A recent question in generalized Ramsey theory is that for fixed positive integers \( s \leq t \), at least how many vertices can be covered by the vertices of no more than \( s \) monochromatic members of the family \( F \) in every edge coloring of \( K_n \) with \( t \) colors. This is related to \( d \)-chromatic Ramsey numbers introduced by Chung and Liu. In this paper, we first compute these numbers for stars generalizing the well-known result of Burr and Roberts. Then we extend a result of Cockayne and Lorimer to compute \( d \)-chromatic Ramsey numbers for stars and one matching.

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**Key words:** \( d \)-chromatic Ramsey number, edge coloring.

1. **INTRODUCTION**

Ramsey theory is an area of combinatorics which uses techniques from many branches of mathematics and is currently among the most active areas in combinatorics. Let \( G_1, \ldots, G_c \) be graphs. The Ramsey number denoted by \( r(G_1, \ldots, G_c) \) is defined to be the least number \( p \) such that if the edges of the complete graph \( K_p \) are arbitrarily colored with \( c \) colors, then for some \( i \) the spanning subgraph whose edges are colored with the \( i \)-th color contains \( G_i \). More information about the Ramsey numbers of known graphs can be found in the survey [10].

There are various types of Ramsey numbers that are important in the study of classical Ramsey numbers and also hypergraph Ramsey numbers. A question recently proposed by Gyárfás et al. in [6]; for fixed positive integers \( s \leq t \), at least how many vertices can be covered by the vertices of no more than \( s \) monochromatic members of the family \( F \) in every edge coloring of \( K_n \) with \( t \) colors. Several problems and interesting conjectures were presented in [6]. A basic problem here is to find the largest \( s \)-colored element of \( F \) that can be found in every \( t \)-coloring of \( K_n \). The answer for matchings when \( s = t - 1 \) was given in [6]; every \( t \)-coloring of \( K_n \) contains a \((t - 1)\)-colored matching of size

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$k$ provided that $n \geq 2k + \left[\frac{k-1}{2^{t-1}-1}\right]$. Note that for $t = 2, 3, 4$, we can guarantee the existence of a $(t-1)$-colored path on $2k$ vertices instead of a matching of size $k$. This was proved in [5, 9] and [8], respectively.

The above mentioned question is related to an old problem of Chung and Liu [3]; for a given graph $G$ and for fixed $s$, $t$, find the smallest $n$ such that in every $t$-coloring of the edges of $K_n$ there is a copy of $G$ colored with at most $s$ colors. More generally, let $1 \leq d < c$ and let $t = \binom{c}{d}$. Assume that $A_1, \ldots, A_t$ are all $d$-subsets of a set containing $c$ distinct colors. Let $G_1, \ldots, G_t$ be graphs. The $d$-chromatic Ramsey numbers denoted by $r^t_d(G_1, \ldots, G_t)$ is the least number $p$ such that, if the edges of the complete graph $K_p$ are arbitrarily colored with $c$ colors, then for some $i$, the subgraph whose edges are colored by colors in $A_i$ contains $G_i$.

For complete graphs these numbers were partially determined in [3] and [7]. However for these graphs, the problem is very few known and there are many open problems. For stars, when $d = 1$ it is a well-know result [1], and for $d = t - 1 = 2$ the value of $r^3_2(K_{1,i}, K_{1,j}, K_{1,t})$ was determined in [2]. For stars and one matching, when $d = 1$ it is again a well-known result; see [4].

In this paper, we first extend the result of [2] for stars to arbitrary $c, d$ with $d = c - 1 \geq 2$. Then we replace one of the stars by a matching generalizing the result of Cockayne and Lorimer to any $c, d$ with $d = c - 1 \geq 2$. To fix the notation, we use $r^t_{t-1}(G_1, \ldots, G_t)$ to denote the minimum $p$ such that any coloring of the edges of $K_p$ with $t$ colors $1, \ldots, t$ contains a copy of $G_i$ for some $i$, missing the color $i$. It is assumed throughout the paper that $m_i \leq m_j$, where $i \leq j$ and graphs are all simple and finite. A matching of size $m$ is denoted by $mP_2$ and a star of order $m + 1$ by $K_{1,m}$.

2. $(t - 1)$-COLORED STARS IN $t$-COLORED COMPLETE GRAPHS

In this section, we denote $\Sigma^t_{i=1}(m_i - 1)$ briefly by $S_t$. Let $\text{ex}(p, H)$ be the maximum number of edges in a graph on $p$ vertices which is $H$-free, i.e. it does not have $H$ as a subgraph. It is easily seen that $\text{ex}(p, K_{1,m}) \leq \frac{p(m-1)}{2}$. We use this fact in the proof of Theorem 2.1.

**Theorem 2.1.** Let $x = \left\lceil \frac{S_t+t-1}{t-1} \right\rceil$. Then $r^t_{t-1}(K_{1,m_1}, \ldots, K_{1,m_t}) \leq x + 1$.

**Proof.** Consider an edge coloring of $K_{x+1}$ with $t$ colors $1, \ldots, t$. Let $l_i$, $1 \leq i \leq t$ be the number of edges in color $i$ and $l = \Sigma_{i=1}^t l_i$. Note that $l = \frac{x(x+1)}{2}$. If for every $i$, we have $l - l_i \leq \frac{(x+1)(m_i-1)}{2}$, then $x+1 \leq \frac{S_t+t-1}{t-1}$, a contradiction. So there exists an $i$ with $l - l_i > \frac{(x+1)(m_i-1)}{2}$. Hence the induced subgraph on the edges with colors $\{1, \ldots, t\} - \{i\}$ contains a $K_{1,m_i}$, as required. □
For graphs $G_1, G_2,$ and $G_3$ with $|G_1| \leq |G_2| \leq |G_3|$ it is shown [3] that $r_2^2(G_1, G_2, G_3) \leq r(G_1, G_2)$ and the equality holds if $|G_3| \geq r(G_1, G_2)$, where $|G|$ is the number of vertices of $G$. Note that for graphs $G_1$ and $G_2$, $r(G_1, G_2) = r_1^2(G_1, G_2)$. So we can replace $|G_3| \geq r(G_1, G_2)$ by $|G_3| \geq r_1^2(G_1, G_2)$. Theorem 2.2, is a trivial generalization of this result.

**Theorem 2.2.** Let $G_1, \ldots, G_t$ be graphs. Then we have $r_{t-1}^t(G_1, \ldots, G_t) \leq r_{t-2}^t(G_1, \ldots, G_{t-1})$ and the equality holds if $|G_t| \geq r_{t-2}^t(G_1, \ldots, G_{t-1})$.

**Proof.** Let $l = r_{t-2}^t(G_1, \ldots, G_{t-1})$ and $c : E(G) \to \{1, 2, \ldots, t\}$ be a coloring of $G = K_t$. Define a new coloring $c'$ of $G$ with $t-1$ colors $1, 2, \ldots, t-1$ with $c'(e) = i$ if $c(e) = i$, $1 \leq i \leq t - 2$, and $c'(e) = t-1$ if $c(e) = t - 1$ or $c(e) = t$. By definition, $G$ contains a copy of $G_i$, for some $1 \leq i \leq t - 1$, in colors $\{1, \ldots, t-1\} - \{i\}$ which implies that $G$ contains a copy of $G_i$, for some $1 \leq i \leq t$, in colors $\{1, \ldots, t\} - \{i\}$, as required.

Now suppose that $|G_t| \geq r_{t-2}^t(G_1, \ldots, G_{t-1})$. By definition, there exists a coloring of $K_{t-1}$ with $t-1$ colors such that $K_{t-1}$ does not contain $G_i$, for some $1 \leq i \leq t - 1$, in colors $\{1, \ldots, t-1\} - \{i\}$. This is also a coloring of $K_{t-1}$ with $t$ colors without $G_i$, $1 \leq i \leq t$, in colors $\{1, \ldots, t\} - \{i\}$. Thus

$l - 1 < r_{t-1}^t(G_1, \ldots, G_t) \leq l = r_{t-2}^t(G_1, \ldots, G_{t-1}),$

completing the proof. $\square$

For abbreviation, we let $R_t = r_{t-1}^t(K_1,m_1,\ldots, K_1,m_t)$ and $x_t = \left[\frac{(\Sigma_{i=1}^t m_i)}{t-1}\right]$. Then by Theorem 2.2, we can assume that $m_t + 1 \leq R_{t-1}$. On the other hand, $R_t \leq R_{t-1} \leq \ldots \leq R_2$ and by Theorem 2.1, $R_t \leq x_t+1$. Hence $m_t \leq R_{t-1} - 1 \leq x_{t-1}$, which implies that $(t-2)m_t \leq (\Sigma_{i=1}^{t-1} m_i) - 1$. The last inequality is equivalent to $x_{t-1} \geq x_t$. Similarly, $m_{t-1} \leq m_t \leq R_{t-1} - 1 \leq R_{t-2} - 1 \leq x_{t-2}$ implies $x_{t-2} \geq x_{t-1}$. We continue in this way, obtaining that $x_i \leq x_j$ for $j < i$. Using this observation, we next find a lower bound for $r_{t-1}^t(K_1,m_1,\ldots, K_1,m_t)$.

**Theorem 2.3.** Let $x = \left[\frac{S_{t+1} t}{t-1}\right]$ and $m_t \leq R_{t-1} - 1$. Then

$r_{t-1}^t(K_1,m_1,\ldots, K_1,m_t) > x - 1.$

**Proof.** Let $p = x - \epsilon$ where $\epsilon = 1$ if $x$ is odd and $\epsilon = 0$, otherwise. By Vizing’s Theorem, there exists a proper edge coloring of $K_p$ with $p - 1$ colors. Let $r$, $1 \leq r < t$ be the smallest index such that $p - m_r \geq 0$ and $p - m_{r+1} < 0$ if it exists, and $r = t - 1$ otherwise. Partition these $p - 1$ colors into $r + 1$ new color classes as follows. Consider $p - m_i$ colors as the new color $i$, for $1 \leq i \leq r$ and all of the remaining colors as the new color $r + 1$. Note that since $p \leq x = x_r = \left[\frac{(\Sigma_{i=1}^r m_i)}{r-1}\right]$, we have $\Sigma_{i=1}^r (p - m_i) \leq p - 1$. This
yields an edge coloring of $K_p$ with $t$ colors $\{1, \ldots, t\}$ such that for each $i \leq r$, every vertex $v$ is adjacent to at least $p - m_i$ edges in color $i$ which rules out the existence of $K_{1,m_i}$ in colors $\{1, \ldots, t\} - \{i\}$. Moreover for $i \geq r + 1$, no $K_{1,m_i}$ occurs since $p < m_i$. Hence $r^{t}_{t-1}(K_{1,m_1}, \ldots, K_{1,m_t}) > p$, which is our assertion. \[\square\]

The above proof gives more, namely if $x = \left\lceil \frac{S_t + t - 1}{t - 1} \right\rceil$ is even, then

$$r^{t}_{t-1}(K_{1,m_1}, \ldots, K_{1,m_t}) > x.$$ 

Combining this with Theorem 2.1, we conclude the following.

**Corollary 2.4.** Let $x = \left\lceil \frac{S_t + t - 1}{t - 1} \right\rceil$ be even and $m_t \leq R_{t-1} - 1$. Then

$$r^{t}_{t-1}(K_{1,m_1}, \ldots, K_{1,m_t}) = x + 1.$$

**Remark.** Let $v_1, \ldots, v_x$ be vertices of $K_x$, where $x$ is odd. Eliminating $v_x$, there exists corresponding matching $M_{v_x}$ containing $(x - 1)/2$ independent edges $v_1v_{x-1}, v_2v_{x-2}, \ldots, v_{(x-1)/2}v_{(x+1)/2}$. Order these edges as above. Similarly, for each vertex $v_i$, $1 \leq i \leq x - 1$, there exists a matching $M_{v_i}$ containing $(x - 1)/2$ ordered edges. These matchings are used to construct certain edge colorings of $K_x$, for example as in the proof of Theorem 2.5.

**Theorem 2.5.** Let $x = \left\lceil \frac{S_t + t - 1}{t - 1} \right\rceil$, $m_t \leq R_{t-1} - 1$ and $S_t = q(t - 1) + h$, where $0 \leq h \leq t - 2$. Then

$$r^{t}_{t-1}(K_{1,m_1}, \ldots, K_{1,m_t}) = \begin{cases} x & \text{if } x \text{ is odd, } h = 0 \text{ and some } m_i \text{ is even,} \\ x + 1 & \text{otherwise.} \end{cases}$$

**Proof.** If $x$ is even, then by Corollary 2.4, $r^{t}_{t-1}(K_{1,m_1}, \ldots, K_{1,m_t}) = x + 1$. So we may assume that $x$ is odd. We consider three cases as follows.

**Case 1.** $h \geq 1$. Then $r = x + S_t + t - tx = S_t + t - (t - 1)x \geq 2$. Partition the vertices of $K_x$ as $v_1, v_2, \ldots, v_r$ plus $x - m_1$ classes $T_1, \ldots, T_{x-m_1}$ such that for $1 \leq i \leq t$, we have $T_i = \{u_{ij} : 1 \leq j \leq n_i\}$, where $n_i$ is the largest value $\lambda$ for which $i \leq x - m_\lambda$. For each vertex $u_{ij}$, $1 \leq j \leq t$, paint with $j$ all edges in $M_{u_{ij}}$. Let $v_1$ and $v_r$ be the vertices next to $T_1$ and $T_{x-m_1}$, respectively (see Fig. 1(a)).

For the vertex $v_1$ (respectively $v_r$) paint the edge $e = u_{ij}v_1 \in M_{v_1}$ (respectively $M_{v_r}$) with $j$ and paint the edge $e = u_{ij}u_{i'j'} \in M_{v_1}$ (respectively $M_{v_r}$) with $j$ if either $i < i'$ or $i = i'$ and $j < j'$ (respectively if either $i > i'$ or $i = i'$ and $j > j'$). The result is an edge coloring of $K_x$ with the property that for each vertex, every color $i$ appears on at least $x - m_i$ edges; that is, $r^{t}_{t-1}(K_{1,m_1}, \ldots, K_{1,m_t}) > x$, and so by Theorem 2.1, our assertion follows.

**Case 2.** $h = 0$, and every $m_i$ is odd. Then $S_t = q(t - 1)$, and $(t - 1)(q - x) + t = 1$. Partition the vertices of $K_x$ as a single vertex $v_x$ plus
\[ (x - m_1)/2 \text{ classes } T_1, \ldots, T_{(x-m_1)/2}, \text{ and } (x - m_1)/2 \text{ classes } T'_1, \ldots, T'_{(x-m_1)/2} \]

such that \( T_i = \{u_{ij} \colon 1 \leq j \leq n_i\} \) and \( T'_i = \{u'_{ij} \colon 1 \leq j \leq n_i\} \), where \( n_i \) is the largest value \( \lambda \) for which \( 2i \leq x - m_\lambda \). Set the classes \( T_1, \ldots, T_{(x-m_1)/2} \) one side of \( v_x \) and the classes \( T'_1, \ldots, T'_{(x-m_1)/2} \) on the other side of \( v_x \), respectively (see Fig. 1(b)). For each vertex \( u_{ij} \) (also \( u'_{ij} \)), \( 1 \leq j \leq t \), paint with \( j \) all edges in \( M_{u_{ij}} \) (also \( M_{u'_{ij}} \)). Moreover, for the vertex \( v_x \), paint with \( j \) the edge \( e = u_{ij}u'_{ij} \in M_{v_x} \). The result is an edge coloring of \( K_x \) with the property that for each vertex, every color \( i \) appears on exactly \( x - m_i \) edges; that is, \( r^t_{t-1}(K_{1,m_1}, \ldots, K_{1,m_t}) > x \), and so by Theorem 2.1, our assertion follows.

Case 3. \( h = 0 \), and some \( m_i \) is even. Let \( m_{i_0} \) be even. Then \( x - m_{i_0} \) is odd. Suppose, contrary to our claim, that \( r^t_{t-1}(K_{1,m_1}, \ldots, K_{1,m_t}) > x \). Consider the correspondent edge coloring of \( K_x \) with \( t \) colors \( 1, \ldots, t \). As a sufficient condition, the degree of each vertex in color \( i \), \( 1 \leq i \leq t \), is exactly \( x - m_i \). Then the induced subgraph with the edges in color \( i_0 \), is \( (x - m_{i_0}) \)-regular on \( x \) vertices, a contradiction. Hence \( r^t_{t-1}(K_{1,m_1}, \ldots, K_{1,m_t}) \leq x \), and so by Theorem 2.3, our assertion follows. \( \Box \)

It may be worth reminding the reader that Theorem 2.5 is consistent with the well-known result of [1] that \( r(K_{1,n}, K_{1,m}) = m + n - \epsilon \) where \( \epsilon = 1 \) if both \( n \) and \( m \) are even and \( \epsilon = 0 \), otherwise.

3. \((t - 1)\)-COLORED STARS-MATCHING

IN \( t \)-COLORED COMPLETE GRAPHS

In this section, we calculate \( r^t_{t-1}(K_{1,m_1}, \ldots, K_{1,m_{t-1}}, sP_2) \). In [4] the value of \( r^2_1(K_{1,m_1}, sP_2) = r(K_{1,m_1}, sP_2) \) has been determined, so we can assume that \( t \geq 3 \). Continuing the notation of Section 2, we denote \( \Sigma_{i=1}^{t-1}(m_i - 1) \) briefly by \( S_{t-1} \) and write \( R \) instead of \( r^t_{t-1}(K_{1,m_1}, \ldots, K_{1,m_{t-1}}, sP_2) \). If \( 2s \geq r^{t-1}_{t-2}(K_{1,m_1}, \ldots, K_{1,m_{t-1}}) \), then by Theorem 2.2, \( R = r^{t-1}_{t-2}(K_{1,m_1}, \ldots, K_{1,m_{t-1}}) \). Therefore in the following two lemmas we assume \( 2s < r^{t-1}_{t-2}(K_{1,m_1}, \ldots, K_{1,m_{t-1}}) \).
Lemma 3.1. If $t \geq 3$, $S_{t-1} < (2t-3)s - t + 2$, and $2s < r_{t-2}^{t-1}(K_{1,m_1}, \ldots, K_{1,m_{t-1}})$, then $R = 2s$.

Proof. Since $2s < r_{t-2}^{t-1}(K_{1,m_1}, \ldots, K_{1,m_{t-1}})$, there exists an edge coloring of $K_{2s-1}$ with colors $1, \ldots, t-1$, such that for each $i$, $1 \leq i \leq t-1$, the induced subgraph on the edges with colors $\{1, \ldots, t-1\} - i$ does not contain $K_{1,m_i}$. This also can be considered as an edge coloring of $K_{2s-1}$ with $t$ colors $1, \ldots, t$ such that in addition, the induced subgraph on the edges with colors $\{1, \ldots, t-1\}$ does not contain $sP_2$; that is, $R > 2s - 1$.

We now show that $R \leq 2s$. Consider an edge coloring of $K_{2s}$ with colors $1, \ldots, t$. Let $M$ be the maximal matching of edges with colors $1, \ldots, t-1$. Then $M$ has at most $s' \leq s-1$ independent edges, since otherwise we are done. Let $W$ be the set of those vertices that are not incident with these $s'$ edges. Note that $|W| \geq 2$, and every edge incident with two vertices in $W$ has color $t$. Moreover, every vertex is incident with at least $2s - m_i$ edges in color $i$, $1 \leq i \leq t-1$, since otherwise we are done. Thus every vertex is incident with at least $2(t-1)s - S_{t-1} - (t-1)$ edges in colors $1, \ldots, t-1$. Since $S_{t-1} < (2t-3)s - t + 2$, each of the vertices $w_1, w_2 \in W$ is incident with at least $s$ edges in colors $1, \ldots, t-1$; that is, there exists $e = uv \in M$ such that the color of both $w_1u$, and $w_2v$ belongs to $\{1, \ldots, t-1\}$, which contradicts the maximality of $M$. \[\square\]

Lemma 3.2. If $t \geq 3$, $S_{t-1} \geq (2t-3)s - t + 2$, and $2s < r_{t-2}^{t-1}(K_{1,m_1}, \ldots, K_{1,m_{t-1}})$, then $R = \left\lceil \frac{S_{t-1}+s}{t-1} \right\rceil + 1$.

Proof. Let $l = \left\lfloor \frac{S_{t-1}+s}{t-1} \right\rfloor$. To prove $R \leq l + 1$, consider an edge coloring of $K_{l+1}$ with $t$ colors $1, \ldots, t$. Let $M$ be the maximal matching of edges with colors $1, \ldots, t-1$. Then $M$ has at most $s' \leq s-1$ independent edges, since otherwise we are done. Let $W$ be the set of those vertices that are not incident with these $s'$ edges. Note that $|W| \geq 2$, and every edge incident with two vertices in $W$ has color $t$. Moreover, every vertex is incident with at least $l+1-m_i$ edges in color $i$, $1 \leq i \leq t-1$. Thus every vertex is incident with at least $(t-1)(l+1) - S_{t-1} - (t-1)$ edges in colors $1, \ldots, t-1$. Let $w_1, w_2 \in W$. Since $l > \frac{S_{t-1}+s-1}{t-1}$, $(t-1)(l+1) - S_{t-1} - (t-1) > s-1$ and so each of the vertices $w_1, w_2$ is incident with at least $s$ edges in colors $1, \ldots, t-1$. Therefore, there exists $e = uv \in M$ such that the color of both $w_1u$, and $w_2v$ belong to $\{1, \ldots, t-1\}$, which contradicts the maximality of $M$.

We now turn our attention to the lower bound. Set $n_i = l - m_i$, $1 \leq i \leq t-1$. Partition the vertices of $K_l$ into $t-1$ classes $X_i$, $1 \leq i \leq t-1$, with $|X_i| = n_i$ plus the set $X$ consists of the rest of the vertices. Note that $n_i \geq 0$ and $\sum_{i=1}^{t-1} n_i < l$. First let $z = \sum_{i=1}^{t-1} n_i$ be odd and suppose that $x \in X$. \[\square\]
By Vizing’s Theorem, there exists an edge coloring of the complete graph on $z + 1$ vertices $\{x\} \cup \bigcup_{i=1}^{t-1} X_i$ with $z$ colors. Set these $z$ colors into $t - 1$ color classes by considering $n_i$ colors as the new color $i$, $1 \leq i \leq t - 1$. This yields an edge coloring of $K_z$ with $t - 1$ colors $\{1, \ldots, t - 1\}$ such that every vertex $v \in \{x\} \cup \bigcup_{i=1}^{t-1} X_i$ is adjacent to $n_i = l - m_i$ edges in color $i$, $1 \leq i \leq t - 1$. Moreover, for $1 \leq i \leq t - 1$, paint with $i$ the edges having one vertex in $X_i$ and one vertex in $X - \{x\}$. Finally, paint with $t$ all the remaining edges. In this coloring of $K_t$, every vertex is adjacent to at least $n_i$ edges in color $i$, $1 \leq i \leq t - 1$, which rules out the existence of $K_{1,m_i}$ in colors $\{1, \ldots, t\} - \{i\}$. Moreover, the subgraph on the edges with colors $1, \ldots, t - 1$ contains at most $s - 1$ independent edges. We now suppose that $z = \Sigma_{i=1}^{t-1} n_i$ is even. Let $x, y \in X$. By Vizing’s Theorem, there exists an edge coloring of the complete graph on $z + 2$ vertices $\{x, y\} \cup \bigcup_{i=1}^{t-1} X_i$ with $z + 1$ colors. Without loss of generality we can assume that $xy$ has color 1. Partition these $z + 1$ colors into $t - 1$ color classes by considering $n_1 + 1$ colors as the new color 1 and $n_i$ colors as the new color $i$, $2 \leq i \leq t - 1$. This yields an edge coloring of $K_{z+2}$ with $t - 1$ colors $\{1, \ldots, t - 1\}$ such that every vertex $v \in \{x, y\} \cup \bigcup_{i=1}^{t-1} X_i$ is adjacent to at least $n_i = l - m_i$ edges in color $i$, $1 \leq i \leq t - 1$. Moreover, for $1 \leq i \leq t - 1$, paint with $i$ the edges having one vertex in $X_i$ and one vertex in $X - \{x, y\}$. Finally, paint with $t$ all the remaining edges and change the color of $xy$ into $t$. Again in this coloring of $K_t$, every vertex is adjacent to at least $n_i$ edges in color $i$, $1 \leq i \leq t - 1$, which rules out the existence of $K_{1,m_i}$ in colors $\{1, \ldots, t\} - \{i\}$. Moreover, the subgraph on the edges with colors $1, \ldots, t - 1$ contains at most $s - 1$ independent edges. Therefore, $R > l$, completing the proof. \hfill \Box

Combining Lemmas 3.1, and 3.2 with the above discussion we have the following theorem.

**Theorem 3.3.** Let $t \geq 3$. Then

i. If $2s \geq R_{t-1}$, then $R = R_{t-1}$.

ii. If $2s < R_{t-1}$ and $S_{t-1} < (2t - 3)s - t + 2$, then $R = 2s$.

iii. If $2s < R_{t-1}$ and $S_{t-1} \geq (2t - 3)s - t + 2$, then $R = \left\lceil \frac{S_{t-1} + s}{t-1} \right\rceil + 1$.

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