THE FOURTH POWER MEAN OF TWO-TERM EXPONENTIAL SUMS AND ITS APPLICATION

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The main purpose of this paper is using the analytic methods and a relation between the two-term cubic exponential sums and general Kloosterman sums to study the computational problem of one kind fourth power mean of two-term exponential sums, and give an exact computational formula for it. As an application of our result, we proved an interesting conclusion for the number of zeros of diagonal cubic forms.

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1. INTRODUCTION

Let $q \geq 3$ be a positive integer. For any integers m and n, the two-term exponential sum $C(m,n,k;q)$ and general Kloosterman sums $K(m,n;q)$ are defined as follows:

$$C(m,n,k;q) = \sum_{a=1}^{q} e\left(\frac{ma^k + na}{q}\right),$$

and

$$K(m,n;q) = {\sum_{a=1}^{q}}' e\left(\frac{ma + n\overline{a}}{q}\right),$$

where ${\sum_{a=1}^{q}}'$ denotes the summation over all $1 \leq a \leq q$ such that $(a,q) = 1$, $e(y) = e^{2\pi i y}$, and $\overline{a}$ denotes the multiplicative inverse of a mod q, that is, $a \cdot \overline{a} \equiv 1 \bmod q$.

Many authors have studied the various properties of $C(m,n,k;q)$ and $K(m,n;q)$, and obtained a series of results in [2, 3, 5–11] and [13–17]. For example, T. Cochrane and Z. Zheng [6] show for the general sum that

$$|C(m,n,k;q)| \leq k^{\omega(q)} q^{\frac{1}{2}},$$

where $\omega(q)$ denotes the number of all distinct prime divisors of q.

B.J. Birch [2] proved that for $1 \leq R \leq 4$, one has the identities

$$\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\left|\sum_{x=0}^{p-1} e\left(\frac{ax^3+bx}{p}\right)\right|^{2R} = p^{2R} + \frac{(2R-1)!(p-1)}{(R-1)!(R+1)!}p^R(2p-R+1). \tag{1}$$

W. Zhang [15], J. Li and Y. Liu [13] proved the identity

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma+n\bar{a}}{p}\right)\right|^{4} \tag{2}$$

$$= \begin{cases} 2p^3-3p^2-3p-1, & \text{if } \chi \text{ be the principal character mod } p; \\ 3p^3-8p^2, & \text{if } \chi \text{ be the Legendre mod } p; \\ p^2(2p-7), & \text{if } \chi \text{ be a non-real character mod } p, \end{cases}$$

where $(n,p)=1$.

M. Zhu and D. Han [17] used analytic methods to prove the identity

$$\sum_{m=1}^{p}\left|\sum_{a=1}^{p-1} e\left(\frac{ma^3+na}{p}\right)\right|^{4}$$

$$= \begin{cases} 2p^3-3p^2-3p, & \text{if } 3 \nmid p-1; \\ 2p^3-5p^2-15p+4\tau^3(\psi)+4\tau^3\left(\overline{\psi}\right), & \text{if } 3 | p-1, \end{cases}$$

where $(n,p)=1$, ψ be any three order character mod p.

The case $3|p-1$ in Zhu and Han's work is not explicit enough in the sense that it involves characters of order three modulo p.

Recently, W. Zhang and D. Han [16] studied the sixth power mean of the two-term exponential sums and proved that for any prime $p>3$ with $(3,p-1)=1$, one has the identity

$$\sum_{a=1}^{p-1}\left|\sum_{n=0}^{p-1} e\left(\frac{n^3+an}{p}\right)\right|^{6} = 5p^4-8p^3-p^2.$$

However, the method used in [16] seems to be unsuitable for the $2k$-th power mean

$$\sum_{a=1}^{p-1}\left|\sum_{n=0}^{p-1} e\left(\frac{an^3+n}{p}\right)\right|^{2k}, \text{ for all positive integers } k \geq 2.$$

This paper, as a note of [17], we shall combine the analytic methods, W. Zhang's work [15] and an interesting conversion formula of W. Duke and H. Iwaniec [10] to study this problem, and give an exact computational formula for the fourth power mean. That is, we shall prove the following conclusion.

Theorem. *Let $p > 3$ be a prime. Then for any integer n with $(n, p) = 1$, we have the identity*

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+na}{p}\right)\right|^4 = \begin{cases} 2p^3 - p^2, & \text{if } 3\dagger p-1, \\ 2p^3 - 7p^2, & \text{if } 3|p-1. \end{cases}$$

From this theorem and the work of B.J. Birch [2], we may immediately deduce a conclusion of S. Chowla, J. Cowles and M. Cowles [4]. That is, we have the following.

Corollary. *Let M_s be the number of solutions of the equation*

$$X_1^3 + X_2^3 + X_3^3 + \cdots + X_s^3 = 0$$

in the finite field $GF(p)$. For any prime $p \equiv 1 \bmod 3$, one has the identity

$$M_4 = p^3 + 6(p^2 - p).$$

Some notes: For $k \geq 4$, the fourth moments for $\sum_{n=1}^{p-1} |C(m, n, k; p)|^4$ can not be calculated exactly by using our method, unless $(k, p-1) = 1$. Since for $k|(p-1)$, we can not change the two-term k-th exponential sums $C(m, n, k; p)$ into the general Kloosterman sums. So our method is not applicable. For the same reason, our theorem also can not be generalized to any integer q, unless q is a square-free number. That is, $\mu(q) \neq 0$, where $\mu(n)$ is the Möbius function.

Here we propose the following two interesting open problems:

1. For general integers $h \geq 3$, whether there exists an exact expression for

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+na}{p}\right)\right|^{2h} ?$$

where p be an odd prime and $(n, p) = 1$.

2. Can the number of solutions to the cubic equation $x_1^3 + x_2^3 + x_3^3 + x_4^3 \equiv b \bmod p$ be calculated when $b \neq 0$?

2. SEVERAL LEMMAS

In this section, we will give several lemmas which are necessary in the proof of our theorem. First we have the following:

Lemma 1. *Let p be an odd prime with $(p-1, 3) = 1$, then we have the identity*

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+na}{p}\right)\right|^2 = p^2.$$

Proof. Since $(3, p-1) = 1$, so for all integers $1 \le a, b \le p-1$, the congruence $a^3 \equiv b^3 \bmod p$ holds if and only if $a = b$. Thus, from the trigonometric identity

$$\sum_{m=0}^{p-1} e\left(\frac{nm}{p}\right) = \begin{cases} p, & \text{if } (p,n) = p; \\ 0, & \text{if } (p,n) = 1 \end{cases}$$

we have

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+na}{p}\right)\right|^2 = \sum_{m=0}^{p-1}\left|\sum_{a=1}^{p-1} e\left(\frac{ma^3+na}{p}\right)+1\right|^2$$

$$= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{m=0}^{p-1} e\left(\frac{m(a^3-b^3)+n(a-b)}{p}\right) + \sum_{a=1}^{p-1}\sum_{m=0}^{p-1} e\left(\frac{ma^3+na}{p}\right)$$

$$+ \sum_{b=1}^{p-1}\sum_{m=0}^{p-1} e\left(\frac{-mb^3-nb}{p}\right) + p$$

$$= p \sum_{a=1}^{p-1}\sum_{\substack{b=1 \\ a^3\equiv b^3 \bmod p}}^{p-1} e\left(\frac{n(a-b)}{p}\right) + p = p^2.$$

This proves Lemma 1. □

Lemma 2. *Let p be an odd prime with $(p-1, 3) = 1$, and let χ be any non-principal character* mod p. *Then for any integer n with $(n, p) = 1$, we have the identity*

$$\left|\sum_{m=1}^{p-1}\chi(m)\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+na}{p}\right)\right|^2\right| = \begin{cases} p^{\frac{3}{2}}, & \text{if } \chi \neq \chi_2; \\ p, & \text{if } \chi = \chi_2, \end{cases}$$

where $\chi_2 = \left(\frac{*}{p}\right)$ denotes the Legendre symbol mod p.

Proof. It is clear that if a pass through a complete residue system mod p and $(n, p) = 1$, then na also pass through a complete residue system mod$p$. So from this property we have the identity

$$\left|\sum_{m=1}^{p-1}\chi(m)\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+na}{p}\right)\right|^2\right| = \left|\sum_{m=1}^{p-1}\chi(m)\left|\sum_{a=0}^{p-1} e\left(\frac{m\overline{n}^2a^3+a}{p}\right)\right|^2\right|$$

$$= \left|\sum_{m=1}^{p-1}\chi(n^2)\chi(m\overline{n}^2)\left|\sum_{a=0}^{p-1} e\left(\frac{m\overline{n}^2a^3+a}{p}\right)\right|^2\right| = \left|\sum_{m=1}^{p-1}\chi(m)\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^2\right|.$$

Thus, without loss of generality, we can assume $n = 1$. Since χ is a non-principal character mod p and $(3, p-1) = 1$, so χ is not a three order character mod p, from the properties of Gauss sums we have

$$
\begin{aligned}
(3)\quad & \sum_{m=1}^{p-1} \chi(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right|^2 \\
&= \sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{m=1}^{p-1} \chi(m)\, e\left(\frac{m(a^3-b^3)+n(a-b)}{p}\right) \\
&= \tau(\chi) \sum_{a=0}^{p-1}\sum_{b=0}^{p-1} \overline{\chi}\left(a^3 - b^3\right) e\left(\frac{a-b}{p}\right) \\
&= \tau(\chi)\tau\left(\overline{\chi}^3\right) + \tau(\chi) \sum_{a=0}^{p-1}\sum_{b=1}^{p-1} \overline{\chi}\left(a^3-1\right) \overline{\chi}\left(b^3\right) e\left(\frac{b(a-1)}{p}\right) \\
&= \tau(\chi)\tau\left(\overline{\chi}^3\right) + \tau(\chi)\tau\left(\overline{\chi}^3\right) \sum_{a=0}^{p-1} \overline{\chi}\left(a^3-1\right) \chi^3(a-1) \\
&= 2\tau(\chi)\tau\left(\overline{\chi}^3\right) + \tau(\chi)\tau\left(\overline{\chi}^3\right) \sum_{a=2}^{p-1} \overline{\chi}\left(a^3-1\right) \chi\left((a-1)^3\right) \\
&= 2\tau(\chi)\tau\left(\overline{\chi}^3\right) + \tau(\chi)\tau\left(\overline{\chi}^3\right) \sum_{a=1}^{p-2} \overline{\chi}\left(a^3+3a^2+3a\right) \chi\left(a^3\right) \\
&= 2\tau(\chi)\tau\left(\overline{\chi}^3\right) + \tau(\chi)\tau\left(\overline{\chi}^3\right) \sum_{a=1}^{p-2} \overline{\chi}\left(3a^2+3a+1\right) \\
&= \tau(\chi)\tau\left(\overline{\chi}^3\right) \sum_{a=0}^{p-1} \overline{\chi}\left(3a^2+3a+1\right) \\
&= \overline{\chi}(3)\chi(4)\tau(\chi)\tau\left(\overline{\chi}^3\right) \sum_{a=0}^{p-1} \overline{\chi}\left((2a+1)^2 + 4\cdot\overline{3} - 1\right) \\
&= \overline{\chi}(3)\chi(4)\tau(\chi)\tau\left(\overline{\chi}^3\right) \sum_{b=0}^{p-1} \overline{\chi}\left(b^2 + \overline{3}\right).
\end{aligned}
$$

On the other hand, if $(c, P) = 1$, then, from Theorem 7.5.4 of [12], we have

$$
(4)\qquad \sum_{a=0}^{p-1} e\left(\frac{ca^2}{p}\right) = \chi_2(c)\tau(\chi_2).
$$

From (4) and the definition and properties of Gauss sums, we have

$$(5)\quad \sum_{b=0}^{p-1}\overline{\chi}\left(b^2+c\right)=\frac{1}{\tau\left(\chi\right)}\sum_{a=1}^{p-1}\chi(a)\sum_{b=0}^{p-1}e\left(\frac{a(b^2+c)}{p}\right)$$
$$=\frac{1}{\tau\left(\chi\right)}\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ca}{p}\right)\sum_{b=0}^{p-1}e\left(\frac{ab^2}{p}\right)$$
$$=\frac{\tau(\chi_2)}{\tau\left(\chi\right)}\sum_{a=1}^{p-1}\chi(a)\chi_2(a)e\left(\frac{ca}{p}\right)=\overline{\chi}(c)\chi_2(c)\cdot\frac{\tau(\chi_2)\tau(\chi\chi_2)}{\tau\left(\chi\right)}.$$

Since $|\tau(\chi)|=\sqrt{p}$, by combining (3) and (5), we may deduce the identity

$$\left|\sum_{m=1}^{p-1}\chi\left(m\right)\left|\sum_{a=0}^{p-1}e\left(\frac{ma^3+na}{p}\right)\right|^2\right|=\begin{cases} p^{\frac{3}{2}}, & \text{if } \chi\neq\chi_2;\\ p, & \text{if } \chi=\chi_2.\end{cases}$$

This proves Lemma 2.

Lemma 3. *Let p be an odd prime with $3|(p-1)$. Then, for any integer a with $(a,p)=1$ and any three order character ψ, we have the identity*

$$\sum_{n=0}^{p-1}e\left(\frac{an^3+n}{p}\right)=\sum_{n=1}^{p-1}\psi\left(n\overline{a}\right)e\left(\frac{n-\overline{27an}}{p}\right).$$

Proof. This is an interesting conversion formula between the two-term cubic exponential sums and Kloosterman sums. Its proof can be found in W. Duke and H. Iwaniec [10] for general conclusion. □

3. PROOF OF THE THEOREM

In this section, we shall complete the proof of our theorem. First, from the orthogonality of characters mod p, we have

(6)
$$\sum_{\chi \bmod p}\left|\sum_{m=1}^{p-1}\chi\left(m\right)\left|\sum_{a=0}^{p-1}e\left(\frac{ma^3+na}{p}\right)\right|^2\right|^2=(p-1)\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1}e\left(\frac{ma^3+na}{p}\right)\right|^4.$$

On the other hand, if $3\nmid p-1$, then from Lemma 1 and Lemma 2, we also have

$$(7)\quad \sum_{\chi \bmod p}\left|\sum_{m=1}^{p-1}\chi\left(m\right)\left|\sum_{a=0}^{p-1}e\left(\frac{ma^3+na}{p}\right)\right|^2\right|^2$$

$$= \left(\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3+na}{p} \right) \right|^2 \right)^2 + \left| \sum_{m=1}^{p-1} \chi_2(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3+na}{p} \right) \right|^2 \right|^2$$

$$+ \sum_{\substack{\chi \bmod p \\ \chi \neq \chi_0, \chi_2}} \left| \sum_{m=1}^{p-1} \chi(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3+na}{p} \right) \right|^2 \right|^2$$

$$= p^4 + p^2 + (p-3)p^3 = 2p^4 - 3p^3 + p^2 = (p-1)(2p-1)p^2.$$

Combining (6) and (7), we can deduce the identity

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3+na}{p} \right) \right|^4 = 2p^3 - p^2. \tag{8}$$

If $3|p-1$, let ψ be a three order character mod p, then ψ must be a non-real character mod p, from identity (2) and Lemma 3 we have

$$\sum_{a=1}^{p-1} \left| \sum_{x=0}^{p-1} e\left(\frac{ax^3+nx}{p} \right) \right|^4 = \sum_{a=1}^{p-1} \left| \sum_{x=0}^{p-1} e\left(\frac{ax^3+x}{p} \right) \right|^4 \tag{9}$$

$$= \sum_{a=1}^{p-1} \left| \sum_{x=1}^{p-1} \psi(x\overline{a}) e\left(\frac{x - \overline{27ax}}{p} \right) \right|^4 = \sum_{a=1}^{p-1} \left| \sum_{x=1}^{p-1} \psi(x) e\left(\frac{ax+\overline{x}}{p} \right) \right|^4$$

$$= p^2 \cdot (2p-7).$$

Now from (8) and (9) we may immediately deduce

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3+na}{p} \right) \right|^4 = \begin{cases} 2p^3 - p^2, & \text{if } 3\dagger p-1, \\ 2p^3 - 7p^2, & \text{if } 3|p-1. \end{cases}$$

This completes the proof of our theorem.

Now, we are using formula (1) and our theorem to complete the proof of our corollary. First, taking $R = 2$ in (1), we have

(10)

$$\sum_{a=1}^{p-1} \left| \sum_{x=0}^{p-1} e\left(\frac{ax^3}{p} \right) \right|^4 + \sum_{b=1}^{p-1} \left| \sum_{x=0}^{p-1} e\left(\frac{bx}{p} \right) \right|^4 + \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \left| \sum_{x=0}^{p-1} e\left(\frac{ax^3+bx}{p} \right) \right|^4 + p^4$$

$$= p^4 + (p-1)(2p-1)p^2.$$

For all $1 \le b \le p-1$, from the properties of reduced residue system mod p, we have

$$\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \left| \sum_{x=0}^{p-1} e\left(\frac{ax^3+bx}{p} \right) \right|^4 = \sum_{b=1}^{p-1} \sum_{a=1}^{p-1} \left| \sum_{x=0}^{p-1} e\left(\frac{a(x\overline{b})^3 + b(x\overline{b})}{p} \right) \right|^4 \tag{11}$$

$$= \sum_{b=1}^{p-1}\sum_{a=1}^{p-1}\left|\sum_{x=0}^{p-1} e\left(\frac{a\bar{b}^3x^3+x}{p}\right)\right|^4 = (p-1)\sum_{a=1}^{p-1}\left|\sum_{x=0}^{p-1} e\left(\frac{ax^3+x}{p}\right)\right|^4.$$

From the properties of trigonometric sums, we have

$$\sum_{b=1}^{p-1}\left|\sum_{x=0}^{p-1} e\left(\frac{bx}{p}\right)\right|^4 = 0 \tag{12}$$

and

$$\sum_{a=1}^{p-1}\left|\sum_{x=0}^{p-1} e\left(\frac{ax^3}{p}\right)\right|^4 = pM_4 - p^4. \tag{13}$$

Combining (10), (11), (12), (13) and our theorem, we may immediately deduce the identity

$$p^4 + (p-1)(2p-1)p^2 = pM_4 - p^4 + (p-1)(2p^3 - 7p^2) + p^4,$$

or

$$M_4 = p^3 + 6p(p-1).$$

This completes the proof of our corollary. □

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