i \in I$, then, H_i is compact and non-empty for each $i \in I$. For this purpose, let us choose $i \in I$ such that $H_i \neq G_i$. Then, $y \overset{G}{\succ_i} x$ for some $x, y \in G_i$ and consequently, the set H_i is non-empty.

Furthermore, we will prove that H_i is compact. Let $y \in H_i$ and let us define $Z(y) = C_{\overline{G}} P_i^{-1}(y)$. According to the assumption i), $Z(y) \neq \emptyset$. According to iii), $P_i^{s_{-i}}$ has compactly open lower sections in G_i for each $s_{-i} \in G_{-i}$, then, P_i has compactly open lower sections in $\overline{G}$. Hence, $Z(y)$ is closed in the compact set $\overline{G}$, and therefore, it is compact.

Let us define $Z_i(y) :=\mathrm{pr}_i Z(y)$. We have that $Z_i(y)$ is a non-empty and closed set in G_i, with $y \in Z_i(y)$.

Now, we prove that $H_i = \cap_{y \in H_i} Z_i(y)$. In order to show that $H_i \subseteq \cap_{y \in H_i} Z_i(y)$, we consider $z \in G_i$. If for every $y \in H_i$, $z \notin Z_i(y)$, it follows that $y \in P_i(z, s_{-i})$ for each $s_{-i} \in G_{-i}$, then $z \notin H_i$ and therefore, $C_{G_i}(\cap_{y \in H_i} Z_i(y)) \subset C_{G_i} H_i$, which implies $H_i \subseteq \cap_{y \in H_i} Z_i(y)$. Now, we want to show that $\cap_{y \in H_i} Z_i(y) \subset H_i$, that is $C_{G_i} H_i \subseteq C_{G_i}(\cap_{y \in H_i} Z_i(y))$.

If $z \notin H_i$, then there exists $x \in G_i$ such that $x \in P_i(z, s_{-i})$ for each $s_{-i} \in G_{-i}$.

According to Remark 7, there exists $x^* \in G_i$ such that $x^* \in P_i(z, s_{-i})$ for each $s_{-i} \in G_{-i}$. It follows that $z \notin Z_i(x^*)$, therefore, $z \notin \cap_{y \in G_i} Z_i(y)$, and, hence, $C_{G_i} H_i \subseteq C_{G_i}(\cap_{y \in H_i} Z_i(y))$. Since $H_i = \cap_{y \in H_i} Z_i(y)$, H_i is closed in G_i and therefore, it is compact.

2) Let R^t, $t = 0, 1, ...$ denote the unique sequence of games of G such that $R^0 = G$ and $R^t \rightarrow R^{t+1}$ is fast for each t. Result 1) implies that R^t is compact and non-empty for each t, so that, for each $i \in I$, $M_i = \cap_t R_i^t$ is compact and non-empty. According to iii), it follows that $P_i^{-1}(y)$ is open in M_i. We still have to show that M is a maximal $\rightarrow^*$reduction of G.

Let's consider $i \in I$ and $x \in M_i$. We will prove that x is not dominated by any $y \in M_i$. Let $y \in M_i$ and let $A = C_{G_{-i}} B$, where $B = \{s_{-i} \in G_{-i} : y \in P_i(x, s_{-i})\}$. Then $A = C_{G_{-i}}\{s_{-i} \in G_{-i} : (x, s_{-i}) \in P_i^{-1}(y)\}$.

If $A \cap R_{-i}^t = \emptyset$ for every t such that $R^t \neq M$, then $y \in P_i(x, s_{-i})$ for each $s_{-i} \in R_{-i}^t$, that is $y \overset{R^t}{\succ_i} x$, which contradicts $x \in M_i$. Therefore, $A \cap R_{-i}^t$

is non-empty and compact for every t, such that $R^t \neq M$. This fact implies $A \cap M_{-i}$ is non-empty and then, $y \notin P_i(x, s_{-i})$ for each $s_{-i} \in M_{-i}$, that is $y \overset{M}{\not\succ}_i x$. $\square$

Remark 7. As we have seen in Remark 5, Theorem 3 subsumes the compact continuous payoff games and, therefore, it is indeed an extension of Theorem 1 due to Dufwenberg and Stegeman [10].

Remark 8. Theorem 3 handles discontinuous payoff games as it can be seen in Example 6.

5. OTHER CONDITIONS WHICH IMPLY THE UNIQUENESS OF MAXIMAL REDUCTIONS

In this section, we will establish other versions of Theorem 2. Theorem 5 and Theorem 7 prove the uniqueness of the maximal ($\to^*$) reduction of a game G (if it exists) in the case the preference correspondences are $U_\theta-$majorized or Q_θ-majorized, that is, if they have topological properties which generalize upper semicontinuity or lower semicontinuity. The main tools for the proofs are the maximal element theorem for qualitative games. The hypotheses of these results are different from those of the ones presented above, so that the new variants deserve to be stated.

5.1. U-MAJORIZED CORRESPONDENCES

We will begin by presenting the notions of generalized topological properties of the correspondences and the maximal element theorem which will be used in the proof of the theorem established in the next subsection.

Let X, Y be topological spaces and let $T : X \to 2^Y$ be a correspondence. T is said to be *upper semicontinuous* if, for each $x \in X$ and each open set V in Y with $T(x) \subset V$, there exists an open neighborhood U of x in X such that $T(y) \subset V$ for each $y \in U$.

The notion of U-majorized correspondence is given below. It generalizes the classical upper semicontinuous correspondences.

Definition 9 (Yuan and Tarafdar, [23]). Let X be a topological space and Y be a non-empty subset of a topological vector space E, $\theta : X \to E$ a function and $P : X \to 2^Y$ a correspondence.

1) P is *of class* U_θ (or U) if:

i) for each $x \in X$, $\theta(x) \notin P(x)$ and

ii) P is upper semicontinuous with closed convex values in Y;

2) A correspondence $P_x : X \to 2^Y$ is a *U_θ-majorant of P at x* if there exists an open neighborhood $N(x)$ of x such that

i) for each $z \in N(x)$, $P(z) \subset P_x(z)$ and $\theta(z) \notin P_x(z)$;

ii) P_x is upper semicontinuous with closed convex values;

3) P is *$U_\theta-$majorized* if for each $x \in X$ with $P(x) \neq \emptyset$, there exists a U_θ-majorant P_x of P at x.

When we deal with the case $X = Y$, which is a non-empty and convex subset of a topological vector space E and $\theta = I_X$, the identity map on X, we write U in place of U_θ.

The following theorem is Ding's result on the existence of maximal elements for $U-$majorized correspondences. It will be used in the next subsection to prove Theorem 5, which states the uniqueness of maximal reductions for qualitative games with $U-$majorized correspondences.

THEOREM 4 (Ding, [9]). *Let X be a non-empty subset of a Hausdorff locally convex topological vector space and D a non-empty and compact subset of X. Let $P : X \to 2^D$ be a $U-$majorized correspondence. Then, there exists $x^* \in coD$ such that $P(x^*) = \emptyset$.*

5.2. THE UNIQUENESS OF MAXIMAL REDUCTIONS FOR GAMES WITH U-MAJORIZED CORRESPONDENCES

The main result of this subsection is Theorem 5. Its proof is based on Ding's Theorem, which gives conditions for the existence of the maximal elements for $U-$majorized correspondences. We notice that the upper semicontinuity of the correspondences is widely used in many economic applications which are modelled as games.

THEOREM 5. *Let I be a non-empty, countable set of players. Let $G = (G_i, P_i)_{i\in I}$ be a qualitative game, $\overline{G} = \prod_{i\in I} G_i$, where for each $i \in I$, G_i is a non-empty compact subset of a Hausdorff topological vector space and let us suppose that there exists $Q_i : \overline{G} \to 2^{G_i}$ such that the pair (P_i, Q_i) satisfies the property T and the following assumptions are fulfilled:*

i) $s_i \in Q_i(s)$ for each $s \in G$;

ii) Q_i has convex and closed values;

iii) $P_i(., s_{-i})$ is $U-$majorized on G_i for each $s_{-i} \in G_{-i}$.

Then,

a) If $G \to^* H$ is a game reduction and if there exists $y \overset{H}{\succ}_i x$, for some $x, y \in G_i$ and $i \in I$, there exists $x^* \in H_i$ such that $z \overset{H}{\not\succ}_i x^* \overset{H}{\succ}_i x$ for each *$z \in G_i$;*

b) a non-empty maximal $(\to^*)$ *reduction of* G *is the unique maximal* $(\to^*)$ *reduction of* G.

Proof. a) *Let* R^t be the sequence of restrictions of G, $t = 0, 1, 2...$, such that $R^0 = G$, $R^t \to R^{t+1}$ for each $t \geq 0$ and $H_i = \cap_t R_i^t$ for each $i \in I$.

Let $i \in I$ be arbitrarily fixed. Assume that there exists $x, y \in G_i$ such that $y \overset{H}{\succ}_i x$. Let us define $Z_i = \cap_{s_{-i} \in H_{-i}} Q(y, s_{-i})$. According to i), we have that $Z_i \neq \emptyset$. The set Z_i is convex and closed, so it is compact. Since $y \overset{H}{\succ}_i x$, we have that $H_{-i} \neq \emptyset$. Let $s_{-i}^* \in H_{-i}$ be fixed and $F_i : Z_i \to 2^{Z_i}$, $F_i = (P_i \cap Z_i)_{|Z_i \times \{s_{-i}^*\}}$. According to Ding's Theorem, which is applied for $X = D = G_i$ and $P = F_i$, there exists $x^* \in Z_i$ such that $F_i(x^*) = \emptyset$, and consequently, $P_i(x^*, s_{-i}^*) \cap Z_i = \emptyset$.

We have that $x^* \in Q_i(y, s_{-i})$ for each $s_{-i} \in H_{-i}$. The relation $y \overset{H}{\succ}_i x$ implies that $y \in P_i(x, s_{-i})$ for each $s_{-i} \in H_{-i}$ and since the pair (P_i, Q_i) has the property T on $\prod_{k \in I} H_k$, it follows that $x^* \in P_i(x, s_{-i})$ for each $s_{-i} \in H_{-i}$.

If there exists $z \in G_i$ such that $z \overset{H}{\succ}_i x^*$, that is, $z \in P_i(x^*, s_{-i}) \subset Q_i(x^*, s_{-i})$ for each $s_{-i} \in H_{-i}$, then $z \in Z_i$ and $z \in P_i(x^*, s_{-i}^*)$, which is a contradiction. It remains that $z \overset{H}{\not\succ}_i x^* \overset{H}{\succ}_i x$ for each $z \in G_i$.

Now, we claim that $z \in H_i$. Indeed, since for each $t \in T$, $H_{-i} \subseteq R_{-i}^t$ and $\cap_{s_{-i} \in H_{-i}} P(x^*, s_{-i}) = \emptyset$, then, $\cap_{s_{-i} \in R_{-i}^t} P(x^*, s_{-i}) \cap R_i^t = \emptyset$. We conclude that $x^* \in R_i^t$ for each $t \geq 0$ and this implies $x^* \in H_i$. The claim is shown.

b) The proof is similar to the proof of Theorem 2, b). □

We obtain the following corollary for qualitative games having upper semicontinuous correspondences P_i, $i \in I$.

Corollary 4. *Let* I *be a non-empty, countable set of players. Let* $G = (G_i, P_i)_{i \in I}$ *be a qualitative game,* $\overline{G} = \prod_{i \in I} G_i$, *where for each* $i \in I$, G_i *is a non-empty and compact subset of a Hausdorff topological vector space and let us suppose that there exists* $Q_i : \overline{G} \to 2^{G_i}$ *such that the pair* (P_i, Q_i) *satisfies the property* T *and the following assumptions are fulfilled:*

i) $s_i \in Q_i(s)$ *and* $s_i \notin P_i(s)$ *for each* $s \in G$;

ii) Q_i *has convex and closed values;*

iii) P_i *is upper semicontinuous, with closed and convex values in* G_i.

Then,

a) If $G \to^* H$ *is a game reduction and if there exists* $y \overset{H}{\succ}_i x$, *for some* $x, y \in G_i$ *and* $i \in I$, *there exists* $x^* \in H_i$ *such that* $z \overset{H}{\not\succ}_i x^* \overset{H}{\succ}_i x$ for each $z \in G_i$;

b) a non-empty maximal $(\to^*)$ *reduction of* G *is the unique maximal* $(\to^*)$ *reduction of* G.

The following example shows that, under the hypotheses of the above corollary, the game G has a non-empty unique maximal $(\to^*)$ reduction.

Example 6. Let $I = \{1,2\}$, $G_1 = G_2 = [0,2]$, and let the symmetric functions $u_i : [0,2] \times [0,2] \to \mathbb{R}$, $i \in I$, be defined in the following way:

$$u_1(x,y) = \begin{cases} \frac{1}{2} \text{ if } x \in [0,1] \text{ and } y \in [0,2]; \\ x \text{ if } x \in (1,2] \text{ and } y \in [0,2] \end{cases} \text{ and}$$

$$u_2(x,y) = \begin{cases} \frac{1}{2} \text{ if } x \in [0,2] \text{ and } y \in [0,1]; \\ y \text{ if } x \in [0,2] \text{ and } y \in (1,2]. \end{cases}$$

We note that $u_1(x,\cdot)$ and $u_2(\cdot,y)$ are not upper semicontinuous for each $x \in [0,2]$, respectively for each $y \in [0,2]$.

Then,

$$P_1(x,y) = \{z \in [0,2] : u_1(z,y) > u_1(x,y)\} = \begin{cases} (1,2], \text{ if } (x,y) \in [0,1] \times [0,2]; \\ (x,2], \text{ if } (x,y) \in (1,2] \times [0,2] \end{cases}$$

and

$$P_2(x,y) = \{z \in [0,2] : u_2(x,z) > u_2(x,y)\} = \begin{cases} (1,2], \text{ if } (x,y) \in [0,2] \times [0,1]; \\ (y,2], \text{ if } (x,y) \in [0,2] \times (1,2]. \end{cases}$$

$P_1(\cdot,y)$ and $P_2(x,\cdot)$ are upper semicontinuous for each $y \in [0,2]$, respectively for each $x \in [0,2]$.

Let us set the correspondences $Q_1, Q_2 : [0,2] \times [0,2] \to 2^{[0,2]}$ defined by

$$Q_1(x,y) = \{z \in [0,2] : u_1(z,y) \geq u_1(x,y)\} = \begin{cases} [0,2] \text{ if } (x,y) \in [0,1] \times [0,2]; \\ [x,2] \text{ if } (x,y) \in (1,2] \times [0,2] \end{cases}$$

and

$$Q_2(x,y) = \{z \in [0,2] : u_2(x,) \geq u_2(x,y)\} = \begin{cases} [0,2] \text{ if } (x,y) \in [0,2] \times [0,1]; \\ [y,2] \text{ if } (x,y) \in [0,2] \times (1,2]. \end{cases}$$

Note that Q_1 and Q_2 have convex and closed values; $x \in Q_1(x,y)$ and $y \in Q_2(x,y)$ for each $x,y \in [0,2]$.

Now, we prove that, for each $i \in \{1,2\}$, the pair (P_i, Q_i) has the property T.

Let $i = 1$ and $(x,y) \in [0,2] \times [0,2]$.

If $x \in [0,1)$, $P_1(x,y) \subset Q_1(x,y)$ and if $z \in P_1(x,y) = (1,2]$, then, $Q_1(z,y) = [z,2] \subseteq (1,2] = P_1(x,y)$.

If $x \in (1,2]$, $P_1(x,y) \subset Q_1(x,y)$ and if $z \in P_1(x,y) = (x,2]$, then, $Q_1(z,y) = [z,2] \subseteq (x,2] = P_1(x,y)$.

Therefore, for each $(x,y) \in [0,2] \times [0,2]$, $P_1(x,y) \subset Q_1(x,y)$ and $z \in P_1(x,y)$ imply $Q_1(z,y) \subset P_1(x,y)$, and then, the pair (P_1, Q_1) has the property T.

We can show, similarly, that the pair (P_2, Q_2) has the property T.

All the assumptions of Corollary 4 are satisfied. By applying this result, we can assert that there exists a unique non-empty maximal $\rightarrow^*$reduction of G.

By eliminating $[0,1]$ for $i \in \{1,2\}$, we obtain $R_1^1 = R_2^1 = (1,2]$.

$P_1(x,y) = (x,2]$ if $(x,y) \in (1,2] \times (1,2]$ and

$P_2(x,y) = (x,2]$ if $(x,y) \in (1,2] \times (1,2]$.

$G \rightarrow^* R^1$ is a game reduction and there exists $y \overset{R^1}{\succ}_i x$, for some $x, y \in G_i$ and $i = 1,2$. There also exists $x^* = 2 \in R_i^1$ such that $z \overset{R^1}{\not\succ}_i x^* \overset{R^1}{\succ}_i x$ for each $z \in G_i$.

We eliminate again $(1,2) \times (1,2)$, and we obtain $R_1^2 = R_2^2 = \{2\}$.

$P_1(x,y) = \{2\}$ if $(x,y) = (2,2)$ and

$P_2(x,y) = \{2\}$ if $(x,y) = (2,2)$.

$H = R^2$, the non-empty maximal $\rightarrow^*$reduction of G, is the unique maximal reduction of G.

The IESDS procedure is an order independent one.

5.3. Q_θ-MAJORIZED CORRESPONDENCES

In this subsection, we will deal with the correspondences of class Q_θ and the Q_θ-majorized correspondences, defined by Liu and Cai [13]. These types of correspondences generalize the lower semicontinuous ones.

Let X, Y be topological spaces and let $T : X \rightarrow 2^Y$ be a correspondence. T is said to be *lower semicontinuous* if, for each x$\in X$ and each open set V in Y with $T(x) \cap V \neq \emptyset$, there exists an open neighborhood U of x in X such that $T(y) \cap V \neq \emptyset$ for each $y \in U$.

Now, we are presented the correspondences of class Q_θ and the Q_θ-majorized correspondences.

Definition 10 (Liu and Cai, [13]). Let X be a topological space and let Y be a non-empty subset of a vector space E, $\theta : X \rightarrow E$ a function and $P : X \rightarrow 2^Y$ a correspondence.

1) *P is of class Q_θ* (or Q) if:

i) for each $x \in X$, $\theta(x) \notin \mathrm{cl}P(x)$ and

ii) P is lower semicontinuous, with open and convex values in Y;

2) A correspondence $P_x : X \rightarrow 2^Y$ is a *Q_θ-majorant of* P at x, if there exists an open neighborhood $N(x)$ of x such that:

i) for each $z \in N(x)$, $P(z) \subset P_x(z)$ and $\theta(z) \notin \mathrm{cl}P_x(z)$;

ii) P_x is lower semicontinuous, with open and convex values;

3) *P is Q_θ-majorized* if for each $x \in X$ with $P(x) \neq \emptyset$, there exists a Q_θ-majorant P_x of P at x.

The next result is also due to Liu and Cai and states the maximal element existence for $Q_\theta-$majorized correspondences. It will be used in the next subsection to prove Theorem 7, which states the uniqueness of the maximal reductions for qualitative games with $Q_\theta-$majorized correspondences.

THEOREM 6 (Liu and Cai, [13]). *Let X be a convex paracompact subset of a locally convex Hausdorff topological vector space E, let D be a non-empty and compact metrizable subset of X. Let $P : X \to 2^D$ be a $Q_\theta-$majorized correspondence. Then, there exists $x^* \in X$ such that $P(x^*) = \emptyset$.*

5.4. THE UNIQUENESS OF MAXIMAL REDUCTIONS FOR GAMES WITH Q_θ-MAJORIZED CORRESPONDENCES

Theorem 7 concerns the games with $Q_\theta-$majorized correspondences.

THEOREM 7. *Let I be a non-empty, countable set of players. Let $G = (G_i, P_i)_{i\in I}$ be a qualitative game, $\overline{G} = \prod_{i\in I} G_i$, where for each $i \in I$, G_i is a non-empty and compact subset of a Hausdorff topological vector space and let us suppose that there exists $Q_i : \overline{G} \to 2^{G_i}$ such that the pair (P_i, Q_i) satisfies the property T and the following assumptions are fulfilled:*

i) $s_i \in Q_i(s)$ for each $s \in G$;

ii) Q_i has convex and closed values;

iii) $P_i(., s_{-i})$ is $Q_\theta-$majorized on G_i for each $s_{-i} \in G_{-i}$.

Then,

a) If $G \to^* H$ is a a game reduction and if there exists $y \overset{H}{\succ}_i x$, for some $x, y \in G_i$ and $i \in I$, there exists $x^* \in H_i$ such that $z \overset{H}{\not\succ}_i x^* \overset{H}{\succ}_i x$ for each *$z \in G_i$;*

b) a non-empty maximal $\to^*$reduction of G is the unique maximal reduction of G.

Proof. a) Let R^t be the sequence of restrictions of G, $t = 0, 1, 2...$, such that $R^0 = G$, $R^t \to R^{t+1}$ for each $t \geq 0$ and $H_i = \cap_t R_i^t$ for each $i \in I$.

Let $i \in I$ be arbitrarily fixed. Assume that there exists $x, y \in G_i$ such that $y \overset{H}{\succ}_i x$. We apply Liu and Cai's Theorem to the correspondence $F_i : Z_i \to 2^{Z_i}$, $F_i = (P_i \cap Z_i)_{|Z_i\times\{s^*_{-i}\}}$, where $Z_i = \cap_{s_{-i}\in H_{-i}} Q(y, s_{-i})$ is non-empty, convex and compact and $s^*_{-i} \in H_{-i}$ is fixed. We obtain that there exists $x^* \in Z_i$ such that $P_i(x^*, s^*_{-i}) = \emptyset$. For the rest, the proof follows the same line as in the proof of Theorem 2. □

Since a correspondence of class Q_{I_X} is Q_{I_X}-majorized ($I_X : X \to X$ is the identity map), we obtain the following corollary.

Corollary 5. *Let I be a non-empty, countable set of players. Let $G = (G_i, P_i)_{i\in I}$ be a qualitative game, $\overline{G} = \prod_{i\in I} G_i$, where for each $i \in I$, G_i is a non-empty compact subset of a Hausdorff topological vector space and let us suppose that there exists $Q_i : \overline{G} \to 2^{G_i}$ such that the pair (P_i, Q_i) satisfies the property T and the following assumptions are fulfilled:*

i) $s_i \in Q_i(s)$ and $s_i \notin clP_i(s)$ for each $s \in G$;

ii) Q_i has convex and closed values;

iii) P_i is lower semicontinuous, with open and convex values in G_i.

Then,

a) If $G \to^* H$ is a game reduction and if there exists $y \overset{H}{\succ}_i x$, for some $x, y \in G_i$ and $i \in I$, there exists $x^* \in H_i$ such that $z \overset{H}{\not\succ}_i x^* \overset{H}{\succ}_i x$ for each *$z \in G_i$;*

b) a non-empty maximal $\to^*$reduction of G is the unique maximal reduction of G.

6. MAXIMAL ELEMENTS FOR QUALITATIVE GAMES

This subsection is meant to prove that the set of maximal elements is preserved in any game by the process of iterated elimination of strictly dominated strategies.

Notation. Let $G = (G_i, P_i)_{i\in I}$ be a qualitative game and let H be a restriction of G, $H = (H_i, P_{i|\prod_{k\in I} H_k})_{i\in I}$, where $H_i \subseteq G_i$ for each $i \in I$. We will denote H' the qualitative game associated with H, that is $H' = (H_i, (P_i \cap H_i)_{|\prod_{k\in I} H_k})_{i\in I}$.

Theorem 8. *Let I be a non-empty, countable set of players. Let $G = (G_i, P_i)_{i\in I}$ be a qualitative game and $\overline{G} = \prod_{i\in I} G_i$. For each $i \in I$, let us suppose that there exists $Q_i : \overline{G} \to 2^{G_i}$ such that the pair (P_i, Q_i) satisfies the property T and $s_i \notin P_i(s)$ for each $s \in \overline{G}$. Let us also assume that for each $s \in \overline{G}$, there exists $z^* \in \overline{G}$ such that $z_i^* \in Q_i(z_i, s_{-i})$ for all $z \in \overline{G}$ and $i \in I$. If H is a $(\to^*)-$reduction of G, then the games G and H' have the same maximal elements.*

Proof. Let R^t, $t = 0, 1, ...$ denote the unique sequence of games of G such that $R^0 = G$, $R^t \Rightarrow R^{t+1}$ is fast for each t and $H_i = \cap_t R_i^t$ for each $i \in I$. Let us suppose that $s^* \in \overline{G}$ is a maximal element in the game G, that is $P_i(s^*) = \emptyset$ for each $i \in I$ and then, s_i^* is never eliminated in the sequence R^t for each $i \in I$. It follows that $s^* \in \prod_{i\in I} H_i$, so that $P_{i|\prod_{i\in I} H_i}(s^*) = \emptyset$ and, therefore, $P_{i|\prod_{i\in I} H_i}(s^*) \cap H_i = \emptyset$ and s is also a maximal element in H'.

Conversely, let $s^* \in \prod_{i\in I} H_i$ be a maximal element in H' (that is, $P_i(s^*) \cap H_i = \emptyset$ for each $i \in I$) and consider z^* as in the hypothesis: $z^* \in \overline{G}$ such that, for each $i \in I$, $z_i^* \in Q_i(z, s_{-i}^*)$ for all $z \in \overline{G}$. We will prove that $P_i(z_i^*, s_{-i}^*) = \emptyset$ for each $i \in I$. If, on the contrary, we assume that there exists $i_0 \in I$ and $s_{i_0}' \in P_{i_0}(z_{i_0}^*, s_{-i_0}^*)$, according to property T, it follows that $Q_{i_0}(s_{i_0}', s_{-i_0}^*) \subset P_{i_0}(z_{i_0}^*, s_{-i_0}^*)$. However, $z_{i_0}^* \in Q_{i_0}(z_{i_0}, s_{-i_0}^*)$ for all $z \in \overline{G}$, particularly $z_{i_0}^* \in Q_{i_0}(s_{i_0}', s_{-i_0}^*)$ and then, $z_{i_0}^* \in P_{i_0}(z_{i_0}^*, s_{-i_0}^*)$, which contradicts the hypothesis. Since $P_i(z_i^*, s_{-i}^*) = \emptyset$, z_i^* is never eliminated in the sequence R^t for each $i \in I$, and $z^* \in \prod_{i \in I} H_i$. The last assertion implies $z_i^* \in Q_i(z_i, s_{-i}^*) \cap H_i$ for all $z \in \overline{G}$ and $i \in I$. We will prove that $P_i(s^*) = \emptyset$ for each $i \in I$. On the contrary, let us assume that there exists $i_0 \in I$ and $s' \in \overline{G}$ such that $s_{i_0}' \in P_{i_0}(s^*)$. Then, Property T implies $Q_{i_0}(s_{i_0}', s_{-i_0}^*) \subset P_{i_0}(s^*)$. However, we have that $z_{i_0}^* \in Q_{i_0}(s_{i_0}', s_{-i_0})$ from the hypothesis, so that $z_{i_0}^* \in P_{i_0}(s^*)$. In addition, $z_{i_0}^* \in H_{i_0}$ and, then, $z_{i_0}^* \in P_{i_0}(s^*) \cap H_i$, which contradicts the fact that $P_{i_0}(s^*) \cap H_{i_0} = \emptyset$. In conclusion, $P_i(s^*)$ must be the empty set for each $i \in I$ and thus, s^* is a maximal element for the game G. $\square$

Remark 9. Let us suppose that the preference correspondences are defined by using the functions $u_i : \overline{G} \to \mathbb{R}$, $i \in I$. Then, for each $i \in I$, $P_i(s) = \{x \in R : u_i(x, s_{-i}) > u_i(s)\}$ and $s_i \notin P_i(s)$ for each $s \in \prod_k G_k$. If $Q_i(s) = \{x \in R : u_i(x, s_{-i}) \geq u_i(s)\}$, then, the condition that there exists $z^* \in \overline{G}$ such that $z_i^* \in Q_i(z_i, s_{-i})$ for all $z = (z_i, s_{-i}) \in \overline{G}$ and $i \in I$ is equivalent with the following one: there exists $z^* \in \overline{G}$ such that $u_i(z_i^*, s_{-i}) \geq u_i(z_i, s_{-i})$ for all $z = (z_i, s_{-i}) \in \overline{G}$ and $i \in I$. In this way, we obtain Theorem 2 in Dufwenberg and Stegeman [10]. Therefore, we established conditions under which the iterated elimination of the strictly dominated strategies preserves the set of maximal elements of the qualitative games, which represents the set of Nash equilibria in a particular case.

7. CONCLUDING REMARKS

We have reconsidered the problem of the existence of non-empty maximal reductions. Our motivation has been to introduce the concept of rationalizability to a class of qualitative games that feature discontinuous preferences. We have defined the key concepts which can lead to a unified framework that encompasses a variety of models and an open problem is meant to rigorously formalize the concepts in different classes of games, and to prove the existence of the “rationalizable” results.

8. APPENDIX

We add a list with the main notations used in this paper, in order to make the reading easier.

LIST OF NOTATIONS

Correspondence (set valued map): $T : X \to 2^Y$.

Lower section of $T : X \to 2^Y$: $T^{-1}(y) := \{x \in X : y \in T(x)\}$, $y \in Y$.

Strategic game: $\Gamma = (G_i, u_i)_{i\in I}$, where $\overline{G} := \prod_{i\in I} G_i$ and $u_i : \overline{G} \to \mathbb{R}$.

Qualitative game: $G = (G_i, P_i)_{i\in I}$, where $P_i : \overline{G} \to 2^{G_i}$.

Preference correspondence: $P_i : \overline{G} \to 2^{G_i}$,

$$P_i(s) = \{x \in G_i : u_i(x, s_{-i}) > u_i(s)\} \text{ for each } s \in \overline{G}.$$

$G_{-i} := \prod_{j\in I\setminus\{i\}} G_j$.

$s_{-i} = (s_1, ..., s_{i-1}, s_{i+1}, ...) \in G_{-i}$, if $s \in \overline{G}$.

Restriction of $G : H = (H_i, P_{i|\prod_{k\in I} H_k})_{i\in I}$, where $H_i \subseteq G_i$.

The game associated to the restriction $H : H' = (H_i, (P_i \cap H_i)_{|\prod_{k\in I} H_k})_{i\in I}$.

$y \overset{H}{\succ}_i x$: $x, y \in G_i$, $H_{-i} \neq \emptyset$ and $y \in \cap_{s_{-i}\in H_{-i}} P_i(x, s_{-i})$.

$K \to H$: for each $i \in I$ and $x \in K_i \setminus H_i$, there exists $y \in K_i$ such that $y \overset{K}{\succ}_i x$, (equivalently, for each $i \in I$, $K_{-i} \neq \emptyset$ and $\cap_{s_{-i}\in K_{-i}} P_i(x, s_{-i}) \cap K_i \neq \emptyset$).

$K \to H$ is fast: for each $i \in I$, $K_{-i} \neq \emptyset$ and $\cap_{s_{-i}\in K_{-i}} P_i(x, s_{-i}) \cap K_i \neq \emptyset$ for some $x \in K_i$ implies $x \notin H_i$.

$K \to^* H$: there exists a sequence of restrictions R^t of H, $t = 0, 1, 2...$, such that

$R^0 = K$, $R^t \to R^{t+1}$ fast for each $t \geq 0$ and $H_i = \cap_t R_i^t$ for each $i \in I$.

$K \to^* H$ is maximal: $K \to^* H$ and $H \to H'$ only for $H = H'$.

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