ON JOINT CONVEXITY AND CONCAVITY OF SOME KNOWN TRACE FUNCTIONS

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In this paper, we provide a new and simple proof for joint convexity and concavity of some known trace functions due to Bekjan. Indeed, we will do this by making use of the operator monotone functions and perspective of convex functions.

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1. INTRODUCTION

We are concerned with the joint concavity and convexity of some known trace functions. The concepts and theorems on operator monotone functions and perspective of convex functions play an essential role in the method of proving our main results.

Let H be a finite dimensional Hilbert space, $B(H)$ be the space of all bounded linear operators from H to H, $B^+(H)$ be the set of all positive operators in $B(H)$, $B^s(H)$ be the set of all Hermitian operators in $B(H)$ and $B^{++}(H)$ be the set of all strictly positive operators in $B(H)$, respectively.

A real valued function $f(A, B)$ defined on $B(H) \times B(H)$ is said to be *jointly convex* in (A, B) if

$$f(\lambda A_1 + (1-\lambda)A_2, \lambda B_1 + (1-\lambda)B_2) \leqslant \lambda f(A_1, B_1) + (1-\lambda) f(A_2, B_2)$$

for all $A_i, B_i \in B(H)$, $1 \leqslant i \leqslant 2$ and $\lambda \in [0, 1]$. $f(A, B)$ is said to be *jointly concave* if $-f(A, B)$ is jointly convex in (A, B). A function $f : (0, \infty) \to \mathbb{R}$ is said to be *operator monotone* if $A \geqslant B > 0$ implies that $f(A) \geqslant f(B)$.

We remark that $A \geqslant B$ for all $A, B \in B^s(H)$ if $A - B \geqslant 0$. For more information on the operator monotone functions and their relations with joint convexity and concavity, the reader is referred to [1, 2, 7] and [12].

Let f be a real valued function defined on a convex set $\mathbf{C} \subseteq \mathbb{R}^n$. The perspective function [10] associated to f is a function of two variables on the subset

$$L := \left\{ (t, s) | s > 0, \left(\frac{t}{s} \right) \in \mathbf{C} \right\} \subseteq \mathbb{R}^{n+1}$$

defined by

$$P_f(t, s) = f\left(\frac{t}{s} \right) s.$$

If $f(x)$ is convex on $\mathbf{C}$, then $P_f(t, s)$ is jointly convex in (t, s). In [5], Effros introduced an operator version of perspective functions for commuting positive operators L and R by

$$P_f(L, R) = f\left(\frac{L}{R} \right) R,$$

and proved the following theorem:

Theorem 1.1 ([5], Theorem 2.2). *If $f(t)$ is operator convex and $[L, R] = 0$, then the perspective*

$$P_f(L, R) = f\left(\frac{L}{R} \right) R,$$

is jointly convex.

In [4], Carlen and Lieb proved the joint concavity of $Tr\,(A_1^p + A_2^p + \cdots + A_n^p)^{\frac{1}{p}}$ for all $0 < p \leqslant 1$ and $A, B \in B^{++}(H)$. In 2014, Hansen [8] gave a simple proof for joint convexity of $Tr(A^p + B^p)^{\frac{1}{p}}$ when $1 \leqslant p \leqslant 2$.

In this paper, we give a new and simple proof for the joint concavity of $Tr(A^{-p} + B^{-p})^{-\frac{1}{p}}$ when $0 < p \leqslant 1$. We note that Bekjan [3] has proved the joint concavity of $Tr(A^{-p} + B^{-p})^{-\frac{1}{p}}$ by Epstein's method, while we give easier proof by applying the perspective of convex functions and Effros's convexity theorem.

In [9], Hiai and Petz considered the trace function

$$I_f^\theta(A, B, K) = Tr K^* \left[f(L_A R_B^{-1}) R_B \right]^\theta (K)$$

for a positive function f on $(0, \infty)$, a non zero real parameter θ, an arbitrary operator K on H and $A, B \in B^{++}(H)$. They proved various properties concerning joint concavity and convexity of the function $I_f^\theta(A, B, K)$ in three variables (A, B, K) or in two variables (A, B). Their main theorems clarified what conditions of f and θ are sufficient and necessary for joint convexity and concavity of $I_f^\theta(A, B, K)$. For any $0 < p \leqslant 1$, we prove the joint convexity of $Tr(A^{-p} + B^{-p})^{\frac{1}{p}}$ and $Tr(A^p + B^p)^{-\frac{1}{p}}$ by applying the joint convexity conditions of $I_f^\theta(A, B, K)$.

Notice that the joint convexity of

$$Tr\left[(A_1^{-p}+A_2^{-p}+\cdots+A_n^{-p})^{\frac{1}{p}}\right]$$

and

$$Tr\left[(A_1^{p}+A_2^{p}+\cdots+A_n^{p})^{-\frac{1}{p}}\right]$$

was considered in [3], but in this paper, we introduce a simple and new proof.

Throughout this paper, we suppose that $H=M_n$ have the usual Hilbert space structure with inner product $\langle A,B\rangle := TrAB^*$. For all $A,B\in B^{++}(H)$, we define $L_A(X):=AX$ and $R_B(X):=XB$. They have the following properties [11]:

(1) L_A and R_B are commuting operators since

$$R_B[L_A(X)] = AXB = L_A[R_B(X)];$$

(2) L_A and R_A are invertible, $L_{A^{-1}}=L_A^{-1}$ and $R_{A^{-1}}=R_A^{-1}$;
(3) $L_{A^p}=L_A^p$ and $R_{A^p}=R_A^p$ for all $p\in\mathbb{R}$.

2. MAIN RESULTS

Now, we give our main results in this paper.

THEOREM 2.1 (Bekjan). *If* $0<p\le 1$, *then* $f(A,B)=Tr\,(A^{-p}+B^{-p})^{-\frac{1}{p}}$ *is jointly concave for* $A,B\in B^{++}(H)$.

Proof. Let $g(t)=(t^{-p}+1)^{-\frac{1}{p}}$ defined for $t>0$ and $0<p\le 1$. By simple algebraic calculation, $g(t)$ can be rewritten as follows:

$$g(t)=\frac{t}{(t^p+1)^{\frac{1}{p}}}.$$

Since the function $(t^p+1)^{\frac{1}{p}}$ is operator monotone ([1], Corollary 4.3), we have

$$g(t)=\left(t^{-p}+1\right)^{-\frac{1}{p}}=\frac{t}{(t^p+1)^{\frac{1}{p}}}$$

is operator monotone and operator concave ([7], Corollary 2.6 and [12], Corollary 6).

Now, we obtain the perspective of $g(t)=(t^{-p}+1)^{-\frac{1}{p}}$ as follows:

$$P_g(t,s)=g\left(\frac{t}{s}\right)s=\left(t^{-p}+s^{-p}\right)^{-\frac{1}{p}}.$$

So, for commuting operators L_A and R_B, we have

$$P_g(L_A, R_B) = \left(L_A^{-p} + R_B^{-p}\right)^{-\frac{1}{p}}.$$

Since $g(t)$ is operator concave, $P_g(L_A, R_B)$ is jointly concave by Theorem 1.1. Thus we conclude that following mapping

$$(A, B) \mapsto Tr\left[(L_A^{-p} + R_B^{-p})^{-\frac{1}{p}}(K^*)K\right] \tag{2.1}$$

is jointly concave in (A, B) for arbitrary operator K on H ([6], Theorem 1.1).

If we replace K with the identity operator I in (2.1), then the following mapping

$$(A, B) \mapsto Tr\left[\left(L_A^{-p} + R_B^{-p}\right)^{-\frac{1}{p}}(I^*)I\right]$$

is jointly concave. According to the proof of ([8], Theorem 2.1), we have

$$\left[\left(L_A^{p} + R_B^{p}\right)^{\frac{1}{p}}(I)\right] = \left[(A^p + B^p)^{\frac{1}{p}}\right] \tag{2.2}$$

and so, by a simple algebraic calculation, we obtain

$$Tr\left[\left(L_A^{-p} + R_B^{-p}\right)^{-\frac{1}{p}}(I^*)I\right] = Tr\left[\left(L_A^{-p} + R_B^{-p}\right)^{-\frac{1}{p}}(I)\right] = Tr\left[\left(A^{-p} + B^{-p}\right)^{-\frac{1}{p}}\right].$$

Thus

$$f(A, B) = Tr\left[\left(A^{-p} + B^{-p}\right)^{-\frac{1}{p}}\right]$$

is jointly concave in (A, B). This completes the proof. $\square$

Corollary 2.2. *If $-1 \leqslant p \leqslant 1$ and $p \neq 0$, then $Tr\,(A^p + B^p)^{\frac{1}{p}}$ is jointly concave for any $A, B \in B^{++}(H)$.*

Proof. $Tr\,(A^p + B^p)^{\frac{1}{p}}$ is jointly concave for any $0 < p \leqslant 1$ ([8], Theorem 2.1). For any $0 < p \leqslant 1$, $Tr\,(A^{-p} + B^{-p})^{-\frac{1}{p}}$ is jointly concave by Theorem 2.1. So $Tr\,(A^p + B^p)^{\frac{1}{p}}$ is jointly concave for $-1 \leqslant p < 0$ and this completes the proof. $\square$

Remark 2.3. We notice that $f(t) = (t^p + 1)^{-\frac{1}{p}}$ is operator convex for $t > 0$ and $0 < p \leqslant 1$ ([2], Corollary V.2.6). The perspective function

$$P_f(L_A, R_B) = f\left(\frac{L_A}{R_B}\right) R_B = (L_A^p + R_B^p)^{-\frac{1}{p}} R_B^2,$$

is jointly convex from Theorem 1.1 which means that the joint convexity of $g(A, B) = Tr\,(A^p + B^p)^{-\frac{1}{p}}$ is not followed by applying the method which is used in Theorem 2.1. A similar argument is hold for $Tr\,(A^{-p}+B^{-p})^{\frac{1}{p}}$ with $f(t)=(t^{-p}+1)^{\frac{1}{p}}$ (note that $f(t)=(t^{-p}+1)^{\frac{1}{p}}$ is operator convex by ([12], Theorem 4)).

Now, we are ready to give our new proofs for the joint convexity of trace functions $Tr\,(A^{p}+B^{p})^{-\frac{1}{p}}$ and $Tr\,(A^{-p}+B^{-p})^{\frac{1}{p}}$. We do this according to [9] (Theorem 3.2 and Theorem 3.3) and so, first, we state a necessary part of them as a lemma for the convenience of the reader.

LEMMA 2.4. *Let f is an operator monotone function on $(0,\infty)$ and $\theta\in(0,1]$. Then*

$$TrK^{*}\left[f(L_AR_B^{-1})R_B\right]^{-\theta}(K)$$

is jointly convex for all $A,B\in B^{++}(H)$ and an arbitrary operator K on H.

THEOREM 2.5 (Bekjan). *Let $0<p\leqslant 1$ and $A,B\in B^{++}(H)$. Then $Tr\,(A^{p}+B^{p})^{-\frac{1}{p}}$ and $Tr\,(A^{-p}+B^{-p})^{\frac{1}{p}}$ are jointly convex in (A,B).*

Proof. At first, we prove the joint convexity of $Tr\,(A^{p}+B^{p})^{-\frac{1}{p}}$. Let $f(t)=(t^{p}+1)^{\frac{1}{p}}$ for any $t>0$ and $0<p\leqslant 1$. Since $f(t)$ is an operator monotone function, the following trace function

$$TrK^{*}\left[f(L_AR_B^{-1})R_B\right]^{-1}(K)=TrK^{*}\left[(L_A^{p}+R_B^{p})^{\frac{1}{p}}\right]^{-1}(K) \tag{2.3}$$

is jointly convex in (A,B) according to Lemma 2.4 with $\theta=1$.

Now, if we substitute K with the identity operator I in (2.3), then it follows that

$$\begin{aligned}TrI^{*}\left[(L_A^{p}+R_B^{p})^{\frac{1}{p}}\right]^{-1}(I)&=TrI^{*}\left(L_A^{p}+R_B^{p}\right)^{-\frac{1}{p}}(I)\\&=Tr\left(L_A^{p}+R_B^{p}\right)^{-\frac{1}{p}}(I)\end{aligned} \tag{2.4}$$

is jointly convex in (A,B). According to (2.2), (2.4) and a simple calculation, we have

$$Tr\left(L_A^{p}+R_B^{p}\right)^{-\frac{1}{p}}(I)=Tr\,(A^{p}+B^{p})^{-\frac{1}{p}},$$

which means that $Tr\,(A^{p}+B^{p})^{-\frac{1}{p}}$ is jointly convex in (A,B). Similarly, the joint convexity of $Tr\,(A^{-p}+B^{-p})^{\frac{1}{p}}$ can be proved. It is sufficient to replace the operator monotone function $f(t)=(t^{-p}+1)^{-\frac{1}{p}}$ for any $t>0$ and $0<p\leqslant 1$ in Lemma 2.4 with $\theta=1$. This completes the proof. $\square$

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