ON A GRAPH OF HOMOGENOUS SUBMODULES OF GRADED MODULES

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In this paper, we introduce a graph associated to a graded module over a graded ring and study the relationship between the algebraic properties of these modules and their associated graphs. In particular, the modules whose associated graph is complete, complete bipartite or star are studied and several characterizations are given.

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1. INTRODUCTION

Unless otherwise stated, all rings are assumed to be associative rings and any ring R has an identity $1 \in R$. Consider a multiplicatively written group G with identity element $e \in G$. A ring R is called *G-graded*, if there is a family $\{R_g \mid g \in G\}$ of additive subgroups R_g of R such that $R = \bigoplus_{g\in G} R_g$ and $R_g R_h \subset R_{gh}$, for every $g, h \in G$. Throughout this paper R is a G-graded ring for some fixed group G. A (left) *G-graded* R-module is a left R-module M such that $M = \bigoplus_{g\in G} M_g$ where every M_g is an additive subgroup of M, and for every $g \in G$ and $h \in G$ we have $R_g M_h \subset M_{gh}$. Throughout this paper M is a G-graded R-module (see [9] for basic definitions).

For the last few decades several mathematicians studied graphs on various algebraic structures (groups, rings, modules, ...). These interdisciplinary studies allow us to obtain characterizations and representations of special classes of algebraic structures in terms of graphs and vice versa. Various constructions of graphs related to the algebraic structures are found in [1–4, 6, 8, 10]. In the present paper, we introduce a new undirected simple graph (without loops and multiple edges) associated to a graded R-module M denoted by G_M and investigate the relationship between the algebraic properties of M and the properties of the associated graph G_M.

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Definition 1.1. We define the graph G_M as follows: the vertices are nonzero proper homogeneous submodules of M and two distinct vertices A and B are adjacent if $A + B = M$.

In Lemma 2.1 and Theorem 2.2, we introduce graphical characterizations for some classes of modules. More precisely, M is *sum-irreducible if and only if the graph G_M is empty (see Lemma 2.1), and M is a direct sum of two homogeneous *simple modules if and only if G_M is complete (see Theorem 2.2).

Finding a bound for the cardinal number of generators of a finitely generated module is an interesting subject in commutative algebra (for example see [12]). In Theorem 2.6, among other things, we use the properties of the graph G_M to obtain an upper bound for the number of generators of a given G-graded finitely generated module M.

In Theorem 2.8, we use the clique number of G_M as a criterion to find the finiteness of M.

Moreover, some upper bounds for the cardinal number of the set of all attached prime ideals of a *representable module M are introduced in Proposition 2.11 and Corollary 2.13.

In Theorem 2.14, we characterize the G-graded R-modules M whose (jacobson) radical is zero, by means of a property of G_M. More precisely, the radical of M is zero if and only if G_M is connected.

In Section 3, we will show that if we ignore the isolated vertices of G_M, then it always has only one connected component, namely G'_M. We show that some properties of this subgraph have some algebraic consequences. In Theorem 3.4, we obtain that if M is G-graded finitely generated, then M has exactly two homogeneous maximal submodules if and only if G'_M is a complete bipartite graph.

Again, we find a criterion for the finiteness of M according to a property of G'_M. Indeed, if G'_M is a star graph, then M is generated by two homogenous elements (Theorem 3.8).

Finally, Section 4 is devoted to some examples.

2. THE ASSOCIATED GRAPH

Recall that a nonzero G-graded module is called **simple* if it has no nonzero proper homogeneous submodule (see [9, p. 46]). If $M = 0$ or M is a *simple G-graded R-module, then the set of all vertices of G_M is empty. So in the sequel we suppose, unless stated otherwise, that all R-modules are **nonzero** and **non-*simple**. A homogeneous submodule P of a G-graded R-module M is said to be a **maximal* submodule of M, if M/P is a *simple module (see [9, p. 46]). We denote the set of all *maximal submodules of M

by *Max(M). We remark that a submodule of a G-graded R-module M may be a *maximal submodule of M and yet not be a maximal submodule of M. For example, if $R = k[x, x^{-1}]$ is the graded ring of Laurent polynomials with standard grading, where k is a field, then the zero submodule of the graded module R_R (as an R-module) is a *maximal submodule of R_R and yet zero is not a maximal submodule of R_R.

A homogeneous submodule S of a G-graded R-module M is said to be **small* in M if for every homogeneous submodule L of M, $S + L = M$ implies that $L = M$. By the next lemma we can explain *small submodules of a given module according to its associated graph. Recall that a G-graded R-module M is said to be **sum-irreducible* precisely when it is nonzero and all proper homogeneous submodules of M are *small in M (see [11]). The next lemma gives a characterization of *sum-irreducible modules from their associated graphs. The proof is straightforward and is omitted.

LEMMA 2.1.
(1) *A nonzero proper homogeneous submodule S of M is *small if and only if S is an isolated vertex in G_M.*
(2) *M is *sum-irreducible if and only if the graph G_M is empty. (A graph G is said to be* empty *(or* null*) if no two vertices of G are adjacent.)*

A G-graded module whose lattice of homogeneous submodules is a chain, certainly is *sum-irreducible. Thus, for each positive integer t, the $S := k[x_1, \ldots, x_n]$-module $L = S/(x_1, \ldots, x_n)^t$ is *sum-irreducible, where k is an algebraically closed field (here, we assume that S is a positively graded ring, with standard grading).

Lemma 2.1(2) characterized the modules whose associated graph has no edges. Now, we are going to characterize the modules whose associated graph has maximum number of edges. If in a graph, every two distinct vertices are joined by an edge, then the graph is said to be *complete*. A complete graph with p vertices is denoted by K_p. If A is a set (resp. a graph), then $|A|$ denotes the cardinal number (resp. the number of vertices) of A.

THEOREM 2.2. *Let $|G_M| \geq 2$. Then the following statements are equivalent:*
(1) *G_M is complete.*
(2) *There exists a vertex in G_M which is adjacent to every other vertex.*
(3) *M is a direct sum of two homogeneous *simple modules.*

Proof. (1) ⇒ (2): This is trivial.

(2) ⇒ (3): Let N be a vertex in G_M which is adjacent to every other vertex. Clearly, N is both a maximal and a minimal element in the set of all nonzero proper homogeneous submodules of M. Let $L \neq N$ be a vertex of G_M.

So $N + L = M$. If $N \cap L \neq 0$, then $N \cap L$ is a nonzero proper homogeneous submodule of M. So, $N \cap L = N$ by minimality of N. This implies that $N \subseteq L$, which is a contradiction with the maximality of N. Therefore, $N \cap L = 0$ and $N \oplus L = M$. Since $M/N \cong L$, it is clear by the maximality of N that L is *simple, as desired.

(3) $\Rightarrow$ (1): Let $M = S_1 \oplus S_2$ where S_1 and S_2 are *simple submodules of M. Let N and L be two distinct vertices of G_M. Since $N \subsetneqq M$, either $N \cap S_1 = 0$ or $N \cap S_2 = 0$. Without loss of generality, we may assume that $N \cap S_1 = 0$. We claim that $M = S_1 \oplus N$. To do this, it is enough to prove that $S_2 \subseteq S_1 + N$. We have $0 \neq N = N \cap (S_1 + S_2)$. Thus, there are homogeneous elements $s_1 \in S_1$, $s_2 \in S_2$ and $0 \neq n \in N$ such that $s_1 + s_2 = n \in N \cap (S_1 + S_2)$. It is easy to see that $s_2 \neq 0$. Hence, $s_2 = n - s_1 \in S_2 \cap (S_1 + N)$ and so $S_2 \cap (S_1 + N)$ is a nonzero homogeneous submodule of S_2. Therefore, $S_2 \cap (S_1 + N) = S_2$. It follows that $S_2 \subseteq S_1 + N$ and so $M = S_1 \oplus N$. Therefore, $M/N \cong S_1$. Similarly, if $N \cap S_2 = 0$, then $M = S_2 \oplus N$ and $M/N \cong S_2$. In any case, this yields that N is a *maximal submodule of M. Similarly, $M = S_1 \oplus L$ or $M = S_2 \oplus L$, which implies that L is a *maximal submodule of M. So, $N + L = M$. This completes the proof. □

Remark 2.3. Consider $M = \mathbb{Z}_4$ as a graded $\mathbb{Z}_4$-module with trivial grading. In this case, $|G_{\mathbb{Z}_4}| = 1$ and $G_{\mathbb{Z}_4} = K_1$ is a complete graph, while M cannot be presented as a direct sum of two nonzero *simple $\mathbb{Z}_4$-module. This shows that the condition $|G_M| \geq 2$ in the Theorem 2.2 is necessary and cannot be omitted.

An *m-partite graph* is a graph whose vertex set can be partitioned into m subsets so that no two vertices in the same subset are adjacent. The subsets are also called the partite sets of the partition. In an m-partite graph, if each vertex in a partite set is adjacent to all the vertices in every other partite set, then the graph is called a *complete m-partite graph*. A *complete bipartite* (i.e., complete 2-partite) graph with m vertices in one partition and n vertices in the other is denoted by $K_{m,n}$. Recall that the graph $K_{1,n}$ is called a *star graph* (see [7]).

Corollary 2.4. *If $|G_M| \geq 3$, then G_M is not a star graph.*

Proof. Use Theorem 2.2. □

Example 2.5. If $R = M = \mathbb{Z}_6$ with trivial grading, then G_M is a complete graph by Theorem 2.2, because $\mathbb{Z}_6 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$ where $\mathbb{Z}_2$ and $\mathbb{Z}_3$ are *simple modules. Indeed, $G_M = \{2M, 3M\} = K_2$. This shows that $|G_M| \geq 3$ is a necessary condition in Corollary 2.4.

A *k-coloring* of a graph Γ is an assignment of k colors (elements of some set) to the vertices of Γ in such a way that adjacent vertices have received different colors. If Γ has a k-coloring, then Γ is said to be *k-colorable.* The *chromatic number* of Γ, denoted by $\chi(\Gamma)$, is the smallest number k for which Γ is *k-colorable.* A *clique* of a graph Γ is an induced subgraph of Γ, which is complete. The *clique number* denoted by $\omega(\Gamma)$ is the number of vertices in the largest clique in Γ. The *girth* of a graph Γ denoted by $\mathrm{g}(\Gamma)$ is the length of the shortest cycle in Γ. A cycle of length 3 is known as a *triangle* (see [7]). In the next theorem, we explore these three concepts for the associated graph of a graded finitely generated module. Finding a bound for the number of generators of a finitely generated module is an interesting subject in algebra (for example see [12]). We use the properties of the graph G_M to obtain a bound for the number of generators of a given G-graded finitely generated module M.

THEOREM 2.6. *Let M be a G-graded finitely generated R-module. Then the following holds:*

(1) $\omega(G_M) = \chi(G_M) = |{}^*\mathrm{Max}(M)|$.

(2) *Every minimal generating set of M has at most $\omega(G_M)$ elements.*

(3) *G_M has no triangle if and only if M has at most two *maximal submodules.*

(4) *Always* $\mathrm{g}(G_M) = 3$ *except when* $|{}^*\mathrm{Max}(M)| < 3$.

Proof. (1) Let S be an arbitrary complete subgraph of G_M. For any vertex N of S choose a *maximal submodule P_N of M with $N \subseteq P_N$. For any distinct vertices N and L of S, since $N + L = M$, we have $P_N + P_L = M$ and so $P_N \neq P_L$. Thus, the subgraph of G_M induced by $\{P_N \mid N \text{ is a vertex of } S\}$ is a complete graph where its cardinality is exactly the cardinality of S. Since $|S| \leq |{}^*\mathrm{Max}(M)|$, so $\omega(G_M) \leq |{}^*\mathrm{Max}(M)|$. On the other hand, it is clear that the subgraph of G_M generated by elements of ${}^*\mathrm{Max}(M)$ is a complete subgraph of G_M. Therefore, $\omega(G_M) \geq |{}^*\mathrm{Max}(M)|$.

To find the chromatic number of G_M, let $\{P_\lambda \mid \lambda \in \Lambda\}$ be the set of all *maximal submodules of M and suppose that $\prec$ is a well ordering on Λ. For any $\lambda \in \Lambda$, let $G_\lambda(M) = \{N \subseteq M | N \text{ is homogeneous and } 0 \neq N \subseteq P_\lambda \text{ and } N \notin \bigcup_{\lambda' \prec \lambda} G_{\lambda'}(M)\}$. Then for each $\lambda \in \Lambda$, $P_\lambda \in G_\lambda(M)$ and so $G_\lambda(M) \neq \emptyset$. Also, $\{G_\lambda(M) | \lambda \in \Lambda\}$ forms a partition for the set of all vertices of G_M. Since for every $\lambda \in \Lambda$, any two vertices in $G_\lambda(M)$ are not adjacent, all vertices in $G_\lambda(M)$ can have the same color. However, the P_λ's must have different colors. Therefore, the chromatic number of G_M is equal to $|\Lambda|$.

(2) Let $X = \{x_1, \ldots, x_n\}$ be a minimal generating set of M. We may assume that X consists of homogeneous elements. Then $D_j = \sum_{j \neq i=1}^{n} Rx_i$ is a vertex of G_M and the subgraph induced by $\{D_1, \ldots, D_n\}$ is complete. So, $n \leq \omega(G_M)$.

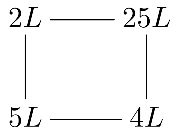
(3) Suppose that G_M has no triangle and Q_1, Q_2, Q_3 are three distinct

*maximal submodules of M. Then it is easy to see that these are vertices of a triangle in G_M, a contradiction. Conversely, if $|{}^*\mathrm{Max}(M)| = 1$, then G_M is empty and there is nothing to prove. So, let P_1 and P_2 be the only two *maximal submodules of M and let N_1, N_2 and N_3 be three arbitrary vertices of G_M. Since M is a G-graded finitely generated R-module, any homogeneous proper submodule of M is contained in a *maximal submodule of M by Zorn's lemma. Therefore, at least two of these three vertices are contained in one of P_1 or P_2, and so they are not adjacent. Therefore, there is no triangle in G_M.

(4) If Q_1, Q_2, Q_3 are three distinct *maximal submodules of M, then these are the vertices of a triangle in G_M. So, $\mathrm{g}(G_M) = 3$. Note that when M has at most two *maximal submodules, G_M has no triangle by (2). Hence, $\mathrm{g}(G_M) \neq 3$. □

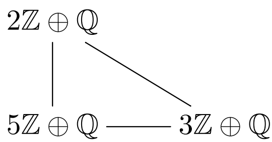
Example 2.7.

(1) Consider the graded ring $\mathbb{Z}$ and the graded finitely generated $\mathbb{Z}$-module $L = \frac{\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{\mathbb{Z}}{25\mathbb{Z}}$ with trivial grading. This module has exactly two *maximal submodules and G_M has a cycle of length four:



By Theorem 2.6(3), G_L has no triangle. Therefore, $\mathrm{g}(G_L) = 4$.

(2) We show that the finitely generatedness of the R-module M in Theorem 2.6(3) is *not* a necessary condition. Consider the graded $\mathbb{Z}$-module $H = \mathbb{Z} \oplus \mathbb{Q}$ with trivial grading. It is easy to see that H is not finitely generated. But we have the below triangle in G_H;



Therefore, $\mathrm{g}(G_M) = 3$.

Now, we are going to show that if G_M is a finite graph, then M is finitely generated.

Theorem 2.8. *If* $\omega(G_M) < \infty$, *then* M *is finitely generated and* $\omega(G_M) = |{}^*\mathrm{Max}(M)|$.

Proof. Suppose that M is not finitely generated and $M = \sum_{\lambda \in \Lambda} Rg_\lambda$, where g_λ is a homogeneous element of M for each $\lambda \in \Lambda$. So, Λ is an infinite set. We may assume that Λ is minimal in the sense that if Λ' is a proper subset of Λ, then $M \neq \sum_{\lambda \in \Lambda'} Rg_\lambda$. For each $\alpha \in \Lambda$, write $N_\alpha = \sum_{\lambda \in \Lambda \setminus \{\alpha\}} Rg_\lambda$. Then for any two distinct elements $\alpha, \beta \in \Lambda$ we have $N_\alpha + N_\beta = M$. Therefore, $S = \{N_\alpha \mid \alpha \in \Lambda\}$ is an infinite clique of G_M, a contradiction.

The last assertion follows from Theorem 2.6(1). □

COROLLARY 2.9. *As a direct consequence of Theorem 2.8, we infer that if G_M is a finite graph, then M is finitely generated.*

We recall that a G-graded R-module M is said to be *left Noetherian* if M satisfies the ascending chain condition for graded left R-submodules.

COROLLARY 2.10. *Let R be a commutative G-graded Noetherian ring and $\omega(G_M) < \infty$. Then M is a G-graded Noetherian R-module.*

We recall some definitions from [11]. Let R be a commutative G-graded ring. Then the G-graded R-module M is said to be *graded-secondary* if $M \neq 0$ and, for each homogeneous element r of R, the endomorphism of M given by multiplication by r is either surjective or nilpotent. If M is graded-secondary, then $\sqrt{(0 :_R M)}$ is a homogeneous prime ideal of R, $\mathfrak{p}$; we say that M is $\mathfrak{p}$-*graded-secondary.* A G-graded R-module M is said to be **representable* if it has a graded-secondary representation, i.e. M may be expressed as a finite sum $M = S_1 + \cdots + S_r$ where each S_t is a graded-secondary homogeneous submodule of M. It is, furthermore, said to be a *minimal graded-secondary representation* for M if, in addition, the r prime ideals $\sqrt{(0 :_R S_i)}$ $(1 \leq i \leq r)$ are all different and, for each $j = 1, ..., r$, $M \neq \sum_{j \neq i=1}^{r} S_i$. A graded-secondary representation for M may be modified to a minimal one. The prime ideals $\sqrt{(0 :_R S_i)}$ $(1 \leq i \leq r)$ in the minimal graded-secondary representation of M are called *attached prime ideals* of M. The set of all attached prime ideals of M is denoted by $Att_R(M)$ (for more details see [11]). For the class of *representable modules we can find a lower bound for the clique number.

PROPOSITION 2.11. *Let R be a commutative G-graded ring and let M be a *representable R-module. Then $|Att_R(M)| \leq \omega(G_M)$.*

Proof. Suppose $M = \sum_{i=1}^{n} S_i$ is a minimal graded-secondary representation of M. If we write $N_j = \sum_{j \neq i=1}^{n} S_i$, then it is easy to see that $\{N_j \mid 1 \leq j \leq n\}$ induces a complete subgraph of G_M. This completes the proof. □

COROLLARY 2.12. *Let R be a commutative G-graded ring and let M be a nonzero R-module such that G_M is empty. Then M is *representable if and only if it is graded-secondary.*

Proof. A graded-secondary module is always *representable. So, we assume that M is *representable. Then by assumption and Proposition 2.11, $|Att_R(M)| \leq \omega(G_M) = 1$. This completes the proof. □

COROLLARY 2.13. *Let R be a commutative G-graded ring and let M be nonzero G-graded finitely generated R-module and *representable. Then*

$$|Att_R(M)| \leq |{}^*\mathrm{Max}(M)|.$$

Proof. Use Proposition 2.11 and Theorem 2.6. □

We define the **radical* of M denoted by $^*\mathrm{Rad}(M)$ to be the intersection of all *maximal submodules of M. If M has no *maximal submodules, we set $^*\mathrm{Rad}(M) = M$ (see [9, p. 52]). For a G-graded R-module M we have $^*\mathrm{Rad}(M) = \sum_{N \text{ is *small in } M} N$.

Recall that a graph Γ is *connected* if there is a path between every pair of distinct vertices and *disconnected* otherwise. The *distance* between two vertices u and v is the length of a shortest path joining them, and is denoted by $d(u, v)$. If there is no path joining u and v, then we define $d(u, v) = \infty$. For a connected graph Γ, we define the *diameter* of Γ denoted by $\mathrm{diam}\,\Gamma$ to be the supremum of the distances between vertices. In the next theorem, again we find a relationship between the algebraic properties of M and its associated graph G_M.

By Theorem 2.2 and Lemma 2.1, if $|G_M| \geq 2$ and G_M is complete, then M has no nonzero proper homogeneous *small submodules, whence $^*\mathrm{Rad}(M) = 0$. Indeed, this is a special case of the next theorem, because every complete graph is connected.

THEOREM 2.14. *Let $|G_M| \geq 2$. Then the following statements are equivalent:*

(1) *G_M is connected.*

(2) $^*\mathrm{Rad}(M) = 0$.

If these conditions are satisfied, then $\mathrm{diam}\, G_M \leq 3$.

Proof. Suppose that $^*\mathrm{Rad}(M) = 0$. Then $|{}^*\mathrm{Max}(M)| \geq 2$. Let N and L be two distinct elements of G_M. If N and L are *maximal, then N and L are adjacent. Otherwise, since $^*\mathrm{Rad}(M) = 0$, there are *maximal submodules P_1 and P_2 of M such that $N \not\subseteq P_1$ and $L \not\subseteq P_2$. If $P_1 = P_2$, then $d(N, L) = 2$ and if $P_1 \neq P_2$, then $d(N, L) = 3$. So always there is a path from N to L in G_M.

Conversely, suppose that G_M is connected and $^*\mathrm{Rad}(M) \neq 0$. If $^*\mathrm{Rad}(M) = M$, then every vertex of G_M is *small in M and so is isolated in G_M, by Lemma 2.1. If $^*\mathrm{Rad}(M) \neq M$, then for each nonzero homogeneous element $x \in {}^*\mathrm{Rad}(M)$, the proper homogeneous submodule Rx of M is *small and therefore, is an isolated vertex of G_M. Both of these cases contradict the connectedness of G_M.

The last assertion follows from the first part of the proof. □

Suppose that $G = \mathbb{Z}$ and $L = \bigoplus_{n\in\mathbb{Z}} L_n$ is a *simple R-module. Then it is easy to see that L is a semisimple R_0-module (see [5, Corollary 9.1.18.]). Therefore, by Theorem 2.14, $G_{L_{R_0}}$ is connected.

Example 2.15. Consider $R=M=\mathbb{Z}_4$ with trivial grading. Then, ${}^*\mathrm{Max}(M) = \{(\bar{2})\}$. Therefore, ${}^*\mathrm{Rad}(M) = (\bar{2}) \neq 0$. But $G_M = K_1$ is connected. This shows that the condition $|G_M| \geq 2$ in Theorem 2.14 is necessary and cannot be omitted.

A non-isolated vertex $N \in G_M$ is called *reduced* if there exists a vertex $L \in G_M$ adjacent to N and no proper submodule of N is adjacent to L.

PROPOSITION 2.16. *Let N be a homogeneous submodule of M.*

(1) *If M is finitely generated and N is a reduced vertex of G_M, then N is finitely generated.*
(2) *If N is finitely generated and* $\deg_{G_M}(N) \neq 0$, *then there exists $L \in G_M$ such that M/L is a G-graded finitely generated R-module.*

Proof. (1) Let $N = \sum_{\lambda\in\Lambda} Rx_\lambda$, where x_λ is a homogeneous element of N for each $\lambda \in \Lambda$. By assumption, there is a vertex $L \in G_M$ such that $L+\sum_{\lambda\in\Lambda} Rx_\lambda = M$. Since M is finitely generated, there is a finite subset $\Lambda' \subseteq \Lambda$ such that $L + \sum_{\lambda\in\Lambda'} Rx_\lambda = M$. Since N is a reduced vertex and $\sum_{\lambda\in\Lambda'} Rx_\lambda \subseteq N$, we infer that $N = \sum_{\lambda\in\Lambda'} Rx_\lambda$. Thus, N is finitely generated.

(2) Since $\deg_{G_M}(N) \neq 0$, there is a homogeneous submodule L of M such that $M = N + L$. By assumption, there are elements $m_1, \ldots, m_t \in N$ such that $M = (\sum_{i=1}^t Rm_i)+L$. Therefore, $M/L = \sum_{i=1}^t R(m_i+L)$. □

3. ON THE CONNECTED COMPONENT

In this section, we will show that if we ignore the isolated vertices of G_M, then it always has only one connected component, namely G'_M. So, in a sense, G'_M is the main part of the graph G_M. We would like to find the relationship between the algebraic structure of M and the properties of G'_M.

Remark 3.1. Let N be a nonzero proper homogeneous submodule of M. It follows from Lemma 2.1 that if N is an isolated vertex in G_M, then $N \subseteq {}^*\mathrm{Rad}(M)$. Assume that ${}^*\mathrm{Max}(M) \neq \emptyset$. Put $\Lambda := \{N \in G_M \mid N \nsubseteq {}^*\mathrm{Rad}(M)\}$. We denote the subgraph induced by the set Λ by G'_M. Note that if $G_M \neq \emptyset$, then $G'_M = G_M$ if and only if ${}^*\mathrm{Rad}(M) = 0$.

LEMMA 3.2. *Let $|G'_M| \geq 1$. Then the graph G'_M is connected and* $\operatorname{diam} G'_M \leq 3$.

Proof. In the case $|G'_M| < 2$ there is nothing to prove. So, we may assume that $|G'_M| \geq 2$. Let N and L be two distinct elements of G'_M. By assumption, there are *maximal submodules P_1 and P_2 of M such that $N \not\subseteq P_1$ and $L \not\subseteq P_2$. Either $P_1 = P_2$ or $P_1 \neq P_2$. In either case, we have a path from N to L in G'_M. Also, we infer that $\operatorname{diam} G'_M \leq 3$. □

Example 3.3. Consider $S = \mathbb{Z}_2 \times \mathbb{Z}_4$ with trivial grading. Then $^*\mathrm{Rad}(S_S) = (0) \times 2\mathbb{Z}_4$ and $G'_{S_S} = \{(0) \times \mathbb{Z}_4, \mathbb{Z}_2 \times (0), \mathbb{Z}_2 \times 2\mathbb{Z}_4\}$. We have

$$\mathbb{Z}_2 \times 2\mathbb{Z}_4 \text{ —— } (0) \times \mathbb{Z}_4 \text{ —— } \mathbb{Z}_2 \times (0).$$

Therefore, $\operatorname{diam} G'_{S_S} = 2$.

THEOREM 3.4. *Let M be a G-graded finitely generated R-module. Then the following statements are equivalent:*
(1) *G'_M is a complete bipartite graph.*
(2) *$|^*\mathrm{Max}(M)| = 2$.*

Proof. (1) ⇒ (2). Suppose that G'_M is a complete bipartite graph with two parts V_1 and V_2. Since M is a G-graded finitely generated R-module, $|^*\mathrm{Max}(M)| \geq 2$. Suppose that $|^*\mathrm{Max}(M)| > 2$. Then by the Pigeon Hole Principle, two of the *maximal submodules of M should belong to one of the V_i's, a contradiction. Therefore, $|^*\mathrm{Max}(M)| = 2$.

(2) ⇒ (1). Suppose that $^*\mathrm{Max}(M) = \{P_1, P_2\}$. Since M is a G-graded finitely generated R-module, every proper homogeneous submodule of M is contained in P_1 or P_2. Set $V_1 = \{N \in G'_M | N \subseteq P_1\}$ and $V_2 = \{N \in G'_M \mid N \subseteq P_2\}$. Clearly, $V_1 \cap V_2 = \emptyset$, $G'_M = V_1 \cup V_2$ and the elements of V_i are not adjacent. Now, suppose that $L \in V_1$ and $N \in V_2$. Hence, $N + L$ is a homogeneous submodule of M such that $N + L \not\subseteq {}^*\mathrm{Rad}(M)$. Since $N + L \not\subseteq P_1$ and $N + L \not\subseteq P_2$ and M is finitely generated, we must have $N + L = M$. This implies that G'_M is a complete bipartite graph. □

The next example shows that for any two positive integers n and m, there is a G-graded R-module M such that $G'_M = K_{n,m}$.

Example 3.5. Consider $S = (F[x]/(x^n)) \oplus (F[y]/(y^m))$ with standard grading, where F is a field. Then $^*\mathrm{Rad}(S_S) = (\bar{x}) \times (\bar{y})$ and so $G'_{S_S} = K_{n,m}$.

PROPOSITION 3.6. *Let M be a G-graded finitely generated R-module and $n > 1$. If $|^*\mathrm{Max}(M)| = n < \infty$, then G'_M is n-partite.*

Proof. Let $^*\mathrm{Max}(M) = \{P_1, \ldots, P_n\}$ and set $A_i = \{N \in G'_M \mid N \subseteq P_i\}$. Suppose that $V_1 = A_1$ and $V_i = A_i \setminus \bigcup_{j=1}^{i-1} A_j$ for each $i \geq 2$. Clearly, for each i, $P_i \in V_i$ and so $V_i \neq \emptyset$. Since M is finitely generated, $G'_M = V_1 \cup \ldots \cup V_n$.

And by construction, $V_i \cap V_j = \emptyset$, for $i \neq j$. Now, let $L, N \in V_i$ for some i. If L and N are adjacent, then $M = L + N \subseteq P_i$, a contradiction. $\square$

Example 3.7. Suppose that $S = \mathbb{Z}$ and consider the S-module $M = \mathbb{Z}_{60}$ with trivial grading. Then $|{}^*\mathrm{Max}(M)| = 3$ and therefore, G'_M is 3-partite, by Proposition 3.6.

According to the Corollary 2.4, if M is a G-graded R-module with $|G_M| \geq 3$, then G_M is not a star graph. But, it is possible for G'_M to be a star graph. Now, we are going to obtain some algebraic properties of M when G'_M is a star graph.

Theorem 3.8. *If G'_M is a star graph, then $|{}^*\mathrm{Max}(M)| = 2$ and M is generated by two homogenous elements and so is finitely generated.*

Proof. Since G'_M is a star graph, $|{}^*\mathrm{Max}(M)| < 3$, by Theorem 2.6(3), and there exists a vertex $P \in G'_M$ such that P is adjacent to any other vertex of G'_M. We claim that P is a *maximal submodule of M. Let N be a proper homogenous submodule of M such that $P \subseteq N$. Then $P + N \neq M$. So, $N \subseteq {}^*\mathrm{Rad}(M) \subseteq P$. Hence, $N \subseteq P$ and so $P = N$.

If $|{}^*\mathrm{Max}(M)| = 1$, then ${}^*\mathrm{Rad}(M) = P$ and so $P \notin G'_M$, a contradiction. Therefore, $|{}^*\mathrm{Max}(M)| = 2$.

Suppose that ${}^*\mathrm{Max}(M) = \{P, Q\}$. We claim that any proper homogenous submodule of M is contained in a *maximal submodule of M. To do this, let N be a proper homogenous submodule of M. If $N \subseteq {}^*\mathrm{Rad}(M)$, then we are done. Otherwise, $N \in G'_M$ and so $P + N = M$, because G'_M is a star graph. Thus, $N \nsubseteq P$. If $N \nsubseteq Q$, then $N + Q = M$. Hence, $N - Q - P - N$ is a cycle in G'_M, which is impossible. Therefore, $N \subseteq Q$. Now, we show that M is generated by two homogenous elements. Since $P \neq Q$, and they are both *maximal, there exist two homogenous elements $x \in P \setminus Q$ and $y \in Q \setminus P$. So, $0 \neq Rx \nsubseteq {}^*\mathrm{Rad}(M)$ and $0 \neq Ry \nsubseteq {}^*\mathrm{Rad}(M)$. It is clear that $Rx \neq M$ and $Ry \neq M$. Hence, Rx and Ry are two vertices of G'_M. Now, if $Rx \neq P$, since G'_M is a star graph, $Rx + P = M$. But $Rx + P = P \neq M$, which is a contradiction. Therefore, $Rx = P$. Consequently, $Rx + Ry = P + Ry = M$, as desired. $\square$

Corollary 3.9. *If G'_M is a star graph and ${}^*\mathrm{Rad}(M) = 0$, then $|G_M| = |G'_M| = 2$.*

Proof. Since ${}^*\mathrm{Rad}(M) = 0$, by definition, we have $G_M = G'_M$. By Corollary 2.4, $|G_M| < 3$. On the other hand, by Theorem 3.8, $|{}^*\mathrm{Max}(M)| = 2$ and so $|G_M| \geq 2$. Therefore, $|G_M| = |G'_M| = 2$. $\square$

Proposition 3.10. *If M has an infinite decreasing chain of reduced vertices of G'_M, then $\omega(G'_M) = \infty$.*

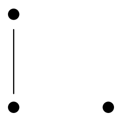
Proof. Let $N_1 \supsetneqq N_2 \supsetneqq \cdots \supsetneqq N_i \supsetneqq \cdots$ be an infinite decreasing chain of reduced vertices of G'_M. By assumption, for each $i \in \mathbb{N}$, there is a vertex L_i in G'_M adjacent to N_i and no proper submodules of N_i is adjacent to L_i. Since $N_{i+1} \subsetneqq N_i$ and N_i is reduced, $L_i + N_{i+1} \neq M$. Hence, $H_i := L_i + N_{i+1}$ is a vertex of G'_M, for each $i \in \mathbb{N}$. We claim that $\{H_i \mid i \in N\}$ is an infinite clique. Let $i \neq j \in \mathbb{N}$. Suppose that $i \geq j+1$. Then

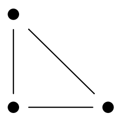
$$H_i + H_j = N_{i+1} + L_j + (L_i + N_{j+1}) = N_{i+1} + L_j + M = M.$$

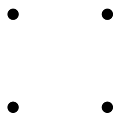
Similarly, $H_i + H_j = M$ when $i < j$. The fact that $H_i + H_j = M$ when $i \neq j$, also proves that $H_i \neq H_j$ for $i \neq j$. This completes the proof. □

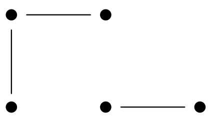
4. SOME EXAMPLES

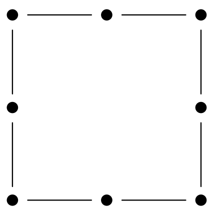
Suppose that a graph Γ is given. Is there a G-graded R-module M such that G_M is isomorphic to Γ? If there is, what can we say about M? For example, consider the following graphs:

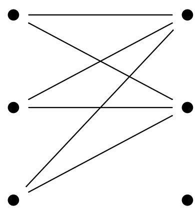

Graph Γ_1


Graph Γ_2


Graph Γ_3


Graph Γ_4


Graph Γ_5

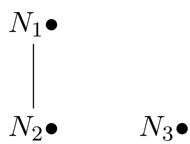

Graph Γ_6

Now, we state the following question:

QUESTION 4.1. *If there exists an R-module β_i such that G_{β_i} is isomorphic to Γ_i, for each i, then what can we say about β_i?*

In this section, we collect our information from previous sections to provide a response to Question 4.1.

Graph Γ_1: We consider the graph Γ_1 and relabel it as follows:

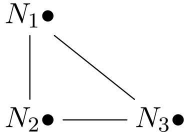


Graph Γ_1

If there exists a G-graded R-module M such that G_M is isomorphic to Γ_1, then the following facts hold:

- M is finitely generated, by Theorem 2.8.
- Every minimal generating set of M has at most 2 elements, by Theorem 2.6(2).
- M is not *sum-irreducible, by Lemma 2.1.
- M is not a direct sum of two *simple submodules, by Theorem 2.2.
- Theorem 2.6(3) gives us that M has at most two *maximal submodules.
- $^*\mathrm{Rad}(M) = N_3$, by Lemma 2.1(1).
- N_1 and N_2 are reduced and so are finitely generated, by Proposition 2.16.

Graph Γ_2: We relabel the graph Γ_2 as follows:



Graph Γ_2

If there exists a graded R-module M such that G_M is isomorphic to Γ_2, then the following facts hold:

- M is finitely generated, by Theorem 2.8.
- Every minimal generating set of M has at most 3 elements, by Theorem 2.6(2).
- M is not *sum-irreducible, by Lemma 2.1.
- N_1, N_2 and N_3 are finitely generated, by Proposition 2.16.
- $|^*\mathrm{Max}(M)| = 3$, by Theorem 2.6(1).
- M is direct sum of two *simple submodules, by Theorem 2.2. Thus, $^*\mathrm{Rad}(M) = 0$.

Graph Γ_3: If there exists a graded R-module M such that G_M is isomorphic to Γ_3, then it is finitely generated and *sum-irreducible, by Theorem 2.8 and Lemma 2.1. Hence, M is cyclic. For example, if $H := S/(x^5)$, where $S = \mathbb{C}[x]$ with standard grading, then G_{H_S} is isomorphic to Γ_3.

Graph Γ_4: We claim that there is no graded R-module M such that G_M is isomorphic to Γ_4. Contrary, suppose that there is a graded R-module M such that G_M is isomorphic to Γ_4. Since G_M has no isolated vertices, we have $G_M = G'_M$. Therefore, by Lemma 3.2, G_M must be a connected graph, which is not true.

Graph Γ_5: Suppose that there is a graded R-module M such that G_M is isomorphic to Γ_5. Since Γ_5 is connected, it follows from Theorem 2.14 that $\operatorname{diam} G_M \leq 3$, a contradiction. Therefore, there is no graded R-module M such that G_M is isomorphic to Γ_5.

Graph Γ_6: By Example 3.5, there exists a graded R-module M such that G'_M is isomorphic to Γ_6. Every such module M is finitely generated, by Theorem 2.8. Moreover, since G'_M is a complete bipartite graph, $|{}^*\mathrm{Max}(M)| = 2$, by Theorem 3.4. Also, every minimal generating set of M has at most 2 elements, by Theorem 2.6(2).

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