A GENERALIZATION OF RAMSEY THEORY FOR STARS AND ONE MATCHING

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Communicated by Ioan Tomescu

A recent question in generalized Ramsey theory is that for fixed positive integers $s \leq t$, at least how many vertices can be covered by the vertices of no more than s monochromatic members of the family $\mathcal{F}$ in every edge coloring of K_n with t colors. This is related to d-chromatic Ramsey numbers introduced by Chung and Liu. In this paper, we first compute these numbers for stars generalizing the well-known result of Burr and Roberts. Then we extend a result of Cockayne and Lorimer to compute d-chromatic Ramsey numbers for stars and one matching.

AMS 2010 Subject Classification: 05C55, 05D10.

Key words: d-chromatic Ramsey number, edge coloring.

1. INTRODUCTION

Ramsey theory is an area of combinatorics which uses techniques from many branches of mathematics and is currently among the most active areas in combinatorics. Let $G_1, \ldots, G_c$ be graphs. The *Ramsey number* denoted by $r(G_1, \ldots, G_c)$ is defined to be the least number p such that if the edges of the complete graph K_p are arbitrarily colored with c colors, then for some i the spanning subgraph whose edges are colored with the i-th color contains G_i. More information about the Ramsey numbers of known graphs can be found in the survey [10].

There are various types of Ramsey numbers that are important in the study of classical Ramsey numbers and also hypergraph Ramsey numbers. A question recently proposed by Gyárfás *et al.* in [6]; for fixed positive integers $s \leq t$, at least how many vertices can be covered by the vertices of no more than s monochromatic members of the family $\mathcal{F}$ in every edge coloring of K_n with t colors. Several problems and interesting conjectures were presented in [6]. A basic problem here is to find the largest s-colored element of $\mathcal{F}$ that can be found in every t-coloring of K_n. The answer for matchings when $s = t - 1$ was given in [6]; every t-coloring of K_n contains a $(t-1)$-colored matching of size

[1] This research was in part supported by a grant from IPM (No.93030059).

k provided that $n \geq 2k + [\frac{k-1}{2^{t-1}-1}]$. Note that for $t = 2, 3, 4$, we can guarantee the existence of a $(t-1)$-colored path on $2k$ vertices instead of a matching of size k. This was proved in [5, 9] and [8], respectively.

The above mentioned question is related to an old problem of Chung and Liu [3]; for a given graph G and for fixed s, t, find the smallest n such that in every t-coloring of the edges of K_n there is a copy of G colored with at most s colors. More generally, let $1 \leq d < c$ and let $t = \binom{c}{d}$. Assume that $A_1, \ldots, A_t$ are all d-subsets of a set containing c distinct colors. Let $G_1, \ldots, G_t$ be graphs. The *d-chromatic Ramsey numbers* denoted by $r_d^c(G_1, \ldots, G_t)$ is the least number p such that, if the edges of the complete graph K_p are arbitrarily colored with c colors, then for some i, the subgraph whose edges are colored by colors in A_i contains G_i.

For complete graphs these numbers were partially determined in [3] and [7]. However for these graphs, the problem is very few known and there are many open problems. For stars, when $d = 1$ it is a well-know result [1], and for $d = t - 1 = 2$ the value of $r_2^3(K_{1,i}, K_{1,j}, K_{1,l})$ was determined in [2]. For stars and one matching, when $d = 1$ it is again a well-known result; see [4].

In this paper, we first extend the result of [2] for stars to arbitrary c, d with $d = c - 1 \geq 2$. Then we replace one of the stars by a matching generalizing the result of Cockayne and Lorimer to any c, d with $d = c - 1 \geq 2$. To fix the notation, we use $r_{t-1}^t(G_1, \ldots, G_t)$ to denote the minimum p such that any coloring of the edges of K_p with t colors $1, \ldots, t$ contains a copy of G_i for some i, missing the color i. It is assumed throughout the paper that $m_i \leq m_j$, where $i \leq j$ and graphs are all simple and finite. A matching of size m is denoted by mP_2 and a star of order $m + 1$ by $K_{1,m}$.

2. $(t-1)$-COLORED STARS IN t-COLORED COMPLETE GRAPHS

In this section, we denote $\Sigma_{i=1}^t(m_i - 1)$ briefly by S_t. Let $\mathrm{ex}(p, H)$ be the maximum number of edges in a graph on p vertices which is H-free, *i.e.* it does not have H as a subgraph. It is easily seen that $\mathrm{ex}(p, K_{1,m}) \leq \frac{p(m-1)}{2}$. We use this fact in the proof of Theorem 2.1.

THEOREM 2.1. *Let* $x = \left[\frac{S_t+t-1}{t-1}\right]$. *Then* $r_{t-1}^t(K_{1,m_1}, \ldots, K_{1,m_t}) \leq x + 1$.

Proof. Consider an edge coloring of K_{x+1} with t colors $1, \ldots, t$. Let l_i, $1 \leq i \leq t$ be the number of edges in color i and $l = \Sigma_{i=1}^t l_i$. Note that $l = \frac{x(x+1)}{2}$. If for every i, we have $l - l_i \leq \frac{(x+1)(m_i-1)}{2}$, then $x + 1 \leq \frac{S_t+t-1}{t-1}$, a contradiction. So there exists an i with $l - l_i > \frac{(x+1)(m_i-1)}{2}$. Hence the induced subgraph on the edges with colors $\{1, \ldots, t\} - \{i\}$ contains a K_{1,m_i}, as required. □

For graphs G_1, G_2, and G_3 with $|G_1| \le |G_2| \le |G_3|$ it is shown [3] that $r_2^3(G_1, G_2, G_3) \le r(G_1, G_2)$ and the equality holds if $|G_3| \ge r(G_1, G_2)$, where $|G|$ is the number of vertices of G. Note that for graphs G_1 and G_2, $r(G_1, G_2) = r_1^2(G_1, G_2)$. So we can replace $|G_3| \ge r(G_1, G_2)$ by $|G_3| \ge r_1^2(G_1, G_2)$. Theorem 2.2, is a trivial generalization of this result.

THEOREM 2.2. *Let $G_1, \dots, G_t$ be graphs. Then we have $r_{t-1}^t(G_1, \dots, G_t) \le r_{t-2}^{t-1}(G_1, \dots, G_{t-1})$ and the equality holds if $|G_t| \ge r_{t-2}^{t-1}(G_1, \dots, G_{t-1})$.*

Proof. Let $l = r_{t-2}^{t-1}(G_1, \dots, G_{t-1})$ and $c : E(G) \to \{1, 2, \dots, t\}$ be a coloring of $G = K_l$. Define a new coloring c' of G with $t-1$ colors $\mathtt{1}, \mathtt{2}, \dots, \mathtt{t\text{-}1}$ with $c'(e) = \mathtt{i}$ if $c(e) = i$, $1 \le i \le t-2$, and $c'(e) = \mathtt{t\text{-}1}$ if $c(e) = t-1$ or $c(e) = t$. By definition, G contains a copy of G_i, for some $1 \le i \le t-1$, in colors $\{\mathtt{1}, \dots, \mathtt{t\text{-}1}\} - \{\mathtt{i}\}$ which implies that G contains a copy of G_i, for some $1 \le i \le t$, in colors $\{1, \dots, t\} - \{i\}$, as required.

Now suppose that $|G_t| \ge r_{t-2}^{t-1}(G_1, \dots, G_{t-1})$. By definition, there exists a coloring of K_{l-1} with $t-1$ colors such that K_{l-1} does not contain G_i, for some $1 \le i \le t-1$, in colors $\{1, \dots, t-1\} - \{i\}$. This is also a coloring of K_{l-1} with t colors without G_i, $1 \le i \le t$, in colors $\{1, \dots, t\} - \{i\}$. Thus

$$l - 1 < r_{t-1}^t(G_1, \dots, G_t) \le l = r_{t-2}^{t-1}(G_1, \dots, G_{t-1}),$$

completing the proof. □

For abbreviation, we let $R_t = r_{t-1}^t(K_{1,m_1}, \dots, K_{1,m_t})$ and $x_t = \left[\frac{(\Sigma_{i=1}^t m_i)-1}{t-1}\right]$. Then by Theorem 2.2, we can assume that $m_t + 1 \le R_{t-1}$. On the other hand, $R_t \le R_{t-1} \le \dots \le R_2$ and by Theorem 2.1, $R_t \le x_t + 1$. Hence $m_t \le R_{t-1} - 1 \le x_{t-1}$, which implies that $(t-2)m_t \le (\Sigma_{i=1}^{t-1} m_i) - 1$. The last inequality is equivalent to $x_{t-1} \ge x_t$. Similarly, $m_{t-1} \le m_t \le R_{t-1} - 1 \le R_{t-2} - 1 \le x_{t-2}$ implies $x_{t-2} \ge x_{t-1}$. We continue in this way, obtaining that $x_i \le x_j$ for $j < i$. Using this observation, we next find a lower bound for $r_{t-1}^t(K_{1,m_1}, \dots, K_{1,m_t})$.

THEOREM 2.3. *Let $x = \left[\frac{S_t+t-1}{t-1}\right]$ and $m_t \le R_{t-1} - 1$. Then*

$$r_{t-1}^t(K_{1,m_1}, \dots, K_{1,m_t}) > x - 1.$$

Proof. Let $p = x - \epsilon$ where $\epsilon = 1$ if x is odd and $\epsilon = 0$, otherwise. By Vizing's Theorem, there exists a proper edge coloring of K_p with $p-1$ colors. Let r, $1 \le r < t$ be the smallest index such that $p - m_r \ge 0$ and $p - m_{r+1} < 0$ if it exists, and $r = t-1$ otherwise. Partition these $p-1$ colors into $r+1$ new color classes as follows. Consider $p - m_i$ colors as the new color i, for $1 \le i \le r$ and all of the remaining colors as the new color $r+1$. Note that since $p \le x = x_t \le x_r = \left[\frac{(\Sigma_{i=1}^r m_i)-1}{r-1}\right]$, we have $\Sigma_{i=1}^r (p - m_i) \le p - 1$. This yields an edge coloring of K_p with t colors $\{1,\dots,t\}$ such that for each $i \le r$, every vertex v is adjacent to at least $p - m_i$ edges in color i which rules out the existence of K_{1,m_i} in colors $\{1,\dots,t\} - \{i\}$. Moreover for $i \ge r+1$, no K_{1,m_i} occurs since $p < m_i$. Hence $r^t_{t-1}(K_{1,m_1},\dots,K_{1,m_t}) > p$, which is our assertion. $\square$

The above proof gives more, namely if $x = \left[\frac{S_t+t-1}{t-1}\right]$ is even, then

$$r^t_{t-1}(K_{1,m_1},\dots,K_{1,m_t}) > x.$$

Combining this with Theorem 2.1, we conclude the following.

COROLLARY 2.4. *Let* $x = \left[\frac{S_t+t-1}{t-1}\right]$ *be even and* $m_t \le R_{t-1} - 1$. *Then*

$$r^t_{t-1}(K_{1,m_1},\dots,K_{1,m_t}) = x + 1.$$

Remark. Let $v_1,\dots,v_x$ be vertices of K_x, where x is odd. Eliminating v_x, there exists corresponding matching M_{v_x} containing $(x-1)/2$ independent edges $v_1v_{x-1}, v_2v_{x-2},\dots,v_{(x-1)/2}v_{(x+1)/2}$. Order these edges as above. Similarly, for each vertex v_i, $1 \le i \le x-1$, there exists a matching M_{v_i} containing $(x-1)/2$ ordered edges. These matchings are used to construct certain edge colorings of K_x, for example as in the proof of Theorem 2.5.

THEOREM 2.5. *Let* $x = \left[\frac{S_t+t-1}{t-1}\right]$, $m_t \le R_{t-1} - 1$ *and* $S_t = q(t-1) + h$, *where* $0 \le h \le t-2$. *Then*

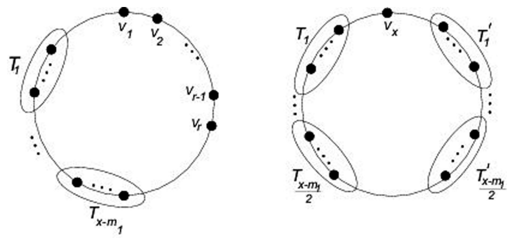
$$r^t_{t-1}(K_{1,m_1},\dots,K_{1,m_t}) = \begin{cases} x & \text{if } x \text{ is odd}, h = 0 \text{ and some } m_i \text{ is even}, \\ x+1 & \text{otherwise.} \end{cases}$$

Proof. If x is even, then by Corollary 2.4, $r^t_{t-1}(K_{1,m_1},\dots,K_{1,m_t}) = x+1$. So we may assume that x is odd. We consider three cases as follows.

Case 1. $h \ge 1$. Then $r = x + S_t + t - tx = S_t + t - (t-1)x \ge 2$. Partition the vertices of K_x as $v_1, v_2,\dots,v_r$ plus $x - m_1$ classes $T_1,\dots,T_{x-m_1}$ such that for $1 \le i \le t$, we have $T_i = \{u_{ij} : 1 \le j \le n_i\}$, where n_i is the largest value λ for which $i \le x - m_\lambda$. For each vertex u_{ij}, $1 \le j \le t$, paint with j all edges in $M_{u_{ij}}$. Let v_1 and v_r be the vertices next to T_1 and T_{x-m_1}, respectively (see Fig. 1(a)).

For the vertex v_1 (respectively v_r) paint the edge $e = u_{ij}v_l \in M_{v_1}$ (respectively M_{v_r}) with j and paint the edge $e = u_{ij}u_{i'j'} \in M_{v_1}$ (respectively M_{v_r}) with j if either $i < i'$ or $i = i'$ and $j < j'$ (respectively if either $i > i'$ or $i = i'$ and $j > j'$). The result is an edge coloring of K_x with the property that for each vertex, every color i appears on at least $x - m_i$ edges; that is, $r^t_{t-1}(K_{1,m_1},\dots,K_{1,m_t}) > x$, and so by Theorem 2.1, our assertion follows.

Case 2. $h = 0$, and every m_i is odd. Then $S_t = q(t-1)$, and $(t-1)(q-x)+t = 1$. Partition the vertices of K_x as a single vertex v_x plus



(a) (b)

Fig. 1. Graph K_x.

$(x-m_1)/2$ classes $T_1, \dots, T_{(x-m_1)/2}$, and $(x-m_1)/2$ classes $T_1', \dots, T'_{(x-m_1)/2}$ such that $T_i = \{u_{ij} : 1 \le j \le n_i\}$ and $T_i' = \{u'_{ij} : 1 \le j \le n_i\}$, where n_i is the largest value λ for which $2i \le x - m_\lambda$. Set the classes $T_1, \dots, T_{(x-m_1)/2}$ one side of v_x and the classes $T_1', \dots, T'_{(x-m_1)/2}$ on the other side of v_x, respectively (see Fig. 1(b)). For each vertex u_{ij} (also u'_{ij}), $1 \le j \le t$, paint with j all edges in $M_{u_{ij}}$ (also $M_{u'_{ij}}$). Moreover, for the vertex v_x, paint with j the edge $e = u_{ij}u'_{ij} \in M_{v_x}$. The result is an edge coloring of K_x with the property that for each vertex, every color i appears on exactly $x - m_i$ edges; that is, $r^t_{t-1}(K_{1,m_1}, \dots, K_{1,m_t}) > x$, and so by Theorem 2.1, our assertion follows.

Case 3. $h = 0$, and some m_i is even. Let m_{i_0} be even. Then $x - m_{i_0}$ is odd. Suppose, contrary to our claim, that $r^t_{t-1}(K_{1,m_1}, \dots, K_{1,m_t}) > x$. Consider the correspondent edge coloring of K_x with t colors $1, \dots, t$. As a sufficient condition, the degree of each vertex in color i, $1 \le i \le t$, is exactly $x - m_i$. Then the induced subgraph with the edges in color i_0, is $(x - m_{i_0})$-regular on x vertices, a contradiction. Hence $r^t_{t-1}(K_{1,m_1}, \dots, K_{1,m_t}) \le x$, and so by Theorem 2.3, our assertion follows. □

It may be worth reminding the reader that Theorem 2.5 is consistent with the well-known result of [1] that $r(K_{1,n}, K_{1,m}) = m + n - \epsilon$ where $\epsilon = 1$ if both n and m are even and $\epsilon = 0$, otherwise.

3. $(t-1)$-COLORED STARS-MATCHING IN t-COLORED COMPLETE GRAPHS

In this section, we calculate $r^t_{t-1}(K_{1,m_1}, \dots, K_{1,m_{t-1}}, sP_2)$. In [4] the value of $r^2_1(K_{1,m_1}, sP_2) = r(K_{1,m_1}, sP_2)$ has been determined, so we can assume that $t \ge 3$. Continuing the notation of Section 2, we denote $\Sigma_{i=1}^{t-1}(m_i - 1)$ briefly by S_{t-1} and write R instead of $r^t_{t-1}(K_{1,m_1}, \dots, K_{1,m_{t-1}}, sP_2)$. If $2s \ge r^{t-1}_{t-2}(K_{1,m_1}, \dots, K_{1,m_{t-1}})$, then by Theorem 2.2, $R = r^{t-1}_{t-2}(K_{1,m_1}, \dots, K_{1,m_{t-1}})$. Therefore in the following two lemmas we assume $2s < r^{t-1}_{t-2}(K_{1,m_1}, \dots, K_{1,m_{t-1}})$.

Lemma 3.1. *If $t \geq 3$, $S_{t-1} < (2t-3)s - t + 2$, and $2s < r_{t-2}^{t-1}(K_{1,m_1}, \ldots, K_{1,m_{t-1}})$, then $R = 2s$.*

Proof. Since $2s < r_{t-2}^{t-1}(K_{1,m_1}, \ldots, K_{1,m_{t-1}})$, there exists an edge coloring of K_{2s-1} with colors $1, \ldots, t-1$, such that for each i, $1 \leq i \leq t-1$, the induced subgraph on the edges with colors $\{1, \ldots, t-1\} - i$ does not contain K_{1,m_i}. This also can be considered as an edge coloring of K_{2s-1} with t colors $1, \ldots, t$ such that in addition, the induced subgraph on the edges with colors $\{1, \ldots, t-1\}$ does not contain sP_2; that is, $R > 2s - 1$.

We now show that $R \leq 2s$. Consider an edge coloring of K_{2s} with colors $1, \ldots, t$. Let M be the maximal matching of edges with colors $1, \ldots, t-1$. Then M has at most $s' \leq s-1$ independent edges, since otherwise we are done. Let W be the set of those vertices that are not incident with these s' edges. Note that $|W| \geq 2$, and every edge incident with two vertices in W has color t. Moreover, every vertex is incident with at least $2s - m_i$ edges in color i, $1 \leq i \leq t-1$, since otherwise we are done. Thus every vertex is incident with at least $2(t-1)s - S_{t-1} - (t-1)$ edges in colors $1, \ldots, t-1$. Since $S_{t-1} < (2t-3)s - t + 2$, each of the vertices $w_1, w_2 \in W$ is incident with at least s edges in colors $1, \ldots, t-1$; that is, there exists $e = uv \in M$ such that the color of both w_1u, and w_2v belongs to $\{1, \ldots, t-1\}$, which contradicts the maximality of M. □

Lemma 3.2. *If $t \geq 3$, $S_{t-1} \geq (2t-3)s - t + 2$, and $2s < r_{t-2}^{t-1}(K_{1,m_1}, \ldots, K_{1,m_{t-1}})$, then $R = \left\lceil \frac{S_{t-1}+s}{t-1} \right\rceil + 1$.*

Proof. Let $l = \left\lceil \frac{S_{t-1}+s}{t-1} \right\rceil$. To prove $R \leq l+1$, consider an edge coloring of K_{l+1} with t colors $1, \ldots, t$. Let M be the maximal matching of edges with colors $1, \ldots, t-1$. Then M has at most $s' \leq s-1$ independent edges, since otherwise we are done. Let W be the set of those vertices that are not incident with these s' edges. Note that $|W| \geq 2$, and every edge incident with two vertices in W has color t. Moreover, every vertex is incident with at least $l + 1 - m_i$ edges in color i, $1 \leq i \leq t-1$. Thus every vertex is incident with at least $(t-1)(l+1) - S_{t-1} - (t-1)$ edges in colors $1, \ldots, t-1$. Let $w_1, w_2 \in W$. Since $l > \frac{S_{t-1}+s-1}{t-1}$, $(t-1)(l+1) - S_{t-1} - (t-1) > s-1$ and so each of the vertices w_1, w_2 is incident with at least s edges in colors $1, \ldots, t-1$. Therefore, there exists $e = uv \in M$ such that the color of both w_1u, and w_2v belong to $\{1, \ldots, t-1\}$, which contradicts the maximality of M.

We now turn our attention to the lower bound. Set $n_i = l - m_i$, $1 \leq i \leq t-1$. Partition the vertices of K_l into $t-1$ classes X_i, $1 \leq i \leq t-1$, with $|X_i| = n_i$ plus the set X consists of the rest of the vertices. Note that $n_i \geq 0$ and $\Sigma_{i=1}^{t=1} n_i < l$. First let $z = \Sigma_{i=1}^{t-1} n_i$ be odd and suppose that $x \in X$.

By Vizing's Theorem, there exists an edge coloring of the complete graph on $z+1$ vertices $\{x\} \cup \bigcup_{i=1}^{t-1} X_i$ with z colors. Set these z colors into $t-1$ color classes by considering n_i colors as the new color i, $1 \leq i \leq t-1$. This yields an edge coloring of K_z with $t-1$ colors $\{1, \ldots, t-1\}$ such that every vertex $v \in \{x\} \cup \bigcup_{i=1}^{t-1} X_i$ is adjacent to $n_i = l - m_i$ edges in color i, $1 \leq i \leq t-1$. Moreover, for $1 \leq i \leq t-1$, paint with i the edges having one vertex in X_i and one vertex in $X - \{x\}$. Finally, paint with t all the remaining edges. In this coloring of K_l, every vertex is adjacent to at least n_i edges in color i, $1 \leq i \leq t-1$, which rules out the existence of K_{1,m_i} in colors $\{1, \ldots, t\} - \{i\}$. Moreover, the subgraph on the edges with colors $1, \ldots, t-1$ contains at most $s-1$ independent edges. We now suppose that $z = \Sigma_{i=1}^{t-1} n_i$ is even. Let $x, y \in X$. By Vizing's Theorem, there exists an edge coloring of the complete graph on $z+2$ vertices $\{x, y\} \cup \bigcup_{i=1}^{t-1} X_i$ with $z+1$ colors. Without loss of generality we can assume that xy has color 1. Partition these $z+1$ colors into $t-1$ color classes by considering $n_1 + 1$ colors as the new color 1 and n_i colors as the new color i, $2 \leq i \leq t-1$. This yields an edge coloring of K_{z+2} with $t-1$ colors $\{1, \ldots, t-1\}$ such that every vertex $v \in \{x, y\} \cup \bigcup_{i=1}^{t-1} X_i$ is adjacent to at least $n_i = l - m_i$ edges in color i, $1 \leq i \leq t-1$. Moreover, for $1 \leq i \leq t-1$, paint with i the edges having one vertex in X_i and one vertex in $X - \{x, y\}$. Finally, paint with t all the remaining edges and change the color of xy into t. Again in this coloring of K_l, every vertex is adjacent to at least n_i edges in color i, $1 \leq i \leq t-1$, which rules out the existence of K_{1,m_i} in colors $\{1, \ldots, t\} - \{i\}$. Moreover, the subgraph on the edges with colors $1, \ldots, t-1$ contains at most $s-1$ independent edges. Therefore, $R > l$, completing the proof. □

Combining Lemmas 3.1, and 3.2 with the above discussion we have the following theorem.

THEOREM 3.3. *Let $t \geq 3$. Then*

i. If $2s \geq R_{t-1}$, then $R = R_{t-1}$.
ii. If $2s < R_{t-1}$ and $S_{t-1} < (2t-3)s - t + 2$, then $R = 2s$.
iii. If $2s < R_{t-1}$ and $S_{t-1} \geq (2t-3)s - t + 2$, then $R = \left\lceil \frac{S_{t-1}+s}{t-1} \right\rceil + 1$.

Acknowledgments. We would like to thank the anonymous reviewer for his/her comments and suggestions that helped us to improve the manuscript.

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Received 8 July 2015

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