SOME INEQUALITIES BETWEEN FUNCTIONALS RELATED TO GENERALIZED LIMITS

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Communicated by Lucian Beznea

Recall that A-statistical core theorem determines a class of regular matrices for which $\limsup(Tx) \leq st_A - \limsup x$ for all $x \in m$. The main object of this paper is to study an inequality between functionals which is sharper than that of the A-statistical core theorem. We also study the relationship between these functionals and some generalized limits which are called S_A-limits and A-Banach limits.

AMS 2010 Subject Classification: Primary 40A05; Secondary 26A03, 11B05.

Key words: A-statistical convergence, Knopp's and A-statistical core of a sequence, absolute equivalence.

1. INTRODUCTION

Let m and c be the spaces of all bounded and convergent real sequences $x = (x_k)$ normed by $\|x\| = \sup_n |x_n|$, respectively. Let

$$m_0 := \left\{ x \in m : \sup_n \left| \sum_{k=1}^{n} x_k \right| < \infty \right\}.$$

Observe that $(x_n) \in m$ if and only if $(x_{n+1} - x_n) \in m_0$. Let $A = (a_{nk})$ be an infinite matrix with real entries. Given a sequence x the A-transform of x, denoted as $Ax = ((Ax)_n)$, is given by $(Ax)_n = \sum_k a_{nk} x_k$ provided that the series converges for each n. Let $\lim_A x := \lim_n (Ax)_n$ whenever the limit exists. By c_A we denote the summability domain of A, *i.e.*, $c_A = \{x : \lim_A x \text{ exists}\}$. We say that A is regular [3, 22] if $\lim_n (Ax)_n = \lim_k x_k$ for each $x \in c$.

For any nonnegative regular matrix A we define the A-density of a set $K \subseteq \mathbb{N}$, denoted as $\delta_A(K)$ as

$$\delta_A(K) = \lim_n \sum_k a_{nk}\, \chi_K(k) = \lim_n (A\chi_K)_n,$$

provided that the limit exists, where χ_K denotes the characteristic sequence of the set K. When A is the Cesàro matrix, C_1, the resulting C_1-density is called the natural density, which we will denote by $\delta(K)$. Throughout the paper the statement $\delta(K) \neq 0$ will mean either $\delta(K) > 0$ or that the natural density of K does not exist.

Using a density, we say that a sequence $x = (x_k)$ is A-statistically convergent to a number ℓ if, for every $\epsilon > 0$,

$$\delta_A(\{k \in \mathbb{N} : \ |x_k - \ell| \geq \epsilon\}) = 0.$$

We denote this limit by $st_A - \lim x = \ell$. In particular, when $A = C_1$, the resulting notation is simply $st - \lim x = \ell$ [5,12,14,16,18,21,26].

Now we recall the concepts of statistical limit superior and statistical limit inferior from [6,10,15]. Let

$$st_A - \limsup x = \begin{cases} \sup B_x, & \text{if } B_x \neq \emptyset \\ -\infty, & \text{if } B_x = \emptyset, \end{cases}$$

where $B_x = \{b \in \Re : \delta_A(\{k \in \mathbb{N} : \ x_k > b\}) \neq 0\}$. Also the A-statistical limit inferior of x is given by

$$st_A - \liminf x = \begin{cases} \inf A_x, & \text{if } A_x \neq \emptyset \\ +\infty, & \text{if } A_x = \emptyset, \end{cases}$$

where $A_x = \{a \in \Re : \delta_A(\{k \in \mathbb{N} : \ x_k < a\}) \neq 0\}$.

In this paper, we define some functionals and study their properties. We obtain two inequalities which reduce to those given in [24] when $A = C_1$. We also provide a relationship between the A-statistical core theorem and absolute equivalence.

2. THE RELATIONSHIP BETWEEN FUNCTIONALS AND GENERALIZED LIMITS

Let $\mathcal{B}$ be the class of (necessarily continuous) linear functionals β on m which are nonnegative and regular, that is, if $x \geq 0$, (*i.e.*, $x_k \geq 0$ for all $k \in \mathbb{N} := \{1, 2, \cdots\}$) then $\beta(x) \geq 0$, and $\beta(x) = \lim_k x_k$, for each $x \in c$. If β has the additional property that $\beta(\sigma(x)) = \beta(x)$ for all $x \in m$, where σ is the left shift operator, defined by $\sigma(x_1, x_2, \cdots) = (x_2, x_3, \cdots)$ then β is called a Banach limit. The existence of Banach limits has been shown by Banach [1] and another proof may be found in [2]. It is well known [19] that the space of all almost convergent sequences can be represented as the set of all $x \in m$ which have the same value under any Banach limit.

A matrix $A = [a_{nk}]$ is called translative [22] if for any $x \in m$ with $\lim_A x = \ell$ we also get $\lim_A \sigma(x) = \ell$. A necessary and sufficient condition for a regular matrix A to be (boundedly) translative [22] is that $\lim_n \sum_k |a_{n,k+1} - a_{nk}| = 0$.

A regular matrix, A, is (boundedly) translative if and only if A sums all almost convergent sequences and equals their Banach limits [22]. Such methods are called strongly regular.

Following [27] we recall some definitions and notations.

Definition 1. Let L be a linear functional on m that satisfies the following properties:

(1): $L(x) \geq 0$, if $x \geq 0$, (positivity of L),
(2): $L(x) = \lim_k x_k$ for $x \in c$, (regularity of L),
(3): For every $E \subseteq \mathbb{N}$ such that $\delta_A(E) = 0$ implies that $L(\chi_E) = 0$.

Every such L will be called an S_A-limit, and denote their collection by SL_A. In the particular case when $A = C_1$ is the Cesàro matrix, any such L will be called an S-limit and their collection denoted by SL. Freedman [13] proved that the space of all bounded statistically convergent sequences can be represented as the set of all $x \in m$ which have the same value under any S-limit.

Definition 2 (A-Banach limits). Let L be a bounded linear functional on m that satisfies the following conditions:

(1): $L(x) \geq 0$ if $x_k \geq 0$ for all k.
(2): $L(x) = \lim_k x_k$ if $x \in c$,
(3): $L(x) \leq \limsup_n \sup_j \sum_k a_{nk} x_{k+j}$ for every $x \in m$.

Any such L will be called an A-Banach limit, and the collection of all such functionals will be denoted by BL_A. In the particular case when $A = C_1$ one gets $BL_{C_1} = BL$ where BL is the set of all Banach limits.

In [27], the authors have proved that when A is a nonnegative regular matrix, both A-Banach limits and S_A-limits exist. The sublinear functionals that generate or dominate these limits have also been examined.

Following Simons [25], we recall the definitions of functionals that generate and/or dominate generalized limits.

Definition 3. Let R and T be sublinear functionals on m and let $\mathcal{L}$ be a collection of bounded linear functionals on m.

(i): We say that R generates $\mathcal{L}$ if for any $L \in m^*$ and $L(x) \leq R(x)$ for all $x \in m$ together imply that $L \in \mathcal{L}$,
(ii): We say that T dominates $\mathcal{L}$ if for every $L \in \mathcal{L}$ we have $L(x) \leq T(x)$ for all $x \in m$,

where m^* denotes the algebraic dual of m.

A sublinear functional, R, on m generates $\mathcal{L}$ if and only if $R(x) \leq W(x)$ for all x, where

$$W(x) \ := \ \sup\{L(x) : \ L \in \mathcal{L}\}, \text{ for all } x \in m.$$

Trivially a sublinear functional, R, dominates $\mathcal{L}$ if and only if $R(x) \geq W(x)$ for all $x \in m$. Combining these two statements, a sublinear functional R on m generates as well as dominates $\mathcal{L}$-limits if and only if it equals W.

We consider the following functionals on m :

$$\begin{aligned} L(x) &= \limsup x_n &&, \quad P_A(x) = st_A - \limsup x_n, \\ L_A(x) &= \limsup_n \sup_j \sum_k a_{nk} x_{k+j} &&, \\ w(x) &= \inf_{z \in m_o} L(x+z) &&, \quad w_A^*(x) = \inf_{z \in m_o} P_A(x+z). \end{aligned}$$

It should be noted that if $z \in m_o$, then $L(z) \geq 0$ (see, e.g. [4]), hence w is well-defined on m (see [11]). The same argument also applies to show that P_A is also well-defined on m. Since $P_A(x+z) \leq L(x+z)$ (see [10]), we have $P_A(x) \leq w_A^*(x)$. Let

$$\Psi_A^*(x) = \inf_{z \in m_o} \limsup_n \sup_j \sum_k a_{nk} (x+z)_{k+j}.$$

It is also known from [8] that if A is a strongly regular matrix then Ψ_A^* is well defined on m.

Theorem 1. *Ψ_A^* and w_A^* are sublinear functionals on m.*

Proof. Given $x, y \in m$ and $\varepsilon > 0$, there exist respectively $z_1, z_2 \in m_0$ such that

$$P_A(x+z_1) < w_A^*(x) + \varepsilon, \ P_A(y+z_2) < w_A^*(y) + \varepsilon.$$

Since $z_1 + z_2 \in m_0$, it follows from the above inequalities that

$$w_A^*(x+y) \leq P_A(x+y+z_1+z_2) \leq w_A^*(x) + w_A^*(y) + 2\varepsilon.$$

Since ε is arbitrary, w_A^* is subadditive.
Also for $\alpha > 0$, $w_A^*(\alpha x) = \alpha \inf_{z \in m_o} P_A(x + \dfrac{z}{\alpha}) = \alpha w_A^*(x)$ which completes the proof. It is known from [8] that Ψ_A^* is also sublinear. $\square$

Theorem 2. *w_A^* generates S_A-limits.*

Proof. Let L be a linear functional such that $L(x) \leq w_A^*(x)$, for every $x \in m$. Observe that $L(x) \leq w_A^*(x) \leq P_A(x)$ on m since $(0, 0, 0, ...) \in m_0$. It is proved in [27] that the sublinear functional P_A generates S_A-limits which implies that L is an S_A-limit. Thus w_A^* generates S_A-limits. $\square$

THEOREM 3. *Let A be a strongly regular matrix. Then Ψ_A^* both generates and dominates A-Banach limits.*

Proof. Let L be a linear functional such that $L(x) \leq \Psi_A^*(x)$ for every $x \in m$. Since $(0,0,0,...) \in m_0$ again one can see that

$$\liminf_n \inf_j \sum_k a_{nk} x_{k+j} \leq L(x) \leq \limsup_n \sup_j \sum_k a_{nk} x_{k+j}, \; x \in m$$

and A is strongly regular which gives that L is an A-Banach limit. Thus Ψ_A^* generates A-Banach limits.

In order to see that Ψ_A^* dominates A-Banach limits note that if L is an A-Banach limit then $L(x) \leq \limsup_n \sup_j \sum_k a_{nk} x_{k+j}$ holds on m. From [8] it is known that $\limsup_n \sup_j \sum_k a_{nk} x_{k+j} = \inf_{z \in m_o} \limsup_n \sup_j \sum_k a_{nk} (x+z)_{k+j}$ whenever A is strongly regular. Then we have for every $x \in m$ that $L(x) \leq \Psi_A^*(x)$ which means Ψ_A^* dominates A-Banach limits. $\square$

Let Ω be a sublinear functional on m and define $\Psi_\Omega(x) = \inf_{z \in m_0} \Omega(x+z)$. If $\Omega(z) \geq 0$ for every $z \in m_0$ then Ψ_Ω is well defined [8, 11]. Similarly it can be shown that Ψ_Ω is sublinear on m and the following holds.

THEOREM 4. *If Ω generates S_A-limits then Ψ_Ω generates S_A-limits.*

THEOREM 5. *If Ω generates A-Banach limits then Ψ_Ω generates A-Banach limits.*

3. SOME INEQUALITIES BETWEEN FUNCTIONALS

In [10], for every $x \in m$, the inequality

$$\limsup Tx \leq st_A - \limsup x \tag{3.1}$$

is studied. Recall from [10] and [15] that if $\sup_n \sum_k |t_{nk}| < \infty$ then (3.1) holds if and only if

(i) $T \in \tau_A^*$, *i.e.*, T is regular and $\lim_n \sum_{k \in E} |t_{nk}| = 0$ whenever $\delta_A(E) = 0$,

(ii) $\lim_n \sum_k |t_{nk}| = 1$.

Now we are ready to give our inequalities. Note that the first one is sharper than that of (3.1) and if we take $A = C_1$, these inequalities reduce to those given in [24].

THEOREM 6. *Assume that* $\sup_n \sum_k |t_{nk}| < \infty$. *Then*

$$L(Tx) \leq w_A^*(x) \text{ for every } x \in m \tag{3.2}$$

if and only if
(i) T *is strongly regular,*
(ii) $\lim_n \sum_{k\in E} |t_{nk}| = 0$ *whenever* $\delta_A(E) = 0$
(iii) $\lim_n \sum_k |t_{nk}| = 1.$

Proof. Observe that $w_A^*(x) \leq P_A(x)$ on m. So, by hypothesis, for every $x \in m$ we have

$$L(Tx) \leq w_A^*(x) \leq P_A(x).$$

By Theorem 7 of [10], we get that $\lim_n \sum_k |t_{nk}| = 1$ and $T \in \tau_A^*$. On the other hand, since $w_A^*(x) \leq w(x)$ on m, we have $L(Tx) \leq w(x)$. So it follows from Theorem 3 of [11] that T is also strongly regular. This completes the proof of necessity.

For the proof of sufficiency, Theorem 7 of [10] implies that

$$\limsup Tx \leq P_A(x) \text{ for every } x \in m.$$

Hence, we have $\limsup T(x+z) \leq P_A(x+z)$ where $z \in m_o$. This implies

$$\inf_{z\in m_o} \limsup T(x+z) \leq \inf_{z\in m_o} P_A(x+z) = w_A^*(x). \tag{3.3}$$

Now let $\theta(x) := \inf_{z\in m_o} \limsup T(x+z)$. Then

$$\theta(x) \geq \inf_{z\in m_o} \left\{\limsup Tx + \liminf Tz\right\}. \tag{3.4}$$

But by Lemma 4 of [11], $Tz = By$, where $y = \left\{\sum_{i=1}^{n} x_i\right\} \in m$, $z \in m_o$ and $B = (b_{nk})$ is given by $b_{nk} = t_{nk} - t_{n,k+1}$. Since A is strongly regular, $Tz = By$ tends to zero. Hence, by (3.4), we have

$$\theta(x) \geq \limsup Tx. \tag{3.5}$$

Combining (3.3) and (3.5), we conclude, for every $x \in m$, that

$$\limsup Tx \leq w_A^*(x)$$

which completes the proof. □

THEOREM 7. *Let B be a normal matrix and denote its triangular inverse by $B^{-1} = (b_{nk}^{-1})$. For an arbitrary matrix T, in order that, whenever $Bx \in m$, Ax exists, bounded and satisfy*

$$L(Tx) \leq w_A^*(Bx) \tag{3.6}$$

it is necessary and sufficient that the following conditions hold:
(i) $C := TB^{-1}$ *exists,*
(ii) C *is strongly regular,*
(iii) $\lim_n \sum_{k \in E} |c_{nk}| = 0$ *for every* $E \subseteq \mathbb{N}$ *with* $\delta_A(E) = 0$,
(iv) $\lim_n \sum_k |c_{nk}| = 1$,
(v) for fixed n, $\lim_v \sum_{k=0}^{v} \left| \sum_{j=v+1}^{\infty} t_{nj} b_{jk}^{-1} \right| = 0.$

Proof. Recall that by a normal matrix we mean a triangular matrix with non-zero diagonal entries. If $(Tx)_n$ exists for every n whenever $Bx \in m$ then by Lemma 2 of [4] we see that (i) and (v) hold. That Lemma also implies $Tx = Cy$, where $y = Bx$. Since $Tx \in m$ we have $Cy \in m$. Hence (3.6) implies $L(Cy) \leq w_A^*(y)$. Now it follows from Theorem 6 that (ii), (iii) and (iv) hold. For the proof of sufficiency assume that (i-v) hold. Then (i), (ii), (iv) and (v) imply the conditions of Lemma 2 of [4]. So it follows from the same Lemma that $Cy \in m$, hence $Tx \in m$. Theorem 6 yields that $L(Cy) \leq w_A^*(y)$. Since $y = Bx$ and $Cy = Tx$ we conclude that

$$L(Tx) \leq w_A^*(Bx),$$

which completes the proof. □

4. ABSOLUTE EQUIVALENCE AND CORE

In this section we establish a relationship between the A-statistical core theorem and absolute equivalence which is complementary to [10] and [24]. To give our results, we pause to collect some notation.

If x is a bounded real sequence then the Knopp core of x, denoted by $K - core\{x\}$, is given by the closed interval $[\liminf x,\ \limsup x]$ [17]. The inequality

$$\limsup Ax \leq \limsup x$$

for a bounded sequence x is studied in [4, 9, 20, 23, 25]. This inequality implies that

$$\liminf x \leq \liminf Ax \leq \limsup Ax \leq \limsup x,$$

or equivalently

$$K - core\,\{Ax\} \subseteq K - core\,\{x\}\,.$$

The idea of statistical core of a real number sequence is studied in [15].

The sequence x is said to be A-statistically bounded if there is a number B such that $\delta_A(\{k : |x_k| > B\}) = 0$. The A-statistical core of such a sequence x of real numbers is defined to be the closed interval $[st_A - \liminf x,\ st_A - \limsup x]$, *i.e.* $st_A - core\{x\} = [st_A - \liminf x,\ st_A - \limsup x]$.

Recall that the matrices A and B are called absolutely equivalent on a set of sequences if

$$\lim_n \{(Ax)_n - (Bx)_n\} = 0$$

for all x in the set. It is well-known from [7] that the regular matrices A and B are absolutely equivalent for bounded sequences if and only if

$$\lim_n \sum_{k=1}^{\infty} |b_{nk} - a_{nk}| = 0. \tag{4.1}$$

Now we have

THEOREM 8. *Let $x \in m$ and let A be a regular matrix. Then*

$$K - core\,\{Tx\} \subseteq st_A - core\,\{x\}$$

if and only if T is absolutely equivalent to a nonnegative matrix B in τ_A^* for $x \in m$.

Proof. Sufficiency: Since T is absolutely equivalent to a nonnegative matrix B in τ_A^*, we have for every $x \in m$,

$$\lim_n \{(Tx)_n - (Bx)_n\} = 0. \tag{4.2}$$

Now Theorem 6.5.1 of Cooke of [7] implies that

$$K - core\,\{Tx\} \subseteq K - core\,\{x\} \text{ for every } x \in m. \tag{4.3}$$

Since B is a nonnegative matrix in τ_A^*, we have, by Theorem 7 of [10], that

$$K - core\,\{Bx\} \subseteq st_A - core\,\{x\} \text{ for every } x \in m. \tag{4.4}$$

It follows from (4.2) and Theorem 6.3.2 of [7] that

$$K - core\,\{Tx\} = K - core\,\{Bx\}\,. \tag{4.5}$$

Now (4.4) and (4.5) imply that $K - core\,\{Tx\} \subseteq st_A - core\,\{x\}$.

Necessity: Let $x \in m$ and let T be a regular matrix. By hypothesis,

$$K - core\,\{Tx\} \subseteq st_A - core\,\{x\} \subseteq K - core\,\{x\}\,. \tag{4.6}$$

Hence, it follows from Theorem 6.5.1 of [7] that there is a nonnegative regular matrix B such that T and B are absolutely equivalent on m. It remains to show that

$$\lim_n \sum_{k\in E} b_{nk} = 0 \tag{4.7}$$

for every $E \subseteq \mathbb{N}$ such that $\delta_A(E) = 0$. In order to see this let $E \subseteq \mathbb{N}$ and $\delta_A(E) = 0$. Then (4.1) implies

$$\lim_n \sum_{k\in E} |b_{nk} - t_{nk}| = 0. \tag{4.8}$$

Now, for any set of A-density zero, one can write

$$\sum_{k\in E} b_{nk} \le \sum_{k\in E} |b_{nk} - t_{nk}| + \sum_{k\in E} |t_{nk}|\,. \tag{4.9}$$

The first inclusion in (4.6) and Theorem 7 of [10] imply that

$$\lim_n \sum_{k\in E} |t_{nk}| = 0. \tag{4.10}$$

So, it follows from (4.8), (4.9) and (4.10) that (4.7) holds, whence the proof is completed. □

The following result deals with $st_A - \limsup$ equality of sequences.

THEOREM 9. *If $x = (x_n)$ and $y = (y_n)$ are A-statistically bounded sequences and*

$$st_A - \limsup (x_n - y_n) = 0 \tag{4.11}$$

then $st_A - \limsup x = st_A - \limsup y$.

Proof. $st_A - \limsup x \le st_A - \limsup(x - y) + st_A - \limsup y = st_A - \limsup y$. Interchanging the roles of x and y, we also get $st_A - \limsup y \le st_A - \limsup x$, hence we have the result. □

Similarly one can get that $st_A - \liminf x = st_A - \liminf y$.

So we immediately conclude the following which is an analogue of Theorem 6.3.2 of [7].

COROLLARY 1. *If x and y are A-statistically bounded sequences such that (4.11) holds, then we have $st_A - core\,\{x\} = st_A - core\,\{y\}$.*

If x and y are A-statistically bounded complex sequences satisfying (4.11), we could get the same conclusion of Corollary 1 by using Theorem 6 in [10] which provides an alternate form of the A-statistical core for A-statistically bounded complex sequences.

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Received 6 July 2015

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