ON FUNCTIONAL EQUATION RELATED TO (m, n)-JORDAN TRIPLE CENTRALIZERS

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In this paper, we prove the following result. Let X be a Banach space over the real or complex field $\mathcal{F}$, let $\mathcal{L}(X)$ be the algebra of all bounded linear operators on X, and let $A(X) \subseteq \mathcal{L}(X)$ be a standard operator algebra. Suppose there exists an additive mapping $T : A(X) \to \mathcal{L}(X)$ satisfying the relation

$$2(m+n)^2 T(A^p) = (2m^2+2mn)T(A)A^{p-1} - mnT(A^{p-2})A^2 + 2mnAT(A^{p-2})A - mnA^2T(A^{p-2}) + (2n^2+2mn)A^{p-1}T(A)$$

for all $A \in A(X)$, where $m \geq 1$, $n \geq 1$, $p \geq 2$ are some fixed integers. In this case T is of the form $T(A) = \lambda A$ for all $A \in A(X)$ and some fixed $\lambda \in \mathcal{F}$. In particular, T is continuous.

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1. INTRODUCTION

Throughout, R will represent an associative ring with center $Z(R)$. As usual we denote by $[x, y]$ the commutator $xy - yx$ and use the commutator identity $[x, yz] = [x, y]z + y[x, z]$. Given an integer $n \geq 2$, a ring R is said to be $n-$torsion free, if for $x \in R$, $nx = 0$ implies $x = 0$. Recall that a ring R is prime if for $a, b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is semiprime in case $aRa = (0)$ implies $a = 0$. We denote by $char(R)$ the characteristic of a prime ring R. We denoted by Q_r and C Martindale right ring of quotients and extended centroid of a semiprime ring R. For the explanation of Q_r and C we refer the reader to [1]. An additive mapping $T : R \to R$ is called a left centralizer in case $T(xy) = T(x)y$ holds for all pairs $x, y \in R$. In case R has the identity element $T : R \to R$ is a left centralizer iff T is of the form $T(x) = ax$ for all $x \in R$, where $a \in R$ is some fixed element. For a semiprime ring R all left centralizers are of the form $T(x) = qx$ for all $x \in R$, where q is some fixed element of Q_r (see Chapter 2 in [1]). An additive mapping $T : R \to R$ is called a left Jordan centralizer in case $T(x^2) = T(x)x$ holds for all $x \in R$. The definition of right centralizer and right Jordan centralizer should be self-explanatory. We call $T : R \to R$ a two-sided centralizer in case T is both a left and a right centralizer. In case $T : R \to R$ is a two-sided centralizer, where R is a semiprime ring with extended centroid C, then T is of the form $T(x) = \lambda x$ for all $x \in R$, where $\lambda \in C$ is some fixed element (see Theorem 2.3.2 in [1]). Zalar [17] has proved that any left (right) Jordan centralizer on a 2-torsion free semiprime ring is a left (right) centralizer (Zalar theorem). Let X be a real or complex Banach space and let $\mathcal{L}(X)$ and $\mathcal{F}(X)$ denote the algebra of all bounded linear operators on X and the ideal of all finite rank operators in $\mathcal{L}(X)$, respectively. An algebra $A(X) \subseteq \mathcal{L}(X)$ is said to be standard in case $\mathcal{F}(X) \subset A(X)$. We denote by X^* the dual space of a real or complex Banach space X. Let us point out that any standard operator algebra is prime, which is a consequence of Hahn-Banach theorem. For results concerning centralizers on rings and algebras we refer to [2, 10–12, 14, 15] where further references can be found. Let R be an arbitrary ring and let $m \geq 0$, $n \geq 0$ be some fixed integers with $m + n \neq 0$. An additive mapping $T : R \to R$ is called an (m, n)-Jordan centralizer in case

$$(m + n)T(x^2) = mT(x)x + nxT(x) \tag{1}$$

holds for all $x \in R$. The concept of (m, n)-Jordan centralizer has been introduced by Vukman in [16]. The concept of (m, n)-Jordan centralizer covers the concept of left Jordan centralizer as well as the concept of right Jordan centralizer. Namely, putting in the relation above $m = 1$, $n = 0$ one obtains left Jordan centralizer, in case $m = 0$, $n = 1$ the relation (1) reduces to right Jordan centralizer. In case $m = 1$ we obtain the relation

$$2T(x^2) = T(x)x + xT(x), \quad x \in R. \tag{2}$$

Vukman [10] has proved that in case an additive mapping $T : R \to R$, where R is a 2-torsion free semiprime ring, satisfies the relation (2) for any $x \in R$, then T is a two-sided centralizer. Vukman conjectured that any (m, n)-Jordan centralizer, where $m \geq 1, n \geq 1$ are some fixed integers, which maps a semiprime ring with suitable torsion restrictions into itself, is a two-sided centralizer. This conjecture is in general still an open question. For results concerning $(m, n)-$Jordan centralizers we refer to [7, 8, 16]. Vukman [16] has proved the following result. Let $m \geq 0$, $n \geq 0$ be some integers with $m+n \neq 0$, let R be a ring and let $T : R \to R$ an (m, n)-Jordan centralizer. In this case, we have

$$(3) \quad 2(m + n)^2T(xyx) = mnT(x)xy + m\,(2m + n)\,T(x)yx - mnT(y)x^2$$

$$+\, 2mnxT(y)x - mnx^2T(y) + n\,(m+2n)\,xyT(x) + mnyxT(x),$$

for all pairs $x, y \in R$. We call additive mapping satisfying the relation (3) (m,n)-Jordan triple centralizer. It seems natural to ask under what additional assumptions any (m,n)-Jordan triple centralizer is a (m,n)-Jordan centralizer.

2. THE MAIN RESULTS

Putting x^{p-2} for y in (3) one obtains the following functional equation

$$2(m+n)^2T(x^p) = \left(2m^2+2mn\right)T(x)x^{p-1} - mnT(x^{p-2})x^2$$
$$+\, 2mnxT(x^{p-2})x - mnx^2T(x^{p-2}) + \left(2n^2+2mn\right)x^{p-1}T(x).$$

In case $p = 3$ the above functional equation has been considered by Peršin and Vukman in [8]. It is our aim in this paper to prove the result below, which is related to the above equation.

Theorem 1. *Let X be a Banach space over the real or complex field $\mathcal{F}$, let $\mathcal{L}(X)$ be the algebra of all bounded linear operators on X, and let $A(X) \subseteq \mathcal{L}(X)$ be a standard operator algebra. Suppose there exists an additive mapping $T : A(X) \to \mathcal{L}(X)$ satisfying the relation*

$$2(m+n)^2\,T(A^p) = \left(2m^2+2mn\right)\,T(A)A^{p-1} - mnT(A^{p-2})A^2$$
$$+ 2mnAT(A^{p-2})A - mnA^2\,T(A^{p-2}) + \left(2n^2+2mn\right)\,A^{p-1}\,T(A)$$

for all $A \in A(X)$, where $m \geq 1$, $n \geq 1$, $p \geq 2$ are some fixed integers. In this case T is of the form $T(A) = \lambda A$ for all $A \in A(X)$ and some fixed $\lambda \in \mathcal{F}$.

The result above fairly generalizes Theorem 8 in [16]. In the result above, we obtain as a result the continuity of T under purely algebraic assumptions concerning T, which means that Theorem 1 might be of some interest from the automatic continuity point of view. For results concerning automatic continuity we refer to [5, 6, 9]. In the proof of Theorem 1, we shall use the result below proved by Brešar [3] and Zalar theorem we have mentioned above.

Theorem 2. *Let R be a prime ring with $char(R) \neq 2$ and let $F : R \to R$ be an additive mapping satisfying the relation*

$$[[F(x), x]\,, x] = 0$$

for all $x \in R$. In this case $[F(x), x] = 0$ holds for all $x \in R$.

For generalizations of the result above we refer to [4] and [13].

Proof of Theorem 1. We have therefore the relation

$$(4)\quad 2(m+n)^2T(A^p) = \left(2m^2+2mn\right)T(A)A^{p-1} - mnT(A^{p-2})A^2 + 2mnAT(A^{p-2})A - mnA^2T(A^{p-2}) + \left(2n^2+2mn\right)A^{p-1}T(A), \quad A \in A(X).$$

First we will consider the restriction of T on $\mathcal{F}(X)$. Let A be from $\mathcal{F}(X)$ and let $P \in \mathcal{F}(X)$, be a projection with $AP = PA = A$. From the above relation one obtains

$$2(m+n)^2T(P) = \left(2m^2+mn\right)T(P)P + 2mnPT(P)P + \left(2n^2+mn\right)PT(P).$$

Right and left multiplication of the above relation by P gives

$$(5)\qquad T(P)P = PT(P) = PT(P)P.$$

Putting $A+P$ for A in the relation (4), we obtain

$$(6)\quad 2(m+n)^2\sum_{i=0}^{p}\binom{p}{i}T\left(A^{p-i}P^i\right)$$
$$= \left(2m^2+2mn\right)\left(T\left(A\right)+B\right)\left(\sum_{i=0}^{p-1}\binom{p-1}{i}A^{p-1-i}P^i\right)$$
$$-\, mn\sum_{i=0}^{p-2}\binom{p-2}{i}T\left(A^{p-2-i}P^i\right)\left(A^2+2A+P\right)$$
$$+\, 2mn\left(A+P\right)\left(\sum_{i=0}^{p-2}\binom{p-2}{i}T\left(A^{p-2-i}P^i\right)\right)\left(A+P\right)$$
$$-\, mn\left(A^2+2A+P\right)\sum_{i=0}^{p-2}\binom{p-2}{i}T\left(A^{p-2-i}P^i\right)$$
$$+\left(2n^2+2mn\right)\left(\sum_{i=0}^{p-1}\binom{p-1}{i}A^{p-1-i}P^i\right)\left(T\left(A\right)+B\right),$$

where B stands for $T\left(P\right)$. Using (4) and rearranging the equation (6) in sense of collecting together terms involving equal number of factors of P we obtain

$$(7)\qquad \sum_{i=1}^{p-1} f_i\left(A,P\right) = 0,$$

where $f_i\left(A,P\right)$ stands for the expression of terms involving i factors of P. Replacing A by $A+2P$, $A+3P$, $\ldots$, $A+\left(p-1\right)P$ in turn in the equation (4), and expressing the resulting system of $p-1$ homogeneous equations of variables $f_i\left(A,P\right)$, $i=1,2,...,p-1$, we see that the coefficient matrix of the system is a van der Monde matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^2 & \cdots & 2^{p-1} \\ \vdots & \vdots & \vdots & \vdots \\ p-1 & (p-1)^2 & \cdots & (p-1)^{p-1} \end{bmatrix}.$$

Since the determinant of the matrix is different from zero, it follows that the system has only a trivial solution.

In particular,

$$\begin{aligned} & f_{p-2}(A,P) \\ &= 2(m+n)^2\binom{p}{p-2}T\left(A^2\right)-\left(2m^2+2mn\right)\left[\binom{p-1}{p-2}T(A)A+\binom{p-1}{p-3}BA^2\right] \\ &\qquad + mn\left[\binom{p-2}{p-4}T\left(A^2\right)P+2\binom{p-2}{p-3}T(A)A+BA^2\right] \\ &- 2mn\left[\binom{p-2}{p-3}AT(A)P+ABA+\binom{p-2}{p-3}PT(A)A+\binom{p-2}{p-4}PT\left(A^2\right)P\right] \\ &\qquad + mn\left[A^2B+2\binom{p-2}{p-3}AT(A)+\binom{p-2}{p-4}PT\left(A^2\right)\right] \\ &\qquad\qquad - \left(2n^2+2mn\right)\left[\binom{p-1}{p-2}AT(A)+\binom{p-1}{p-3}A^2B\right]=0 \end{aligned}$$

and

$$\begin{aligned} & f_{p-1}(A,P)=2(m+n)^2\binom{p}{p-1}T(A)-\left(2m^2+2mn\right)\left[T(A)P+\binom{p-1}{p-2}BA\right] \\ &+ mn\left[\binom{p-2}{p-3}T(A)P+2BA\right]-2mn\left[ABP+\binom{p-2}{p-3}PT(A)P+PBA\right] \\ &+mn\left[2AB+\binom{p-2}{p-3}PT(A)\right]-\left(2n^2+2mn\right)\left[\binom{p-1}{p-2}AB+PT(A)\right]=0. \end{aligned}$$

The above equations reduce to

$$\begin{aligned} (8)\quad (m+n)^2p(p-1)T\left(A^2\right) &= \left[2m(m+n)(p-1)-2mn(p-2)\right]T(A)A \\ &+ \left[2n(m+n)(p-1)-2mn(p-2)\right]AT(A) \\ &+ \left[m(m+n)(p-1)(p-2)-mn\right]BA^2+ \\ &\left[n(m+n)(p-1)(p-2)-mn\right]A^2B \\ &- \frac{mn(p-2)(p-3)}{2}\left(T\left(A^2\right)P+PT\left(A^2\right)\right)+ \\ &2mn(p-2)\left(AT(A)P+PT(A)A\right) \\ &+ 2mnABA+mn(p-2)(p-3)PT\left(A^2\right)P, \end{aligned}$$

and

$$(9)\quad 2(m+n)^2pT(A)=\left(2m^2+4mn-mnp\right)T(A)P + \left(2n^2+4mn-mnp\right)PT(A)+2m(m+n)(p-1)BA + 2n(m+n)(p-1)AB+2mn(p-2)PT(A)P.$$

Right multiplication of the relation (9) by P gives

$$(10)\quad \left(2(m+n)^2p-2m^2-4mn+mnp\right)T(A)P=\left(2n^2+mnp\right)PT(A)P + 2m(m+n)(p-1)BA+2n(m+n)(p-1)AB.$$

Subtracting the relation (10) from the relation (9) one obtains

$$(11)\quad 2(m+n)^2pT(A)=2(m+n)^2pT(A)P+\left(2n^2+4mn-mnp\right)(PT(A) - PT(A)P).$$

Left multiplication of the relation (9) by P gives

$$(12)\quad \left(2(m+n)^2p-2n^2-4mn+mnp\right)PT(A)=\left(2m^2+mnp\right)PT(A)P + 2m(m+n)(p-1)BA+2n(m+n)(p-1)AB.$$

Subtracting the relation (12) from the relation (9) one obtains

$$(13)\quad 2(m+n)^2pT(A)=\left(2m^2+4mn-mnp\right)(T(A)P-PT(A)P) + 2(m+n)^2pPT(A).$$

Subtracting the relation (11) from the relation (13) we obtain

$$(14)\quad \left(2n^2+4mn-mnp-2(m+n)^2p\right)PT(A) + \left(2(m+n)^2p-2m^2-4mn+mnp\right)T(A)P + 2\left(m^2-n^2\right)PT(A)P=0.$$

On the other hand comparing the relations (11) and (13) gives

$$(15)\quad 4(m+n)^2pT(A)=\left(2(m+n)^2p+2m^2+4mn-mnp\right)T(A)P + \left(2(m+n)^2p+2n^2+4mn-mnp\right)PT(A) + \left(2mnp-8mn-2m^2-2n^2\right)PT(A)P.$$

Combining the relations (14) and (15) we obtain

$$2(m+n)^2pT(A)=2(m+n)^2pT(A)P + \left(2n^2+4mn-mnp\right)PT(A)+\left(mnp-4mn-2n^2\right)PT(A)P.$$

Multiplying the above relation by P from the left side gives

$$(16)\quad PT(A)=PT(A)P.$$

Using the relation (16) in the relation (14) gives

$$PT(A) = T(A)P. \tag{17}$$

According to the relation (17) the relations (8) and (9) reduce to

$$\begin{aligned}(m+n)^2 p(p-1)T\left(A^2\right) &= 2m\,(m+n)\,(p-1)\,T\,(A)\,A \\ &+ 2n\,(m+n)\,(p-1)\,AT\,(A) + m\,[(m+n)\,(p-1)\,(p-2) - n]\,BA^2 + \\ &\qquad n\,[(m+n)\,(p-1)\,(p-2) - m]\,A^2B + 2mnABA\end{aligned} \tag{18}$$

and

$$(m+n)pT(A) = (m+n)T(A)P + (p-1)\,(mBA + nAB)\,. \tag{19}$$

Right multiplication of the relation (19) by P and subtracting the relation so obtained from the relation (19) gives

$$T(A) = T(A)P. \tag{20}$$

Using the relation (20) in the relation (19) one obtains

$$(m+n)T(A) = mBA + nAB. \tag{21}$$

Putting A^2 for A in the above relation gives

$$(m+n)T(A^2) = mBA^2 + nA^2B. \tag{22}$$

Multiplication of the relation (21) by A from both sides gives

$$(m+n)T(A)A = mBA^2 + nABA \tag{23}$$

and

$$(m+n)AT(A) = mABA + nA^2B. \tag{24}$$

Using the relations (22), (23) and (24) in the relation (18) we obtain after some calculation

$$A^2B + BA^2 - 2ABA = 0. \tag{25}$$

Now using the relations (22) and (25) in the relation (18) gives after some calculation

$$(m+n)T(A^2) = mT(A)A + nAT(A). \tag{26}$$

The relation (25) can be written in the form

$$[[A, B]\,, A] = 0. \tag{27}$$

Applying the relation (21) we obtain

$$(m+n)\,[[T(A), A]\,, A] = m\,[[BA, A]\,, A] + n\,[[AB, A]\,, A]\,.$$

Applying the commutator identity and the relation (27) on the right side of the relation above we obtain

$$\begin{aligned}(m+n)\left[\left[T(A),A\right],A\right] &= m\left[\left[B,A\right]A,A\right]+n\left[A\left[B,A\right],A\right]\\ &= m\left[\left[B,A\right],A\right]A+nA\left[\left[B,A\right],A\right]=0.\end{aligned}$$

We have therefore

$$\left[\left[T(A),A\right],A\right]=0 \tag{28}$$

for all $A\in\mathcal{F}(X)$. From the relation (20) one can conclude that T maps $\mathcal{F}(X)$ into itself. We have therefore an additive mapping $T:\mathcal{F}(X)\to\mathcal{F}(X)$ satisfying the relation (28) for all $A\in\mathcal{F}(X)$. Since $\mathcal{F}(X)$ is prime one can conclude applying Theorem 2 that

$$[T(A),A]=0$$

holds for all $A\in\mathcal{F}(X)$. The relation above makes it possible to replace in the relation (26), $T(A)A$ by $AT(A)$, which gives

$$T(A^2)=T(A)A.$$

Of course, we also have

$$T(A^2)=AT(A).$$

In other words, T is a left and a right Jordan centralizer on $\mathcal{F}(X)$. By Zalar theorem T is a two-sided centralizer. We intend to prove that there exists an operator $C\in\mathcal{L}(X)$, such that

$$T(A)=CA,\quad \text{for all } A\in\mathcal{F}(X). \tag{29}$$

For any fixed $x\in X$ and $f\in X^*$ we denote by $x\otimes f$ an operator from $\mathcal{F}(X)$ defined by $(x\otimes f)y=f(y)x$, for all $y\in X$. For any $A\in\mathcal{L}(X)$ we have $A(x\otimes f)=((Ax)\otimes f)$. Let us choose f and y such that $f(y)=1$ and define $Cx=T(x\otimes f)y$. Obviously, C is linear. Using the fact that T is left centralizer on $\mathcal{F}(X)$ we obtain

$$(CA)x=C(Ax)=T((Ax)\otimes f)y=T(A(x\otimes f))y=T(A)(x\otimes f)y=T(A)x,\qquad x\in X.$$

We have therefore $T(A)=CA$ for any $A\in\mathcal{F}(X)$. Since T right centralizer on $\mathcal{F}(X)$ we obtain $C(AP)=T(AP)=AT(P)=ACP$, where $A\in\mathcal{F}(X)$ and P is arbitrary one-dimensional projection. We have therefore $[A,C]\,P=0$. Since P is arbitrary one-dimensional projection it follows that $[A,C]=0$ for any $A\in\mathcal{F}(X)$. Using closed graph theorem one can easily prove that C is continuous. Since C commutes with all operators from $\mathcal{F}(X)$ one can conclude that $Cx=\lambda x$ holds for any $x\in X$ and some $\lambda\in\mathcal{F}$, which gives together with the relation (29) that T is of the form

$$T(A) = \lambda A,$$

for any $A \in \mathcal{F}(X)$ and some $\lambda \in \mathcal{F}$. It remains to prove that the above relation holds on $A(X)$ as well. Let us introduce $T_1 : A(X) \to \mathcal{L}(X)$ by $T_1(A) = \lambda A$ and consider $T_0 = T - T_1$. The mapping T_0 is, obviously, additive and satisfies the relation (4). Besides, T_0 vanishes on $\mathcal{F}(X)$. It is our aim to prove that T_0 vanishes on $A(X)$ as well. Let $A \in A(X)$, let P be an one-dimensional projection and $S = A + PAP - (AP + PA)$. Note that S can be written in the form $S = (I - P)A(I - P)$, where I denotes the identity operator on X, Since, obviously, $S - A \in \mathcal{F}(X)$, we have $T_0(S) = T_0(A)$. Besides, $SP = PS = 0$. We have therefore the relation

$$\begin{aligned}(30)\quad 2(m+n)^2T_0(A^p) &= \left(2m^2+2mn\right)T_0(A)A^{p-1} - mnT_0(A^{p-2})A^2\\ &+ 2mnAT_0(A^{p-2})A - mnA^2T_0(A^{p-2}) + \left(2n^2+2mn\right)A^{p-1}T_0(A),\end{aligned}$$

for all $A \in A(X)$. Applying the above relation we obtain

$$\begin{aligned}&\left(2m^2+2mn\right)T_0(S)S^{p-1} - mnT_0(S^{p-2})S^2 + 2mnST_0(S^{p-2})S\\ &\quad - mnS^2T_0(S^{p-2}) + \left(2n^2+2mn\right)S^{p-1}T_0(S) = 2(m+n)^2T_0(S^p)\\ &= 2(m+n)^2T_0(S^p+P) = 2(m+n)^2T_0((S+P)^p)\\ &= \left(2m^2+2mn\right)T_0(S+P)(S+P)^{p-1} - mnT_0((S+P)^{p-2})(S+P)^2\\ &\quad + 2mn(S+P)T_0((S+P)^{p-2})(S+P)\\ &\quad - mn(S+P)^2T_0((S+P)^{p-2}) + \left(2n^2+2mn\right)(S+P)^{p-1}T_0(S+P)\\ &= \left(2m^2+2mn\right)T_0(S)(S^{p-1}+P) - mnT_0(S^{p-2})(S^2+P)\\ &\quad + 2mn(S+P)T_0(S^{p-2})(S+P)\\ &\quad - mn(S^2+P)T_0(S^{p-2}) + \left(2n^2+2mn\right)(S^{p-1}+P)T_0(S)\\ &= \left(2m^2+2mn\right)T_0(S)S^{p-1} + \left(2m^2+2mn\right)T_0(S)P - mnT_0(S^{p-2})S^2\\ &\quad - mnT_0(S^{p-2})P + 2mnST_0(S^{p-2})S + 2mnPT_0(S^{p-2})S + 2mnST_0(S^{p-2})P\\ &\quad + 2mnPT_0(S^{p-2})P - mnS^2T_0(S^{p-2}) - mnPT_0(S^{p-2})\\ &\quad + \left(2n^2+2mn\right)S^{p-1}T_0(S) + \left(2n^2+2mn\right)PT_0(S).\end{aligned}$$

We have therefore

(31)

$$\begin{aligned}&\left(2m^2+2mn\right)T_0(A)P - mnT_0(A^{p-2})P + 2mnPT_0(A^{p-2})S + 2mnST_0(A^{p-2})P\\ &\quad + 2mnPT_0(A^{p-2})P - mnPT_0(A^{p-2}) + \left(2n^2+2mn\right)PT_0(A) = 0.\end{aligned}$$

Multiplying the above relation from both sides by P we obtain

$$(32)\qquad PT_0(A)P = 0.$$

Now right multiplication of the relation (31) by P gives because of (32)

$$2\,(m+n)\,T_0(A)P - nT_0(A^{p-2})P + 2nST_0(A^{p-2})P = 0. \tag{33}$$

Putting $2A$ for A in the relation (33) and subtracting the relation so obtained from the relation (33) gives

$$\left(2^{p-3}-1\right)T_0(A^{p-2})P + 2\left(1-2^{p-2}\right)ST_0(A^{p-2})P = 0. \tag{34}$$

Putting again $2A$ for A in the above relation gives after some calculation (see how the relation (34) was obtained from the relation (33))

$$ST_0(A^{p-2})P = 0 \tag{35}$$

and

$$T_0(A^{p-2})P = 0. \tag{36}$$

Using (35) and (36) in the relation (33) one obtains $T_0(A)P = 0$. Since P is an arbitrary one-dimensional projection, one can conclude that $T_0(A) = 0$, for any $A \in A(X)$, which completes the proof of the theorem.

We proceed with the following conjecture.

Conjecture. *Let $m \geq 1$, $n \geq 1$ and $p \geq 2$ be some fixed integers and let R be a semiprime ring with suitable torsion restrictions. Suppose there exists an additive mapping $T : R \to R$ satisfying the relation*

$$2(m+n)^2T(x^p) = \left(2m^2+2mn\right)T(x)x^{p-1} - mnT(x^{p-2})x^2 + 2mnxT(x^{p-2})x - mnx^2T(x^{p-2}) + \left(2n^2+2mn\right)x^{p-1}T(x)$$

for all $x \in R$. In this case T is a two-sided centralizer.

We proved the above conjecture in case R is a semiprime ring with the identity element.

Theorem 3. *Let $m \geq 1$, $n \geq 1$ and $p \geq 2$ be some fixed integers and let R be a 2, $p-1$, $2p-1$, $m+n$ and $mn-$torsion free semiprime ring with the identity element. Suppose there exists an additive mapping $T : R \to R$ satisfying the relation*

$$2(m+n)^2T(x^p) = \left(2m^2+2mn\right)T(x)x^{p-1} - mnT(x^{p-2})x^2 + 2mnxT(x^{p-2})x - mnx^2T(x^{p-2}) + \left(2n^2+2mn\right)x^{p-1}T(x)$$

for all $x \in R$. In this case T is a two-sided centralizer.

Proof. We have the relation

$$2(m+n)^2T(x^p) = \left(2m^2+2mn\right)T(x)x^{p-1} - mnT(x^{p-2})x^2 + 2mnxT(x^{p-2})x - mnx^2T(x^{p-2}) + \left(2n^2+2mn\right)x^{p-1}T(x), \quad x \in R. \tag{37}$$

Using a similar approach as in the proof of Theorem 1, with the exception that we use the identity element e instead of a projection, we obtain from the above relation

$$(38)\quad (m+n)^2p(p-1)T\left(x^2\right) = 2m(m+n)(p-1)T(x)x + 2n(m+n)(p-1)xT(x) + \left[m(m+n)(p-1)(p-2) - mn\right]ax^2 + \left[n(m+n)(p-1)(p-2) - mn\right]x^2a + 2mnxax, \quad x \in R$$

and

$$(39)\quad (m+n)T\left(x\right) = max + nxa, \quad x \in R,$$

where a stands for $T\left(e\right)$. In the procedure mentioned above we used the fact that R is 2, $p-1$ and $(m+n)-$torsion free.

The substitution x^2 for x in (39) gives

$$(40)\quad (m+n)T\left(x^2\right) = max^2 + nx^2a, \quad x \in R.$$

Multiplying the relation (39) first from the right side then from the left side by x we obtain

$$(41)\quad (m+n)T\left(x\right)x = max^2 + nxax, \quad x \in R$$

and

$$(42)\quad (m+n)xT\left(x\right) = mxax + nx^2a, \quad x \in R.$$

Using $(40), (41)$ and (42) in the relation (38) and applying the fact that R is $2p-1$ and $mn-$torsion free we obtain after some calculation

$$x^2a + ax^2 - 2xax = 0, \quad x \in R.$$

Combining the above relation and the relation (38) we obtain since R is $p-1$ and $(m+n)-$torsion free

$$(m+n)pT\left(x^2\right) = 2mT(x)x + 2nxT(x) + m\left(p-2\right)ax^2 + n\left(p-2\right)x^2a, \quad x \in R.$$

Applying (40) in the above relation gives

$$(43)\quad (m+n)T(x^2) = mT(x)x + nxT(x), \quad x \in R.$$

Combining (40) with (43) we obtain

$$mT(x)x + nxT(x) = max^2 + nx^2a, \quad x \in R.$$

Applying the relation (39) in the above relation we obtain

$$2mnxax - mn(ax^2 + x^2a) = 0$$

which can be written in the form

$$(44)\quad \left[\left[a, x\right], x\right] = 0, \quad x \in R.$$

Putting $x+y$ for x in the above relation we obtain

$$[[a,x],y]+[[a,y],x]=0, \quad x,y\in R. \tag{45}$$

Putting xy for y in the above relation and applying the commutator identity and (44) and (45) we obtain

$$0=[[a,x],xy]+[[a,xy],x]=[[a,x],x]y+x[[a,x],y]+[[a,x]y+x[a,y],x]$$
$$=x[[a,x],y]+[[a,x],x]y+[a,x][y,x]+x[[a,y],x]=[a,x][y,x],\ \ x,y\in R.$$

Thus we have

$$[a,x][y,x]=0, \quad x,y\in R.$$

The substitution ya for y in the above relation gives

$$[a,x]y[a,x]=0, \quad x,y\in R.$$

So far, we have not used the assumption that R is semiprime. Since R is semiprime, it follows from the above relation that $[a,x]=0$ for all $x\in R$. In other words, $a\in Z(R)$, which reduces the relation (39) to $T(x)=ax$, $x\in R$, since R is $(m+n)-$torsion free. The proof of the theorem is complete.

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