METRIC PROPERTIES OF N-CONTINUED FRACTIONS

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Communicated by Marius Iosifescu

A generalization of the regular continued fractions was given by Burger et al. in 2008 [3]. In this paper, we give metric properties of this expansion. For the transformation which generates this expansion, its invariant measure and Perron-Frobenius operator are investigated.

AMS 2010 Subject Classification: 11J70, 11K50.

Key words: continued fractions, invariant measure, Perron-Frobenius operator.

1. INTRODUCTION

The modern history of continued fractions started with Gauss who found a natural invariant measure of the so-called *regular continued fraction (or Gauss) transformation*, *i.e.*, $T : [0,1] \to [0,1]$, $T(x) = 1/x - \lfloor 1/x \rfloor$, $x \neq 0$, and $T(0) = 0$. Here $\lfloor \cdot \rfloor$ denotes the floor (or entire) function. Let G be this measure which is called *Gauss measure*. The Gauss measure of an interval $A \in \mathcal{B}_{[0,1]}$ is $G(A) = (1/\log 2) \int_A 1/(x+1)\mathrm{d}x$, where $\mathcal{B}_{[0,1]}$ denotes the σ-algebra of all Borel subsets of $[0,1]$. This measure is T-invariant in the sense that $G(T^{-1}(A)) = G(A)$ for any $A \in \mathcal{B}_I$.

By the very definition, any irrational $0 < x < 1$ can be written as the infinite regular continued fraction

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}} := [a_1, a_2, a_3, \ldots], \tag{1}$$

where $a_n \in \mathbb{N}_+ := \{1, 2, 3, \ldots\}$ [5]. Such a_n's are called *incomplete quotients (or continued fraction digits)* of x and they are given by the formulas $a_1(x) = \lfloor 1/x \rfloor$ and $a_{n+1}(x) = a_1(T^n(x))$, where T^n denotes the nth iterate of T.

Thus, the continued fraction representation conjugates the Gauss transformation and the shift on the space of infinite integer-valued sequence $(a_n)_{n \in \mathbb{N}_+}$.

Other famous probabilists like Paul Lévy and Wolfgang Doeblin also contributed to what is nowadays called the "metric theory of continued fractions".

The first problem in the metric theory of continued fractions was Gauss' famous 1812 problem [2]. In a letter dated 1812, Gauss asked Laplace how fast $\lambda(T^{-n}([0,x]))$ converges to the invariant measure $G([0,x])$, where λ denotes the Lebesgue measure on $[0,1]$. Gauss' question was answered independently in 1928 by Kuzmin [10], and in 1929 by Paul Lévy [13].

Apart from regular continued fractions, there are many other continued fraction expansions: Engel continued fractions, Rosen expansions, the nearest integer continued fraction, the grotesque continued fractions, f-expansions etc. For most of these expansions the Gauss-Kuzmin-Lévy theorem has been proved [7, 9, 12, 15–19]

The purpose of this paper is to show and prove some metric properties of N-continued fraction expansions introduced by Burger *et al.* [3].

In Section 2, we present the current framework. Next we show a Legendre-type result and the Brodén-Borel-Lévy formula by using the probability structure of $(a_n)_{n\in\mathbb{N}_+}$ under the Lebesgue measure. In Section 6, we find the invariant measure G_N of T_N the transformation which generate the N-continued fraction expansions. In Section 7, we consider the so-called natural extension of $([0,1],\mathcal{B}_{[0,1]},G_N,T_N)$ [15]. In Section 8, we derive its Perron-Frobenius operator under different probability measures on $([0,1],\mathcal{B}_{[0,1]})$. Especially, we derive the asymptotic behavior for the Perron-Frobenius operator of $([0,1],\mathcal{B}_{[0,1]},G_N,T_N)$.

2. *N*-CONTINUED FRACTION EXPANSIONS AS DYNAMICAL SYSTEM

In this paper, we consider a generalization of the Gauss transformation. Fix an integer $N \geq 1$. In [3], Burger *et al.* proved that any irrational $0 < x < 1$ can be written in the form

$$x = \cfrac{N}{a_1 + \cfrac{N}{a_2 + \cfrac{N}{a_3 + \ddots}}} := [a_1, a_2, a_3, \ldots]_N, \tag{2}$$

where a_n's are non-negative integers. We will call (2) the *N-continued fraction expansion of x.*

This continued fraction is treated as the following dynamical systems.

Definition 2.1. Fix an integer $N \geq 1$.

(i) The measure – theoretical dynamical system $(I,\mathcal{B}_I,T_N)$ is defined as follows: $I := [0,1]$, $\mathcal{B}_I$ denotes the σ –algebra of all Borel subsets of I, and T_N is the transformation

(3) $$T_N : I \to I; \quad T_N(x) := \begin{cases} \dfrac{N}{x} - \left\lfloor \dfrac{N}{x} \right\rfloor & \text{if } x \in I, \\ 0 & \text{if } x = 0. \end{cases}$$

(ii) In addition to (i), we write $(I, \mathcal{B}_I, G_N, T_N)$ as $(I, \mathcal{B}_I, T_N)$ with the following probability measure G_N on $(I, \mathcal{B}_I)$:

(4) $$G_N(A) := \frac{1}{\log \frac{N+1}{N}} \int_A \frac{\mathrm{d}x}{x+N}, \quad A \in \mathcal{B}_I.$$

Define the *quantized index map* $\eta : I \to \mathbb{N} := \mathbb{N}_+ \cup \{0\}$ by

(5) $$\eta(x) := \begin{cases} \left\lfloor \dfrac{N}{x} \right\rfloor & \text{if } x \neq 0, \\ \infty & \text{if } x = 0. \end{cases}$$

By using T_N and η, the sequence (a_n) in (2) is obtained as follows:

(6) $$a_n = \eta\left(T_N^{n-1}(x)\right), \quad n \geq 1$$

with $T_N^0(x) = x$. Since $x \in (0,1)$ we have that $a_n \geq N$ for any $n \geq 1$.

In this way, T_N gives the algorithm of N-continued fraction expansion which is an obvious generalization of the regular continued fraction.

PROPOSITION 2.2. *Let $(I, \mathcal{B}_I, G_N, T_N)$ be as in Definition 2.1(ii).*

(i) $(I, \mathcal{B}_I, G_N, T_N)$ is ergodic.

(ii) The measure G_N is invariant under T_N.

Proof. See [4] and Section 6. □

By Proposition 2.2(ii), $(I, \mathcal{B}_I, G_N, T_N)$ is a "dynamical system" in the sense of Definition 3.1.3 in [1].

3. SOME ELEMENTARY PROPERTIES OF N-CONTINUED FRACTIONS

Roughly speaking, the metrical theory of continued fraction expansions is the asymptotic analysis of incomplete quotients $(a_n)_{n \in \mathbb{N}_+}$ and related sequences [8]. First, note that in the rational case, the continued fraction expansion (2) is finite, unlike the irrational case, when we have an infinite number of digits. In [20], Van der Wekken showed the convergence of the expansion. For $x \in I \setminus \mathbb{Q}$, define the *n-th order convergent* $[a_1, a_2, \ldots, a_n]_N$ of x by truncating the expansion on the right-hand side of (2), that is,

(7) $$[a_1, a_2, \ldots, a_n]_N \to x, \quad n \to \infty.$$

To this end, for $n \in \mathbb{N}_+$, define integer-valued functions $p_n(x)$ and $q_n(x)$ by

$$p_n(x) := a_n p_{n-1} + N p_{n-2}, \quad n \geq 2 \tag{8}$$

$$q_n(x) := a_n q_{n-1} + N q_{n-2}, \quad n \geq 1 \tag{9}$$

with $p_0(x) := 0$, $q_0(x) := 1$, $p_{-1}(x) := 1$, $q_{-1}(x) := 0$, $p_1(x) := N$, $q_1(x) := a_1$. By induction, we have

$$p_{n-1}(x)q_n(x) - p_n(x)q_{n-1}(x) = (-N)^n, \quad n \in \mathbb{N}. \tag{10}$$

By using (8) and (9), we can verify that

$$x = \frac{p_n(x) + T_N^n(x)p_{n-1}(x)}{q_n(x) + T_N^n(x)q_{n-1}(x)}, \quad n \geq 1. \tag{11}$$

By taking $T_N^n(x) = 0$ in (11), we obtain $[a_1, a_2, \ldots, a_n]_N = p_n(x)/q_n(x)$. From this and by using (10) and (11), we obtain

$$\left| x - \frac{p_n(x)}{q_n(x)} \right| = \frac{N^n \cdot T_N^n(x)}{q_n(x)\left(T_N^n(x)q_{n-1}(x) + q_n(x)\right)}, \quad n \geq 1. \tag{12}$$

Now, since $T_N^n(x) < 1$ and $\left| T_N^n(x)\frac{q_{n-1}(x)}{q_n(x)} + 1 \right| \geq 1$, we have

$$\left| x - \frac{p_n(x)}{q_n(x)} \right| < \frac{N^n}{q_n^2(x)}, \quad n \geq 1. \tag{13}$$

In order to prove (7), it is sufficient to show the following inequality:

$$\left| x - \frac{p_n(x)}{q_n(x)} \right| \leq \frac{1}{N^n}, \quad n \geq 1. \tag{14}$$

From (9), we have that $q_n(x) > Nq_{n-1}(x)$ and because $q_0 = 1$ we have $q_n(x) > N^n$. Finally, (14) follows from (13).

4. DIOPHANTINE APPROXIMATION

Diophantine approximation deals with the approximation of real numbers by rational numbers [5]. We approximate $x \in I \setminus \mathbb{Q}$ by incomplete quotients in (8) and (9).

For $x \in I \setminus \mathbb{Q}$, let a_n be as in (6). For any $n \in \mathbb{N}_+$ and $i^{(n)} = (i_1, \ldots, i_n) \in \mathbb{N}^n$, define the *fundamental interval associated with* $i^{(n)}$ by

$$I_N\left(i^{(n)}\right) = \{x \in I \setminus \mathbb{Q} : a_k(x) = i_k \text{ for } k = 1, \ldots, n\} \tag{15}$$

where we write $I_N(i^{(0)}) = I \setminus \mathbb{Q}$. Remark that $I_N(i^{(n)})$ is not connected by definition. For example, we have

$$I_N(i) = \{x \in I \setminus \mathbb{Q} : a_1 = i\} = (I \setminus \mathbb{Q}) \cap \left(\frac{N}{i+1}, \frac{N}{i}\right) \quad \text{for any } i \in \mathbb{N}. \tag{16}$$

Lemma 4.1. *Let λ denote the Lebesgue measure. Then*

$$\lambda\left(I_N\left(i^{(n)}\right)\right) = \frac{N^n}{q_n(x)(q_n(x)+q_{n-1}(x))}, \tag{17}$$

where (q_n) is as in (9).

Proof. From the definition of T_N and (11), we have

$$I_N\left(i^{(n)}\right) = (I \setminus \mathbb{Q}) \cap \left(u(i^{(n)}),\, v(i^{(n)})\right), \tag{18}$$

where both $u\left(i^{(n)}\right)$ and $v\left(i^{(n)}\right)$ are rational numbers defined as

$$u\left(i^{(n)}\right) := \begin{cases} \dfrac{p_n(x)+p_{n-1}(x)}{q_n(x)+q_{n-1}(x)} & \text{if } n \text{ is odd,} \\[2ex] \dfrac{p_n(x)}{q_n(x)} & \text{if } n \text{ is even,} \end{cases} \tag{19}$$

and

$$v\left(i^{(n)}\right) := \begin{cases} \dfrac{p_n(x)}{q_n(x)} & \text{if } n \text{ is odd,} \\[2ex] \dfrac{p_n(x)+p_{n-1}(x)}{q_n(x)+q_{n-1}(x)} & \text{if } n \text{ is even.} \end{cases} \tag{20}$$

By using (10), we have (17). □

We now give a Legendre-type result for N-continued fraction expansions. For $x \in I \setminus \mathbb{Q}$, we define the *approximation coefficient* $\Theta_N(x,n)$ by

$$\Theta_N(x,n) := \frac{q_n^2}{N^n}\left|x - \frac{p_n}{q_n}\right|, \quad n \geq 1 \tag{21}$$

where p_n/q_n is the nth continued fraction convergent of x in (2).

Proposition 4.2. *For $x \in I \setminus \mathbb{Q}$ and an irreducible fraction $0 < p/q < 1$, assume that p/q is written as follows:*

$$\frac{p}{q} = [i_1, \ldots, i_n]_N \tag{22}$$

where $[i_1, \ldots, i_n]_N$ is as in (7), and the length $n \in \mathbb{N}_+$ of N-continued fraction expansion of p/q is chosen in such a way that it is even if $p/q < x$ and odd otherwise. Then

$$\Theta_N(x,n) < \frac{q}{q+q_{n-1}} \quad \textit{if and only if} \quad \frac{p}{q} \textit{ is the } n\textit{th convergent of } x \tag{23}$$

where $\Theta_N(x,n)$ is as in (21) and the positive integer q_{n-1} is defined as the denominator of the irreducible fraction representation of the rational number $[i_1,\ldots,i_{n-1}]_N$ with $q_0=1$ for the sequence $i_1,\ldots,i_n$.

Proof. Fix $x\in I\setminus\mathbb{Q}$ and $n\geq 1$. Let $\Theta := \Theta_N(x,n)$.
($\Leftarrow$) Assume that p/q is the nth convergent of x. By (12) and the definition of Θ, we have

$$\Theta = \frac{q^2}{N^n}\left|x-\frac{p}{q}\right| = \frac{T_N^n(x)q}{q+T_N^n(x)q_{n-1}(x)} \leq \frac{q}{q+q_{n-1}} \tag{24}$$

where we use $q_{n-1}=q_{n-1}(x)$.
($\Rightarrow$) Conversely,

$$\text{if } \Theta < \frac{q}{q+q_{n-1}}, \quad \text{then} \quad q\left|x-\frac{p}{q}\right| < \frac{N^n}{q+q_{n-1}}. \tag{25}$$

If n is even, then $x>p/q$ and we have

$$x-\frac{p}{q} < \frac{N^n}{q(q+q_{n-1})}. \tag{26}$$

From these,

$$\frac{p}{q} < x < \frac{p}{q} + \frac{N^n}{q(q+q_{n-1})} = \frac{p+p_{n-1}}{q+q_{n-1}} \tag{27}$$

where p_{n-1} is defined as $p_{n-1}/q_{n-1}=[i_1,\ldots,i_{n-1}]_N$. Hence $x\in I_N\left(i^{(n)}\right)$, *i.e.*, $p/q=[i_1,\ldots,i_n]_N$ is a convergent of x. The case when n is odd is treated similarly. $\square$

5. BRODÉN-BOREL-LÉVY FORMULA AND ITS CONSEQUENCES

We derive the so-called Brodén-Borel-Lévy formula [6,8] for N-continued fraction expansion. For $x\in I$, let a_n and q_n be as in (6) and (9), respectively. We define $(s_n)_{n\geq 0}$ by

$$s_0 := 0, \quad s_n := N\frac{q_n}{q_{n-1}}, \quad n\geq 1. \tag{28}$$

From (9), $s_n = N/(a_n+s_{n-1})$ for $n\geq 1$. Hence

$$s_n = \cfrac{N}{a_n+\cfrac{N}{a_{n-1}+\ddots+\cfrac{N}{a_1}}} = [a_n,a_{n-1},\ldots,a_2,a_1]_N, \tag{29}$$

for $n\geq 1$.

PROPOSITION 5.1 (Brodén-Borel-Lévy formula). *Let λ denote the Lebesgue measure on I. For any $n \in \mathbb{N}_+$, the conditional probability $\lambda(T_N^n < x|a_1, \ldots, a_n)$ is given as follows:*

$$\lambda(T_N^n < x|a_1, \ldots, a_n) = \frac{(s_n + N)x}{s_n x + N}, \quad x \in I \tag{30}$$

where s_n is as in (28) *and $a_1, \ldots, a_n$ are as in* (6).

Proof. By definition, we have

$$\lambda\left(T_N^n < x|a_1, \ldots, a_n\right) = \frac{\lambda\left((T_N^n < x) \cap I_N(a_1, \ldots, a_n)\right)}{\lambda\left(I_N(a_1, \ldots, a_n)\right)} \tag{31}$$

for any $n \in \mathbb{N}_+$ and $x \in I$. From (11) and (18) we have

$$\begin{aligned} \lambda\left((T_N^n < x) \cap I_N(a_1, \ldots, a_n)\right) &= \left|\frac{p_n}{q_n} - \frac{p_n + xp_{n-1}}{q_n + xq_{n-1}}\right| \\ &= \frac{N^n x}{q_n(q_n + xq_{n-1})}. \end{aligned} \tag{32}$$

From this and (17), we have

$$\begin{aligned} \lambda\left(T_N^n < x|a_1, \ldots, a_n\right) &= \frac{\lambda\left((T_N^n < x) \cap I_N(a_1, \ldots, a_n)\right)}{\lambda\left(I_N(a_1, \ldots, a_n)\right)} \\ &= \frac{x\left(q_n + q_{n-1}\right)}{q_n + xq_{n-1}} = \frac{(s_n + N)x}{s_n x + N} \end{aligned} \tag{33}$$

for any $n \in \mathbb{N}_+$ and $x \in I$. □

The Brodén-Borel-Lévy formula allows us to determine the probability structure of incomplete quotients $(a_n)_{n \in \mathbb{N}_+}$ under λ.

PROPOSITION 5.2. *For any $i \geq N$ and $n \in \mathbb{N}_+$, we have*

$$\lambda(a_1 = i) = \frac{N}{i(i+1)}, \quad \lambda\left(a_{n+1} = i|a_1, \ldots, a_n\right) = V_{N,i}(s_n) \tag{34}$$

where (s_n) is as in (28), and

$$V_{N,i}(x) := \frac{x + N}{(x+i)\,(x+i+1)}. \tag{35}$$

Proof. From (16), the case $\lambda(a_1 = i)$ holds. For $n \geq N$ and $x \in I \setminus \mathbb{Q}$, we have $T_N^n(x) = [a_{n+1}, a_{n+2}, \ldots]_N$ where (a_n) is as in (6). By using (30), we have

$$\lambda(\, a_{n+1} = i \,|\, a_1, \ldots, a_n \,) \;=\; \lambda\left(T_N^n \in \left(\frac{N}{i+1}, \frac{N}{i}\right] \,|\, a_1, \ldots, a_n\right).$$

$$= \frac{(s_n + N)\frac{N}{i}}{s_n\frac{N}{i} + N} - \frac{(s_n + N)\frac{N}{i+1}}{s_n\frac{N}{i+1} + N} = V_{N,i}(s_n). \quad \square \tag{36}$$

In (34), $\sum_{i=N}^{\infty} \lambda(a_{n+1} = i|a_1, \ldots, a_n)$ must be 1 because λ is a probability measure on $(I, \mathcal{B}_I)$. This can be verified from (34) and (36) by using the partial fraction decomposition. By the same token, we see that

$$\sum_{i=N}^{\infty} V_{N,i}(x) = 1 \quad \text{for any } x \in I. \tag{37}$$

Remark 5.3. Proposition (5.2) is the starting point of an approach to the metrical theory of N-continued fraction expansions via dependence with complete connections (see [6], Section 5.2). We apply this method in [11] to obtain a solution of Gauss-Kuzmin-type problem for N-continued fraction expansions.

COROLLARY 5.4. *The sequence $(s_n)_{n\in\mathbb{N}_+}$ with $s_0 = 0$ is a homogeneous I-valuated Markov chain on $(I, \mathcal{B}_I, \lambda)$ with the following transition mechanism: from state $s \in I$ the only possible one-step transitions are those to states $N/(s+i)$, $i \geq N$, with corresponding probabilities $V_{N,i}(s)$, $i \geq N$.*

6. THE INVARIANT MEASURE OF T_N

Let $(I, \mathcal{B}_I)$ be as in Definition 2.1(i). In this section, we will give the explicit form of the invariant probability measure G_N of the transformation T_N in (3), *i.e.*, $G_N\left(T_N^{-1}(A)\right) = G_N(A)$ for any $A \in \mathcal{B}_I$. From the aspect of metric theory, the digits a_n in (6) can be viewed as random variables on $(I, \mathcal{B}_I)$ that are defined almost surely with respect to any probability measure on $\mathcal{B}_I$ assigning probability 0 to the set of rationals in I. Such a probability measure is Lebesgue measure λ, but a more important one in the present context is the invariant probability measure G_N of the transformation T_N.

PROPOSITION 6.1. *The invariant probability density ρ_N of the transformation T_N is given by*

$$\rho_N(x) = \frac{k_N}{x + N}, \quad x \in I \tag{38}$$

where k_N is the normalized constant such that the invariant measure G_N is a probability measure. Furthermore, the constant k_N is given in (4), i.e., $k_N = \left(\log\left(\frac{N+1}{N}\right)\right)^{-1}$.

Proof. We will give a proof which involves properties of the Perron-Frobenius operator of T_N under G_N. Therefore, the proof will be given in Section 8. $\square$

7. NATURAL EXTENSION AND EXTENDED RANDOM VARIABLES

Fix an integer $N \geq 1$. In this section, we introduce the natural extension $\overline{T_N}$ of T_N in (3) and its extended random variables [15].

Let $(I, \mathcal{B}_I, T_N)$ be as in Definition 2.1(i). Define $(u_{N,i})_{i \geq N}$ by

$$u_{N,i} : I \to I; \quad u_{N,i}(x) := \frac{N}{x+i}, \quad x \in I. \tag{39}$$

For each $i \geq N$, $u_{N,i}$ is a right inverse of T_N, that is,

$$(T_N \circ u_{N,i})(x) = x, \quad \text{for any } x \in I. \tag{40}$$

Furthermore, if $\eta(x) = i$, then $(u_{N,i} \circ T_N)(x) = x$ where η is as in (5).

Definition 7.1. The natural extension $(I^2, \mathcal{B}_{I^2}, \overline{T_N})$ of $(I, \mathcal{B}_I, T_N)$ is the transformation $\overline{T_N}$ of the square space $(I^2, \mathcal{B}_I^2) := (I, \mathcal{B}_I) \times (I, \mathcal{B}_I)$ defined as follows [14]:

$$\overline{T_N} : I^2 \to I^2; \quad \overline{T_N}(x, y) := \left(T_N(x),\, u_{N,\eta(x)}(y)\right), \quad (x, y) \in I^2. \tag{41}$$

From (40), we see that $\overline{T_N}$ is bijective on I^2 with the inverse

$$(\overline{T_N})^{-1}(x, y) = (u_{N,\eta(y)}(x),\, T_N(y)), \quad (x, y) \in I^2. \tag{42}$$

Iterations of (41) and (42) are given as follows for each $n \geq 2$:

$$\left(\overline{T_N}\right)^n (x, y) = (\, T_N^n(x),\, [x_n, x_{n-1}, \ldots, x_2(x),\, x_1 + y]_N\,), \tag{43}$$

$$\left(\overline{T_N}\right)^{-n} (x, y) = (\, [y_n, y_{n-1}, \ldots, y_2,\, y_1 + x]_N,\, T_N^n(y)\,) \tag{44}$$

where $x_i := \eta\left(T_N^{i-1}(x)\right)$ and $y_i := \eta\left(T_N^{i-1}(y)\right)$ for $i = 1, \ldots, n$.

For G_N in (4), Dajani *et al.* [4] define its *extended measure* $\overline{G_N}$ on $(I^2, \mathcal{B}_I^2)$ as

$$\overline{G_N}(B) := \frac{1}{\log\left(\frac{N+1}{N}\right)} \iint_B \frac{N\mathrm{d}x\mathrm{d}y}{(xy+N)^2}, \quad B \in \mathcal{B}_I^2. \tag{45}$$

Then $\overline{G_N}(A \times I) = \overline{G_N}(I \times A) = G_N(A)$ for any $A \in \mathcal{B}_I$. The measure $\overline{G_N}$ is preserved by $\overline{T_N}$ [4].

Define the projection $E : I^2 \to I$ by $E(x, y) := x$. With respect to $\overline{T_N}$ in (41), define *extended incomplete quotients* $\overline{a}_l(x, y)$, $l \in \mathbb{Z}$ at $(x, y) \in I^2$ by

$$\overline{a}_l(x, y) := (\eta \circ E)\left(\,(\overline{T_N})^{l-1}(x, y)\,\right), \quad l \in \mathbb{Z}. \tag{46}$$

Remark 7.2. (i) Remark that $\overline{a}_l(x, y)$ in (46) is also well-defined for $l \leq 0$ because $\overline{T_N}$ is invertible. By (43) and (44), we have

$$\overline{a}_n(x, y) = x_n, \quad \overline{a}_0(x, y) = y_1, \quad \overline{a}_{-n}(x, y) = y_{n+1}, \quad n \in \mathbb{N}_+,\ (x, y) \in I^2 \tag{47}$$

where we use notations in (43) and (44).

(ii) Since the measure $\overline{G_N}$ is preserved by $\overline{T_N}$, the doubly infinite sequence $(\overline{a}_l(x,y))_{l\in\mathbb{Z}}$ is strictly stationary (*i.e.*, its distribution is invariant under a shift of the indices) under $\overline{G_N}$.

THEOREM 7.3. *Fix* $(x,y)\in I^2$ *and let* $\overline{a}_l := \overline{a}_l(x,y)$ *for* $l\in\mathbb{Z}$. *Define* $a := [\overline{a}_0, \overline{a}_{-1}, \ldots]_N$. *Then the following holds for any* $x\in I$*:*

$$\overline{G_N}([0,x]\times I\,|\,\overline{a}_0,\overline{a}_{-1},\ldots) = \frac{(N+a)x}{ax+N} \quad \overline{G_N}\text{-a.s.} \tag{48}$$

Proof. Recall fundamental interval in (15). Let $I_{N,n}$ denote the fundamental interval $I_N(\overline{a}_0,\overline{a}_{-1},\ldots,\overline{a}_{-n})$ for $n\in\mathbb{N}$. We have

$$\overline{G_N}([0,x]\times I\,|\,\overline{a}_0,\overline{a}_{-1},\ldots) = \lim_{n\to\infty}\overline{G_N}([0,x]\times I\,|\,\overline{a}_0,\ldots,\overline{a}_{-n}) \quad \overline{G_N}\text{-a.s.} \tag{49}$$

and

$$\begin{aligned}
\overline{G_N}([0,x]\times I\,|\,\overline{a}_0,\ldots,\overline{a}_{-n}) &= \frac{\overline{G_N}([0,x]\times I_{N,n})}{\overline{G_N}(I\times I_{N,n})} \\
&= \frac{k_m}{G_N(I_{N,n})}\int_{I_{N,n}}\mathrm{d}y\int_0^x\frac{N\mathrm{d}u}{(yu+N)^2} \\
&= \frac{1}{G_N(I_{N,n})}\int_{I_{N,n}}\frac{x(y+N)}{xy+N}\,G_N(\mathrm{d}y) \\
&= \frac{x(y_n+N)}{xy_n+N} \qquad (50)
\end{aligned}$$

for some $y_n\in I_{N,n}$ where k_m is as in Proposition (6.1). Since

$$\lim_{n\to\infty} y_n = [\overline{a}_0,\overline{a}_{-1},\ldots]_N = a, \tag{51}$$

the proof is completed. □

The stochastic property of $(\overline{a}_l)_{l\in\mathbb{Z}}$ under $\overline{G_N}$ is given as follows.

COROLLARY 7.4. *For any* $i\in\mathbb{N}$*, we have*

$$\overline{G_N}(\overline{a}_1 = i|\,\overline{a}_0,\overline{a}_{-1},\ldots) = V_{N,i}(a) \quad \overline{G_N}\text{-a.s.} \tag{52}$$

where $a = [\overline{a}_0,\overline{a}_{-1},\ldots]_N$ *and* $V_{N,i}$ *is as in* (35).

Proof. Let $I_{N,n}$ be as in the proof of Theorem 7.3. We have

$$\overline{G_N}(\overline{a}_1 = i\,|\,\overline{a}_0,\overline{a}_{-1},\ldots) = \lim_{n\to\infty}\overline{G_N}(\overline{a}_1 = i\,|\,I_{N,n}). \tag{53}$$

Now

$$\overline{G_N}\left(\left(\frac{N}{i+1},\frac{N}{i}\right]\times[0,1)\middle|\,I_{N,n}\right) = \frac{\overline{G_N}\left(\left(\frac{N}{i+1},\frac{N}{i}\right]\times I_{N,n}\right)}{\overline{G_N}(I\times I_{N,n})}$$

$$= \frac{1}{G_N(I_{N,n})} \int_{I_{N,n}} V_{N,i}(y)\, G_N(\mathrm{d}y)$$

$$= V_{N,i}(y_n) \tag{54}$$

for some $y_n \in I_{N,n}$. From (51), the proof is completed. $\square$

Remark 7.5. The strict stationarity of $(\overline{a}_l)_{l\in\mathbb{Z}}$, under $\overline{G_N}$ implies that

$$\overline{G_N}(\overline{a}_{l+1} = i \mid \overline{a}_l, \overline{a}_{l-1}, \ldots) = V_{N,i}(a) \quad \overline{G_N}\text{-a.s.} \tag{55}$$

for any $i \in \mathbb{N}$ and $l \in \mathbb{Z}$, where $a = [\overline{a}_l, \overline{a}_{l-1}, \ldots]_N$. The last equation emphasizes that $(\overline{a}_l)_{l\in\mathbb{Z}}$ is a chain of infinite order in the theory of dependence with complete connections [6].

8. PERRON-FROBENIUS OPERATORS

Let $(I, \mathcal{B}_I, G_N, T_N)$ be as in Definition 2.1(ii). In this section, we derive its Perron-Frobenius operator.

Let μ be a probability measure on $(I, \mathcal{B}_I)$ such that $\mu\left(T_N^{-1}(A)\right) = 0$ whenever $\mu(A) = 0$ for $A \in \mathcal{B}_I$. For example, this condition is satisfied if T_N is μ-preserving, that is, $\mu T_N^{-1} = \mu$. Let $L^1(I, \mu) := \{f : I \to \mathbb{C} : \int_I |f| \mathrm{d}\mu < \infty\}$. The *Perron-Frobenius operator* of $(I, \mathcal{B}_I, \mu, G_N)$ is defined as the bounded linear operator U on the Banach space $L^1(I, \mu)$ such that the following holds:

$$\int_A Uf \,\mathrm{d}\mu = \int_{T_N^{-1}(A)} f \,\mathrm{d}\mu \quad \text{for all } A \in \mathcal{B}_I,\ f \in L^1(I, \mu). \tag{56}$$

For more details, see [1,8] or Appendix A in [12].

Proposition 8.1. *Let $(I, \mathcal{B}_I, G_N, T_N)$ be as in Definition 2.1(ii), and let U denote its Perron-Frobenius operator. Then:*

(i) The following equation holds:

$$\{Uf\}(x) = \sum_{i\geq N} V_{N,i}(x)\, f\left(\frac{N}{x+i}\right), \quad f \in L^1(I, G_N), \tag{57}$$

where $V_{N,i}$ is as in (35).

(ii) Let μ be a probability measure on $(I, \mathcal{B}_I)$ such that μ is absolutely continuous with respect to the Lebesgue measure λ and let $h := \mathrm{d}\mu/\mathrm{d}\lambda$ a.e. in I. Then the following holds:

(a) Let S denote the Perron-Frobenius operator of T_N under μ. Then the following holds a.e. in I:

$$\{Sf\}(x) = \frac{N}{h(x)} \sum_{i \geq N} \frac{h\left(\frac{N}{x+i}\right)}{(x+i)^2} f\left(\frac{N}{x+i}\right) \tag{58}$$

$$= \frac{\{U\hat{f}\}(x)}{(x+N)h(x)} \tag{59}$$

for $f \in L^1(I,\mu)$, *where* $\hat{f}(x) := (x+N)f(x)h(x)$, $x \in I$. *In addition, the nth power* S^n *of* S *is written as follows:*

$$\{S^n f\}(x) = \frac{\{U^n \hat{f}\}(x)}{(x+N)h(x)} \tag{60}$$

for any $f \in L^1(I,\mu)$ *and any* $n \geq 1$.

(b) Let K *denote the Perron-Frobenius operator of* T_N *under* λ. *Then the following holds a.e. in* I:

$$\{Kf\}(x) = \sum_{i \geq N} \frac{N}{(x+i)^2} f\left(\frac{N}{x+i}\right), \quad f \in L^1(I,\lambda). \tag{61}$$

In addition, the nth power K^n *of* K *is written as follows:*

$$\{K^n f\}(x) = \frac{\{U^n \hat{f}\}(x)}{x+N}, \quad f \in L^1(I,\lambda), \tag{62}$$

for any $f \in L^1(I,\lambda)$ *and any* $n \geq 1$, *where* $\hat{f}(x) := (x+N)f(x)$, $x \in I$.

(c) For any $n \in \mathbb{N}_+$ *and* $A \in \mathcal{B}_I$, *we have*

$$\mu\left(T_N^{-n}(A)\right) = \int_A \{U^n f\}(x) \mathrm{d}G_N(x) \tag{63}$$

where $f(x) := (\log\left(\frac{N+1}{N}\right))(x+N)h(x)$ *for* $x \in I$.

Proof. (i) Let $T_{N,i}$ denote the restriction of T_N to the subinterval $I_i := \left(\frac{N}{i+1}, \frac{N}{i}\right]$, $i \geq N$, that is,

$$T_{N,i}(x) = \frac{N}{x} - 1, \quad x \in I_i. \tag{64}$$

Let $C(A) := (T_N)(A)$ and $C_i(A) := (T_{N,i})^{-1}(A)$ for $A \in \mathcal{B}_I$. Since $C(A) = \bigcup_i C_i(A)$ and $C_i \cap C_j$ is a null set when $i \neq j$, we have

$$\int_{C(A)} f \, \mathrm{d}G_N = \sum_{i \geq N} \int_{C_i(A)} f \, \mathrm{d}G_N, \quad f \in L^1(I, G_N), \; A \in \mathcal{B}_I. \tag{65}$$

For any $i \geq N$, by the change of variables $x = (T_{N,i})^{-1}(y) = \dfrac{N}{y+i}$, we successively obtain

$$\int_{C_i(A)} f(x)\, G_N(\mathrm{d}x) = \left(\log\left(\frac{N+1}{N}\right)\right)^{-1} \int_{C_i(A)} \frac{f(x)}{N+x}\,\mathrm{d}x$$

$$= \left(\log\left(\frac{N+1}{N}\right)\right)^{-1} \int_A \frac{f\left(\frac{N}{y+i}\right)}{N+\frac{N}{y+i}} \frac{N}{(y+i)^2}\mathrm{d}y$$

$$= \int_A V_{N,i}(y)\, f\left(\frac{N}{y+i}\right) G_N(\mathrm{d}y). \tag{66}$$

Now, (57) follows from (65) and (66).
(ii)(a) From (64), for any $f \in L^1(I, G_N)$ and $A \in \mathcal{B}_I$, we have

$$\int_{C(A)} f(x)\, \mu(dx) = \sum_{i \geq N} \int_{C_i(A)} f(x)\, \mu(dx). \tag{67}$$

Then

$$\int_{C_i(A)} f(x)\mu(\mathrm{d}x) = \int_{C_i(A)} f(x)h(x)\, dx = \int_A f\left(\frac{N}{y+i}\right) h\left(\frac{N}{y+i}\right) \frac{N}{(y+i)^2}\,\mathrm{d}y. \tag{68}$$

From (67) and (68),

$$\int_{C(A)} f(x)\, \mu(\mathrm{d}x) = \int_A \sum_{i \geq N} f\left(\frac{N}{x+i}\right) h\left(\frac{N}{x+i}\right) \frac{N}{(x+i)^2}\,\mathrm{d}x. \tag{69}$$

Since $\mathrm{d}\mu = h\mathrm{d}x$, (58) follows from (69). Now, since $\hat{f}(x) = (x+N)f(x)h(x)$, from (58), we have

$$\{U\hat{f}\}(x) = N(x+N) \sum_{i \geq N} \frac{h\left(\frac{N}{x+i}\right)}{(x+i)^2} f\left(\frac{N}{x+i}\right). \tag{70}$$

From (58) and (70), (59) follows immediately.
(ii)(b) The formula (61) is a consequence of (59) and follows immediately.
(ii)(c) We will use mathematical induction. For $n = 0$, the equation (63) holds by definitions of f and h. Assume that (63) holds for some $n \in \mathbb{N}$. Then

$$\mu\left(T_N^{-(n+1)}(A)\right) = \mu\left(T_N^{-n}\left(T_N^{-1}(A)\right)\right) = \int_{C(A)} \{U^n f\}(x)\, G_N(\mathrm{d}x). \tag{71}$$

Since $U = U_{T_N}$ and (56), we have

$$\int_{C(A)} \{U^n f\}(x)\, G_N(\mathrm{d}x) = \int_A \{U^{n+1} f\}(x)\, G_N(\mathrm{d}x). \tag{72}$$

Therefore,

$$\mu\left(T_N^{-(n+1)}(A)\right) = \int_A \{U^{n+1}f\}(x) G_N(\mathrm{d}x) \tag{73}$$

which ends the proof. □

For a function $f : I \to \mathbb{C}$, define the *variation* $\mathrm{var}_A f$ of f on a subset A of I by

$$\mathrm{var}_A f := \sup \sum_{i=1}^{k-1} |f(t_{i+1}) - f(t_i)| \tag{74}$$

where the supremum being taken over $t_1 < \cdots < t_k$, $t_i \in A$, $1 \leq i \leq k$, and $k \geq 2$ ([8], p. 75). We write simply $\mathrm{var} f$ for $\mathrm{var}_I f$. Let $BV(I) := \{f : I \to \mathbb{C} : \mathrm{var}\, f < \infty\}$ and let $L^\infty(I)$ denote the collection of all bounded measurable functions $f : I \to \mathbb{C}$. It is known that $BV(I) \subset L^\infty(I) \subset L^1(I, \mu)$. Let $L(I)$ denote the Banach space of all complex-valued Lipschitz continuous functions on I with the following norm $\|\cdot\|_L$:

$$\|f\|_L := \sup_{x \in I} |f(x)| + s(f), \tag{75}$$

with

$$s(f) := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}, \quad f \in L(I). \tag{76}$$

In the following proposition we show that the operator U in (57) preserves monotonicity and enjoys a contraction property for Lipschitz continuous functions.

Proposition 8.2. *Let U be as in* (57)*.*

(i) Let $f \in L^\infty(I)$. Then the following holds:

(a) If f is non-decreasing (non-increasing), then Uf is non-increasing (non-decreasing).

(b) If f is monotone, then

$$\mathrm{var}\,(Uf) \leq \frac{1}{N+1} \cdot \mathrm{var} f. \tag{77}$$

(ii) For any $f \in L(I)$, we have

$$s(Uf) \leq q \cdot s(f), \tag{78}$$

where

$$q := N \left(\sum_{i \geq N} \left(\frac{N}{i^3(i+1)} + \frac{i+1-N}{i(i+1)^3} \right) \right). \tag{79}$$

Proof. (i)(a) To make a choice assume that f is non-decreasing. Let $x < y$, $x, y \in I$. We have $\{Uf\}(y) - \{Uf\}(x) = S_1 + S_2$, where

$$S_1 = \sum_{i\geq N} V_{N,i}(y)\left(f\left(\frac{N}{y+i}\right)-f\left(\frac{N}{x+i}\right)\right), \tag{80}$$

$$S_2 = \sum_{i\geq m} \left(V_{N,i}(y)-V_{N,i}(x)\right) f\left(\frac{N}{x+i}\right). \tag{81}$$

Clearly, $S_1 \leq 0$. Now, since $\sum_{i\geq N} V_{N,i}(x) = 1$ for any $x \in I$, we can write

$$S_2 = -\sum_{i\geq N}\left(f\left(\frac{N}{x+N}\right)-f\left(\frac{N}{x+i}\right)\right)\left(V_{N,i}(y)-V_{N,i}(x)\right). \tag{82}$$

As is easy to see, the functions $V_{N,i}$ are increasing for all $i \geq N$. Also, using that $f\left(\frac{N}{x+N}\right) \geq f\left(\frac{N}{x+i}\right)$, we have that $S_2 \leq 0$. Thus $\{Uf\}(y) - \{Uf\}(x) \leq 0$ and the proof is complete.

(i)(b) Assume that f is non-decreasing. Then by (a) we have

$$\operatorname{var} Uf = \{Uf\}(0) - \{Uf\}(1) = \sum_{i\geq N}\left(V_{N,i}(0) f\left(\frac{N}{i}\right) - V_{N,i}(1) f\left(\frac{N}{1+i}\right)\right). \tag{83}$$

By calculus, we have

$$\begin{aligned}
\operatorname{var} Uf &= \sum_{i\geq N}\left(\frac{N}{i(i+1)} f\left(\frac{N}{i}\right) - \frac{N+1}{(i+1)(i+2)} f\left(\frac{N}{i+1}\right)\right)\\
&= \frac{1}{N+1} f(1) - \sum_{i\geq N} \frac{1}{(i+1)(i+2)} f\left(\frac{N}{i+1}\right)\\
&\leq \frac{1}{N+1} f(1) - \sum_{i\geq N}\left(\frac{1}{i+1}-\frac{1}{i+2}\right) f(0)\\
&= \frac{1}{N+1}\left(f(1)-f(0)\right) = \frac{1}{N+1}\operatorname{var} f.
\end{aligned}$$

(ii) For $x \neq y$, $x, y \in I$, we have

$$\begin{aligned}
\frac{\{Uf\}(y)-\{Uf\}(x)}{y-x} &= \sum_{i\geq N} \frac{V_{N,i}(y)-V_{N,i}(x)}{y-x} f\left(\frac{N}{x+i}\right)\\
&- \sum_{i\geq N} V_{N,i}(y) \frac{f\left(\frac{N}{y+i}\right)-f\left(\frac{N}{x+i}\right)}{\frac{N}{x+i}-\frac{N}{x+i}} \cdot \left(\frac{N}{x+i}\right)\left(\frac{N}{y+i}\right).
\end{aligned} \tag{84}$$

Remark that

$$V_{N,i}(u) = \frac{i+1-N}{u+i+1} + \frac{N-i}{u+i}, \quad i \geq N, \tag{85}$$

and then

$$\sum_{i\geq N} \frac{V_{N,i}(y)-V_{N,i}(x)}{y-x} f\left(\frac{N}{x+i}\right)$$

$$(86)\qquad = \sum_{i\geq N} \frac{i+1-N}{(y+i+1)(x+i+1)} \left(f\left(\frac{N}{x+i+1}\right) - f\left(\frac{N}{x+i}\right)\right).$$

Assume that $x > y$. It then follows from (86) and (84) that

$$\begin{aligned} \left|\frac{\{Uf\}(y)-\{Uf\}(x)}{y-x}\right| &\leq s(f)\sum_{i\geq N}\left(\frac{N(i+1-N)}{(y+i)(y+i+1)^3}+\frac{N\cdot V_{N,i}(y)}{(y+i)^2}\right) \\ (87)\qquad &\leq q\cdot s(f) \end{aligned}$$

where q is as in (79). Since

$$(88)\qquad s(Uf) = \sup_{x,y\in I, x\geq y}\left|\frac{\{Uf\}(y)-\{Uf\}(x)}{y-x}\right|$$

then the proof is complete. $\square$

Proof of Proposition 6.1. For $(I, \mathcal{B}_I, G_N, T_N)$ in Definition 2.1(ii), let U denote its Perron-Frobenius operator. Let

$$(89)\qquad \rho_N(x) := \frac{k_N}{x+N}, \quad x\in I,$$

where $k_N = \left(\log\left(\frac{N+1}{N}\right)\right)^{-1}$. From properties of the Perron-Frobenius operator, it is sufficient to show that the function ρ_N defined in (89) satisfies $U\rho_N = \rho_N$.

Since $T_N^{-1}(x) = \left\{\dfrac{N}{x+i}, i\geq N, x\in I\right\}$, we have

$$\begin{aligned} \{U\rho_N\}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\int_{T_N^{-1}([0,x])}\rho_N(t)\,\mathrm{d}t = \sum_{t\in T_N^{-1}(x)}\frac{\rho_N(t)}{|(T_N)'(t)|} \\ (90)\qquad &= \sum_{i\geq N}\frac{N}{(x+i)^2}\,\rho_N\left(\frac{N}{x+i}\right). \end{aligned}$$

By definition of ρ_N, we see that

$$(91)\qquad \{U\rho_N\}(x) = \sum_{i\geq N}\frac{1}{(x+i)(x+i+1)} = \rho_N(x).$$

Hence the statement is proved. $\square$

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Received 7 October 2015

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