

LIMIT CYCLES FOR A CLASS OF ELEVENTH \mathbb{Z}_{12} -EQUIVARIANT SYSTEMS WITHOUT INFINITE CRITICAL POINTS

ADRIAN C. MURZA

Communicated by Vasile Brînzănescu

We analyze the complex dynamics of a family of \mathbb{Z}_{12} -equivariant planar systems, by using their reduction to an Abel equation. We derive conditions in the parameter space that allow uniqueness and hyperbolicity of a limit cycle surrounding either 1, 13 or 25 equilibria.

AMS 2010 Subject Classification: Primary 37C80, 37G40, 34C15; Secondary 34D06, 34C15.

Key words: planar autonomous ordinary differential equations, symmetric polynomial systems, limit cycles.

1. INTRODUCTION AND MAIN RESULTS

Hilbert's 16th problem is one of the open question in mathematics and even in two dimensions is far from being solved. The study of this problem became a new branch of analysis, based on the recent development of the theory of Golubitsky, Stewart and Schaeffer in [8, 9].

In this paper, we analyze the \mathbb{Z}_{12} -equivariant system

$$(1) \quad \dot{z} = pz^5\bar{z}^4 + sz^6\bar{z}^5 - \bar{z}^{11} = f(z),$$

where z is complex, the time t is real, while p and s are complex parameters ($p_1, p_2, s_1, s_2 \in \mathbb{R}$).

The generalized form of the \mathbb{Z}_q -equivariant equation is

$$\dot{z} = zA(|z|^2) + B\bar{z}^{q-1} + O(|z|^{q+1}),$$

where A is a polynomial of degree $[(q-1)/2]$. The analysis of this class of equations is carried out for example in [3, 5], in the case of strong resonances, *i.e.* $q < 4$ or weak ones $q > 4$. The case corresponding to $q = 4$ is also analyzed in other articles, such as [5, 11]. Our aim is studying the phase portrait of system (1). We devote, especial attention to the existence, uniqueness and location of limit cycles surrounding 1, 13 or 25 equilibria. To reach our goal, we want to transform the system (1) into a scalar Abel equation and then reduce the study of the initial problem to the Abel equation.

We state now the main result of the article.

THEOREM 1. *Let us consider the equation (1) with $s_2 > 1$, $p_2 \neq 0$, and define the additional quantities:*

$$\Sigma_A^- = \frac{p_2 s_1 s_2 - \sqrt{p_2^2 (s_1^2 + s_2^2 - 1)}}{s_2^2 - 1}, \quad \Sigma_A^+ = \frac{p_2 s_1 s_2 + \sqrt{p_2^2 (s_1^2 + s_2^2 - 1)}}{s_2^2 - 1}.$$

Then, the following statements hold true:

(a) *If one of the constraints*

$$(i) \quad p_1 \notin (\Sigma_A^-, \Sigma_A^+), \quad (ii) \quad p_1 \notin \left(\frac{\Sigma_A^-}{2}, \frac{\Sigma_A^+}{2} \right)$$

is satisfied, then equation (1) has at most one limit cycle surrounding the origin. Moreover, if the mentioned limit cycle exists it is hyperbolic.

(b) *There exist equations (1) under condition (i) with exactly one hyperbolic limit cycle surrounding either 1 or 13 equilibria and equations (1) under condition (ii) possessing exactly one hyperbolic limit cycle surrounding either 1, 13 or 25 equilibria.*

The paper is organized in the following manner. Section 2 is devoted to the statement of some preliminary results. In Section 3, we analyze the equilibria. Finally, in Section 4 we give the detailed proof of the main theorem of the paper.

2. PRELIMINARY RESULTS

From the pioneering work of Golubitsky, Schaeffer and Stewart [8, 9] a system of differential equations $dx/dt = f(x)$ is said to be Γ -equivariant if it commutes with the group action of Γ , *i.e.* $f(\gamma x) = \gamma f(x)$, $\forall \gamma \in \Gamma$. Here $\Gamma = \mathbb{Z}_{12}$ with the standard action on \mathbb{C} generated by $\gamma_1 = \exp(2\pi i/12)$ acting by complex multiplication. Applying this result to equation (1) we have the following result.

PROPOSITION 1. *Equation (1) is \mathbb{Z}_{12} -equivariant.*

Proof. Let $\gamma_k = \exp(2\pi i k/12)$, $k = 0, \dots, 11$. Then

$$\begin{aligned} f(\gamma_k z) &= f(\exp(2\pi i k/12)z) = \\ &= (p_1 + ip_2) \exp(10\pi i k/12)z^4 \exp(-8\pi i k/12)\bar{z}^3 + \\ &\quad + (s_1 + is_2) \exp(12\pi i k/12)z^5 \exp(-10\pi i k/12)\bar{z}^4 - \\ &\quad - \exp(-22\pi i k/12)\bar{z}^{11} = \\ &= \exp(2\pi i k/12) ((p_1 + ip_2)z^5\bar{z}^4 + (s_1 + is_2)z^6\bar{z}^5 - \bar{z}^{11}) = \\ &= \exp(2\pi i k/12)f(z) = \gamma_k f(z). \quad \square \end{aligned}$$

We obtain equation (1) by perturbing a Hamiltonian, whose expression is

$$\dot{z} = i(p_2 + s_2 z \bar{z})z^5 \bar{z}^4 - \bar{z}^{11}.$$

We express formally this fact in the following theorem.

THEOREM 2. *The corresponding Hamiltonian equation to the \mathbb{Z}_{12} -equivariant equation (1) is*

$$\dot{z} = i(p_2 + s_2 z \bar{z})z^5 \bar{z}^4 - \bar{z}^{11}.$$

Proof. The class of equations $\dot{z} = F(z, \bar{z})$ is Hamiltonian if $\frac{\partial F}{\partial z} + \frac{\partial \bar{F}}{\partial \bar{z}} = 0$. For equation (1) we have

$$\frac{\partial F}{\partial z} = 5(p_1 + ip_2)z^4 \bar{z}^4 + 6(s_1 + is_2)z^5 \bar{z}^5$$

$$\frac{\partial \bar{F}}{\partial \bar{z}} = 5(p_1 - ip_2)z^4 \bar{z}^4 + 6(s_1 - is_2)z^5 \bar{z}^5.$$

Therefore equation (1) is Hamiltonian if and only if $p_1 = s_1 = 0$. \square

An important technique for studying the limit cycles of a two-dimensional \mathbb{Z}_q -equivariant ODE consists in reducing the initial ODE to a scalar equation of Abel type. In applying this technique to (1), firstly we convert it from cartesian to polar coordinates.

LEMMA 1. *The solutions of equation (1) are equivalent to those of the polar system*

$$(2) \quad \begin{cases} \dot{r} = 2rp_1 + 2r^2(s_1 - \cos(12\theta)) \\ \dot{\theta} = p_2 + r(s_2 + \sin(12\theta)). \end{cases}$$

Proof. We perform the change of variables

$$z = \sqrt{r}(\cos(\theta) + i \sin(\theta))$$

and obtain

$$(3) \quad \begin{cases} \dot{r} = 2r^5(p_1 + rs_1 - r \cos(12\theta)) \\ \dot{\theta} = r^4(p_2 + rs_2 + \sin(12\theta)). \end{cases}$$

Then we do the time rescaling, $\frac{dt}{ds} = r^4$, to get the result. \square

As a consequence, we have the following lemma.

LEMMA 2. *The analysis of non contractible solutions that satisfy $x(0) = x(2\pi)$ of the corresponding Abel equation affords the same information about the periodic orbits of equation (1) that surround the origin.*

$$(4) \quad \frac{dx}{d\theta} = A(\theta)x^3 + B(\theta)x^2 + C(\theta)x$$

where

$$\begin{aligned}
 A(\theta) &= \frac{2}{s_2} \left(-s_1 + p_1 p_2 s_2 - p_2^2 s_1 + (p_1 s_2 - 2p_2 s_1) \sin(12\theta) \right) + \\
 &\quad + \frac{2}{s_2} \left(s_2 \sin(12\theta) + s_1 \cos(12\theta) + p_2 s_2 \right) \cos(12\theta), \\
 B(\theta) &= \frac{2}{s_2} \left(-p_1 s_2 - s_2 \cos(12\theta) + 2s_1 \sin(12\theta) + 2p_2 s_1 \right), \\
 C(\theta) &= -\frac{2s_1}{s_2}.
 \end{aligned}
 \tag{5}$$

Proof. From equation (2) we obtain

$$\frac{dr}{d\theta} = \frac{2r(p_1 + r s_1 + \cos(12\theta))}{p_2 + r s_2 + \sin(12\theta)}.$$

Next we use a Cherkas-like technique [4] to convert the scalar equation (2) into (4). More specifically, we use the variable change $x = \frac{r}{p_2 + r s_2 + \sin(12\theta)}$, to get the result. As the non contractible periodic orbits of equation (4) cannot intersect the set $\{\dot{\theta} = 0\}$, their analysis is the same as studying the limit cycles that surround the origin of equation (1). This is explained in more detail in [6]. \square

Our aim is applying the methodology developed in [1] to analyze conditions for existence, location and unicity of the limit cycles surrounding 1, 13 and 25 critical points.

A precise method for proving the existence of a limit cycle is showing that, in the Poincaré compactification, infinity has no critical points; moreover, we have to prove that both origin and infinity have the same stability. We explicitly describe the set of parameters which satisfies these conditions. The first step in our approach is to determine the stability of infinity. This is done in the next lemma.

LEMMA 3. *Apply the Poincaré compactification to equation (1). The following statements hold true.*

- (i) *Infinity has no equilibria if and only if $|s_2| > 1$;*
- (ii) *For $s_2 > 1$, infinity is attractor (resp. a repellor) if $s_1 > 0$ (resp. $s_1 < 0$) and it has the opposite character for $s_2 < -1$.*

Proof. Firstly, we perform the change of variable $R = 1/r$ in system (2).

Then the parametrization by $\frac{dt}{ds} = R$, yields the system

$$(6) \quad \begin{cases} R' = \frac{dR}{ds} = -2s_1R - 2R^2(p_1 + \cos(12\theta)), \\ \theta' = \frac{d\theta}{ds} = R(s_2 + \sin(12\theta)) + p_2. \end{cases}$$

This yields the invariant set $\{R = 0\}$, corresponding to infinity of system (2). Hence, system (2) has no equilibria at infinity provided $|s_2| > 1$. We apply the method originally proposed by [10], to the system (6). That is, we analyze the stability of $\{R = 0\}$. This is obtained from the sign of

$$\int_0^{2\pi} \frac{-2s_1}{s_2 + \sin(12\theta)} d\theta = \frac{-\operatorname{sgn}(s_2)4\pi s_1}{\sqrt{s_2^2 - 1}}.$$

This completes the proof. \square

3. ANALYSIS OF THE CRITICAL POINTS

In this section, we explicitly determine the conditions that allow equation (2) to have one, thirteen or twenty-five equilibria. Our interest in the origin is twofold. Firstly, we show (by simple inspection), that it is always an equilibrium. Secondly, we demonstrate that it is always *monodromic*.

The main tool to carry out our proofs is represented by the generalized Lyapunov constants. To define those, we first recall the solution of the scalar equation

$$(7) \quad \frac{dr}{d\theta} = \sum_{i=1}^{\infty} R_i(\theta) r^i,$$

where $R_i(\theta)$, $i \geq 1$ are T -periodic functions. Next, we define the generalized Lyapunov constants; consider the solution of (7) that for $\theta = 0$ passes through ρ . It can be described by

$$r(\theta, \rho) = \sum_{i=1}^{\infty} u_i(\theta) \rho^i$$

with $u_1(0) = 1$, $u_k(0) = 0, \forall k \geq 2$. Therefore, the first return map of the solution above is written in terms of the series

$$\Pi(\rho) = \sum_{i=1}^{\infty} u_i(T) \rho^i.$$

To analyse the stability of a solution, the only significant term in the return map is the first nonvanishing term that makes it different from the identity map. Therefore, this term is the one that determines the stability of this solution. On the other side, let us take into account a family of equations that depends on parameters, *i.e* each of the terms $u_i(T)$ is a function of these parameters. We define k^{th} generalized Lyapunov constant $V_k = u_k(T)$ the value for the equation above supposing $u_1(T) = 1, u_2(T) = \dots, = u_{k-1}(T) = 0$.

LEMMA 4. *The stability of the origin of system (2) is given by the sign of p_1 if $p_1 \neq 0$ and by the sign of s_1 if $p_1 = 0$. In addition, the character of the origin is always monodromic.*

Proof. We begin with the proof of the last statement in Lemma 4. To prove origin's monodromicity we firstly write system (2) in cartesian calculate. This yields system (8).

$$\begin{aligned}
 \dot{x} = P(x, y) &= p_1x^9 - x^{11} + s_1x^{11} - p_2x^8y - s_2x^{10}y + 4p_1x^7y^2 + \\
 &\quad 55x^9y^2 + 5s_1x^9y^2 - 4p_2x^6y^3 - 5s_2x^8y^3 + \\
 &\quad 6p_1x^5y^4 - 330x^7y^4 + 10s_1x^7y^4 - 6p_2x^4y^5 - \\
 &\quad 10s_2x^6y^5 + 4p_1x^3y^6 + 462x^5y^6 + 10s_1x^5y^6 - \\
 &\quad 4p_2x^2y^7 - 10s_2x^4y^7 + p_1xy^8 - 165x^3y^8 + \\
 &\quad 5s_1x^3y^8 - p_2y^9 - 5s_2x^2y^9 + 11xy^{10} + s_1xy^{10} - s_2y^{11}, \\
 \dot{y} = Q(x, y) &= p_2x^9 + s_2x^{11} + p_1x^8y + 11x^{10}y + s_1x^{10}y + 4p_2x^7y^2 + \\
 &\quad 5s_2x^9y^2 + 4p_1x^6y^3 - 165x^8y^3 + 5s_1x^8y^3 + \\
 &\quad 6p_2x^5y^4 + 10s_2x^7y^4 + 6p_1x^4y^5 + 462x^6y^5 + \\
 &\quad 10s_1x^6y^5 + 4p_2x^3y^6 + 10s_2x^5y^6 + 4p_1x^2y^7 - \\
 &\quad 330x^4y^7 + 10s_1x^4y^7 + p_2xy^8 + 5s_2x^3y^8 + p_1y^9 + \\
 &\quad 55x^2y^9 + 5s_1x^2y^9 + s_2xy^{10} - y^{11} + s_1y^{11}.
 \end{aligned}
 \tag{8}$$

Following [2, Chapter IX], we conclude that any solutions arriving at the origin are tangent to the directions θ that are the roots of the equation $r\theta = R(x, y) = xQ(x, y) - yP(x, y) = p_2(x^2 + y^2)^5$. It is easy to see, by simple inspection that this is always different than zero. Consequently, the origin is either a focus or a center.

We compute the first two Lyapunov constants in order to analyze the stability of the origin. For an Abel equation these may be expressed in the form

$$V_1 = \exp\left(\int_0^{2\pi} C(\theta)d\theta\right) - 1, \quad V_2 = \int_0^{2\pi} B(\theta)d\theta.$$

Combining these with equations (5) yield:

$$V_1 = \exp\left(4\pi\frac{p_1}{p_2}\right) - 1,$$

and if $V_1 = 0$, then $V_2 = 4\pi s_1$. This completes the proof. \square

Remark 1. We have $(V_1, V_2) = (0, 0)$ iff $(p_1, s_1) = (0, 0)$ i.e. equation (1) is Hamiltonian (see Theorem 2). Therefore, as the origin remains being monodromic, it is a center.

Next we analyze the singular points of equation (2) such that $r \neq 0$. These are represented by the non-zero critical points of the system under consideration.

LEMMA 5. *Let $-\pi/12 < \theta < \pi/12$. Then the critical points of system (2) with $r \neq 0$ are expressed by the following equalities:*

$$(9) \quad r = \frac{-p_2}{s_2 + \sin(12\theta_{\pm})}, \quad \theta_{\pm} = \frac{1}{6} \arctan(\Delta_{\pm}),$$

where $\Delta_{\pm} = \frac{p_1 \pm u}{p_2 - p_1 s_2 + p_2 s_1}$ and $u = \sqrt{p_1^2 + p_2^2 - (p_1 s_2 - p_2 s_1)^2}$.

Proof. Let $-\pi/12 < \theta < \pi/12$. To compute the equilibria of system (2), we must find the roots of the following system:

$$(10) \quad \begin{aligned} 0 &= 2rp_1 + 2r^2(s_1 - \cos(12\theta)) \\ 0 &= p_2 + r(s_2 + \sin(12\theta)). \end{aligned}$$

Let $x = 12\theta$ and $t = \tan\left(\frac{x}{2}\right)$, so $t = \tan(6\theta)$. Then tedious but straightforward computations show that

$$(11) \quad \sin(x) = \frac{2t}{1+t^2} \quad \cos(x) = \frac{1-t^2}{1+t^2}.$$

In the following step, we remove r from equations (10). Making use of the preceding equations yield

$$(-p_2 + p_1 s_2 - p_2 s_1)t^2 + 2p_1 t + p_2 + p_1 s_2 - p_2 s_1 = 0.$$

Next we solve the previous equation for t to get the result.

Finally, let us pay attention to the interval $-\pi/12 \leq \theta < \pi/12$. Let $x = 12\theta$ and $\tau = \cot\left(\frac{x}{2}\right)$. Therefore, we get $\tau = \cot(6\theta)$. Expressions (11) for $\sin x$ and $\cos x$ hold true also if t is replaced by $\tau = 1/t$. This completes the proof of the lemma. \square

We will show now that simultaneous solutions of the type $(r, \theta) = (r, 0)$, $(r, \theta) = (\tilde{r}, \pi/12)$ are not possible.

LEMMA 6. *Assume $|s_2| > 1$. We claim that there exist no parameter values which lead to critical points of system (2) for $\theta = 0$ and $\theta = \pi/12$ different that $(0, 0)$.*

Proof. We solve the following system

$$\begin{cases} 0 = p_1 + r(s_1 - \cos(12\theta)) \\ 0 = p_2 + r(s_2 + \sin(12\theta)) \end{cases}$$

for $\theta = 0$, which yields $p_1 s_2 = p_2(s_1 - 1)$ with the additional constraints on the parameters $\text{sign } p_2 = -\text{sign } s_2$. This way, we have r well defined in the second equation. Secondly, we solve the system for $\theta = \pi/12$, which yields $p_1 s_2 = p_2(s_1 + 1)$. We have to put the same constraints on the parameters $\text{sign } p_2 = -\text{sign } s_2$, in consequence $p_2 = 0$; therefore $r = 0$. This completes the proof. \square

At this point, it is useful to bring together the conditions in the parameter space that allow system (2) to have one, thirteen or twenty-five critical points.

LEMMA 7. *Suppose $s_2 > 1$ in system (2). Then if $p_2 \geq 0$, the only critical point is $(0, 0)$. Assume $p_2 < 0$. In this case the number of critical points is conditioned by the quadratic form:*

$$(12) \quad \mathcal{Q}(p_1, p_2) = p_1^2 + p_2^2 - (p_1 s_2 - p_2 s_1)^2 = (1 - s_2^2)p_1^2 + (1 - s_1^2)p_2^2 + 2s_1 s_2 p_1 p_2$$

Specifically:

- (1) *one critical point $(0, 0)$ if $\mathcal{Q}(p_1, p_2) < 0$;*
- (2) *thirteen critical points (origin plus one saddle-node corresponding to $\frac{1}{12}$ of the phase space), if $\mathcal{Q}(p_1, p_2) = 0$;*
- (3) *twenty-five critical points (origin plus a saddle and a node corresponding to $\frac{1}{12}$ of the phase space), if $\mathcal{Q}(p_1, p_2) > 0$.*

Similarly, we obtain a corresponding result to $s_2 < -1$ by switching $p_2 > 0$ by $p_2 < 0$.

Proof. For brevity we perform the proof only for $s_2 > 1$; the other cases are left as an easy exercise for the reader.

System (2) has always in origin one of its critical points. There will be more of those iff equations (9) have solutions for $r > 0$. This is not the case in the following two situations. Firstly, if r is negative, because in this case $p_2 \geq 0$. Secondly if $\Delta_{\pm} < 0$; this is so because in this case $\mathcal{Q}(p_1, p_2) < 0$.

In order for system (2) to have exactly thirteen equilibria, we need two conditions. Firstly $p_2 < 0$ such that $r > 0$ as $s_2 > 1$. Secondly $\Delta_+ = \Delta_-$. But this last condition leads to the discriminant $u = 0$ in Lemma 5. Therefore, we get $r_+ = r_-$, $\theta_+ = \theta_-$.

To show that these twelve equilibria are saddle-nodes, we work in a $1/12$ of the phase-space. This is due to the fact that system (2) is \mathbb{Z}_{12} -equivariant.

The first step is to evaluate the Jacobian matrix of system (2) at the critical points (r_+, θ_+) . The Jacobian matrix J is

$$(13) \quad J_{(r_+, \theta_+)} = \begin{pmatrix} 2p_1 + 4r_+(s_1 - \cos(12\theta_+)) & 24r_+^2 \sin(12\theta_+) \\ s_2 + \sin(12\theta_+) & 12r_+ \cos(12\theta_+) \end{pmatrix}.$$

Next we evaluate J at the twelve equilibria. Taking into account the constraint $p_1^2 + p_2^2 = (p_1 s_2 - p_2 s_1)^2$, the eigenvalues of J are

$$\lambda_1 = 0, \quad \lambda_2 = -\frac{2(p_1^2 + 6p_2^2 + p_1^2 s_1 + 6p_2^2 s_1 - 5p_1 p_2 s_2 + p_1 p_2 s_1 s_2 - p_1^2 s_2^2)}{p_1 + p_1 s_1 + p_2 s_2 + p_2 s_1 s_2 - p_1 s_2^2}.$$

Hence, (r_+, θ_+) has a eigenvalue of value zero. Next we want to prove that these equilibria are saddle-nodes. We know from [2], that the sum of the indices of the equilibria inside the limit cycle equals $+1$. We already proved that infinity has no critical points. Therefore, we deal with a limit cycle of a planar system which has thirteen equilibria inside. One of them is the origin; we have proved that it is a focus and therefore it has index $+1$. Moreover, there are 12 more equilibria, all of the same kind due to the \mathbb{Z}_{12} -symmetry. Therefore, all of these additional equilibria must have zero index. Since we have showed that these equilibria are semi-hyperbolic, it follows that they are saddle-nodes.

The last task in our mission to prove this result consists in deriving the conditions in the parameter space such that equation (2) has exactly twenty-five equilibria. This is easily shown by observing that $r > 0$ (*i.e.* $p_2 < 0$) and the discriminant in $\Delta_{\pm} > 0$ of Lemma 5.

Next we evaluate the Jacobian matrix (13) at the twenty-four equilibria to analyze their stability. Putting the condition $p_1^2 + p_2^2 > (p_1 s_2 - p_2 s_1)^2$, the eigenvalues of the critical points (r_+, θ_+) are $\lambda_{1,2} = R_+ \pm S_+$, while the eigenvalues of (r_-, θ_-) are $\alpha_{1,2} = R_- \pm S_-$, where

$$R_{\pm} = \frac{\pm p_1^2 s_1 \pm 5p_2^2 s_1 \pm 5p_1 p_2 s_2 + 4p_1 u}{\pm p_1 s_1 \pm p_2 s_2 + u} +$$

$$\frac{\frac{1}{2} \sqrt{96(p_1^2 + p_2^2)u(\pm p_1 s_1 \pm p_2 s_2 + u) + 4(p_1^2 s_1 + 6p_2^2 s_1 - 5p_1 p_2 s_2 \pm p_1 u)^2}}{\pm p_1 s_1 \pm p_2 s_2 + u},$$

$$S_{\pm} = \frac{\pm p_1^2 s_1 \pm 6p_2^2 s_1 \pm 5p_1 p_2 s_2 + 5p_1 u}{\pm p_1 s_1 \pm p_2 s_2 + u} -$$

$$\frac{\frac{1}{2} \sqrt{96(p_1^2 + p_2^2)u(\pm p_1 s_1 \pm p_2 s_2 + u) + 4(p_1^2 s_1 + 6p_2^2 s_1 - 5p_1 p_2 s_2 \pm 5p_1 u)^2}}{\pm p_1 s_1 \pm p_2 s_2 + u}.$$

Our final task is proving that one of these equilibria has index +1, and the other is a saddle.

Some tedious but straightforward computations show that the product of the eigenvalues of (r_+, θ_+) and (r_-, θ_-) are, respectively

$$R_+^2 - S_+^2 = \frac{2(p_1^2 s_1 + 6p_2^2 s_1 - 5p_1(p_2 s_2 - u))}{(p_1 s_1 + p_2 s_2 - u)^2}.$$

$$\sqrt{96(p_1^2 + p_2^2)u(-p_1 s_1 - p_2 s_2 - u) + 4(p_1^2 s_1 + 6p_2^2 s_1 + 5p_1(-p_2 s_2 + u))^2},$$

$$R_-^2 - S_-^2 = -\frac{2(p_1^2 + 6p_2^2 s_1 - 5p_1(p_2 s_2 + u))}{(p_1 s_1 + p_2 s_2 + u)^2}.$$

$$\sqrt{96(p_1^2 + p_2^2)u(p_1 s_1 + p_2 s_2 + u) + 4(p_1^2 s_1 + 6p_2^2 s_1 - 5p_1(p_2 s_2 + u))^2}.$$

The denominator of these formulae is positive.

If

$$p_1 < 0 \text{ and } u < \frac{-p_1^2 s_1 - 5p_2^2 s_1 + 5p_1 p_2 s_2}{5p_1}$$

or

$$p_1 > 0 \text{ and } u > \frac{-p_1^2 s_1 - 6p_2^2 s_1 + 5p_1 p_2 s_2}{5p_1}$$

then $R_-^2 - S_-^2 < 0$. Therefore the equilibrium (r_-, θ_-) is a saddle. If $p_1 < 0$ and $u > \frac{p_1^2 s_1 + 6p_2^2 s_1 - 5p_1 p_2 s_2}{5p_1}$ or $p_1 > 0$ and $u < \frac{p_1^2 s_1 + 6p_2^2 s_1 - 5p_1 p_2 s_2}{5p_1}$ then $R_-^2 - S_-^2 > 0$. Hence, (r_+, θ_+) has index +1. This completes the proof. \square

LEMMA 8. *Let us assume $|s_2| > 1$ and consider the following quantities:*

$$\Sigma_A^- = \frac{p_2 s_1 s_2 - \sqrt{p_2^2 (s_1^2 + s_2^2 - 1)}}{s_2^2 - 1}, \quad \Sigma_A^+ = \frac{p_2 s_1 s_2 + \sqrt{p_2^2 (s_1^2 + s_2^2 - 1)}}{s_2^2 - 1}.$$

We denote $A(\theta)$ the function given in Lemma 2. It follows that $A(\theta)$ changes sign iff $p_1 \in (\Sigma_A^-, \Sigma_A^+)$.

Proof. Let $x = \sin(12\theta)$, $y = \cos(12\theta)$. Then $A(\theta)$ defined in (5) writes

$$A(x, y) = \frac{2}{p_2} (p_1 - p_2 s_1 s_2 + p_1 s_2^2 + (2p_1 s_2 - p_2 s_1)x + (p_2 x - p_1 y + p_2 s_2)y).$$

Next we solve the nonlinear system

$$\begin{aligned} A(x, y) &= 0, \\ x^2 + y^2 &= 1. \end{aligned}$$

which yields the solutions

$$\begin{aligned} x_1 &= -s_2, \quad y_1 = \sqrt{1 - s_2^2} \\ x_2 &= -s_2, \quad y_2 = -\sqrt{1 - s_2^2} \\ x_{\pm} &= \frac{p_1 p_2 s_1 - p_1^2 s_2 \pm p_2 \sqrt{p_1^2 + p_2^2 - (p_2 s_1 - p_1 s_2)^2}}{p_1^2 + p_2^2} \\ y_{\pm} &= \frac{p_2^2 s_1 - p_1 p_2 s_2 \mp p_1 \sqrt{p_1^2 + p_2^2 - (p_2 s_1 - p_1 s_2)^2}}{p_1^2 + p_2^2} \end{aligned}$$

Since $x = \sin(12\theta) = -s_2 < -1$, it follows that two pairs of solutions (x_1, y_1) , (x_2, y_2) cannot be solutions $A(\theta) = 0$.

On the other hand, we have two choices for the intervals for which $A(\theta)$ doesn't change sign:

- (1) $|x_{\pm}| > 1$ (hence $|y_{\pm}| > 1$)
- (2) $\Delta = p_1^2 + p_2^2 - (p_2 s_1 - p_1 s_2)^2 \leq 0$.

If $\Delta < 0$, we get complex non-real solutions, while if $\Delta = 0$, the function has a double zero; in this case it doesn't change sign.

Finally, while $|x_{\pm}| > 1$ is impossible in the parameter space subject to our mentioned constraints, the second case yields the region $p_1 \in \mathcal{R} \setminus (\Sigma_A^+, \Sigma_A^-)$. \square

LEMMA 9. *Let us assume $|s_2| > 1$ and consider the following quantities:*

$$\Sigma_B^{\pm} = \frac{\Sigma_A^{\pm}}{2}.$$

We denote $B(\theta)$ the function given in Lemma 2. It follows that $B(\theta)$ changes sign iff $p_1 \in (\Sigma_B^-, \Sigma_B^+)$.

Proof. We carry out the proof by performing calculations similar to the ones in the proof of the previous lemma. We denote $x = \sin(12\theta)$, $y = \cos(12\theta)$, and calculate the roots of the nonlinear system

$$\begin{aligned} B(x, y) &= 0, \\ x^2 + y^2 &= 1. \end{aligned}$$

These are

$$\begin{aligned} x_{\pm} &= \frac{2p_1 p_2 s_1 - 4p_1^2 s_2 \pm p_2 \sqrt{4p_1^2 + p_2^2 - (p_2 s_1 - 2p_1 s_2)^2}}{4p_1^2 + p_2^2}, \\ y_{\pm} &= \frac{p_2^2 s_1 - 2p_1 p_2 s_2 \mp 2p_1 \sqrt{4p_1^2 + p_2^2 - (p_2 s_1 - 2p_1 s_2)^2}}{4p_1^2 + p_2^2}. \end{aligned}$$

We get that $B(\theta)$ doesn't change sign iff $p_1 \notin (\Sigma_B^+, \Sigma_B^-)$. \square

Remark 2. As a result of the previous analysis,

- we can find examples of systems such that $A(\theta)$ changes sign but $B(\theta)$ does not and vice-versa;
- the constraint that forces function $A(\theta)$ not to change sign is related to the number of equilibria of system 2. Condition $\Delta < 0$ in the proof of Lemma 8 coincides with the constraint for the existence of a unique equilibrium in system (2). Moreover, the constraint $\Delta = 0$ forces the system to have exactly 13 critical points, together with $p_2 < 0$.

Next we will recall a result on scalar Abel equations due to Llibre [7].

THEOREM 3 (Llibre, [7]). *Consider the Abel equation (4) and assume that either $A(\theta) \not\equiv 0$ or $B(\theta) \not\equiv 0$ does not change sign. Then it has at most three solutions satisfying $x(0) = x(2\pi)$, taking into account their multiplicities.*

4. LIMIT CYCLES

Proof of Theorem 1. Let us begin our proof by defining the function $c(\theta) = s_2 + \sin(12\theta)$ and the set $\Theta := \{(r, \theta) : \dot{\theta} = p_2 + (s_2 + \sin(12\theta))r = 0\}$. We have $c(\theta) \neq 0, \forall \theta \in [0, 2\pi]$. This is so because $|s_2| > 1$.

(a) To prove the first item of the Theorem 1, we assume that condition (i) is satisfied. By applying Lemma 2, it is enough to study the non contractible periodic orbits of the Abel equation (4). We have that $A(\theta)$ doesn't change sign. This is so because $p_1 \notin (\Sigma_A^-, \Sigma_A^+)$, by condition (i) of Lemma 8. Moreover, the upper bound for the solutions satisfying $x(0) = x(2\pi)$ in system (4) taking into account their multiplicities, is three. This results directly by applying Llibre's Theorem 3. One of these solutions is $x = 0$. In addition, the curve $x = 1/c(\theta)$ is a second solution satisfying this condition, since $c(\theta) \neq 0$. It is easy to show that the equation $x = 1/c(\theta)$ is mapped into infinity of the differential equation. One possibility to do this is by undoing the Cherkas transformation. Therefore, by applying Lemma 2, the upper bound for the limit cycles of equation (1) is one. The same lemma proves that it is hyperbolic.

(b) To prove the second item of Theorem 1 we fix parameters $p_2 = -1, s_1 = -0.5, s_2 = 1.2$. We get $\Sigma_A^- = -0.52423, \Sigma_A^+ = 3.25151$. If $p_1 > \Sigma_A^+$, by Lemmas 3 and 4, both the origin and infinity have positive sign in the Poincaré compactification. More specifically, the origin is an unstable focus. By applying Lemma 8, the function $A(\theta)$ still exists for $p_1 > \Sigma_A^+$. Under these circumstances the origin is the unique critical point. Now we have $p_2 A(\theta) > 0$. Therefore the exterior of the closed curve Θ is positively invariant. To prove that there is exactly one hyperbolic limit cycle surrounding the curve Θ , we apply the Poincaré-Bendixson Theorem. In addition it is stable.

For $p_1 = \Sigma_A^+$, twelve more semi-elementary additional equilibria appear in the phase space. They are located on Θ . We have showed in Lemma 7 that these equilibria are saddle-nodes. Now our strategy is showing that for this p_1 the periodic orbit surrounding the thirteen critical points still exists. Our tool will be a polygonal line with no contact with the flow of the differential equation. On this polygonal line, the vector field points outside, and since the infinity is repeller, the ω -limit set of the unstable separatrices of the saddle-nodes has to be a limit cycle surrounding the curve Θ . This polygonal line with no contact with the flow is shown in Figure 1.

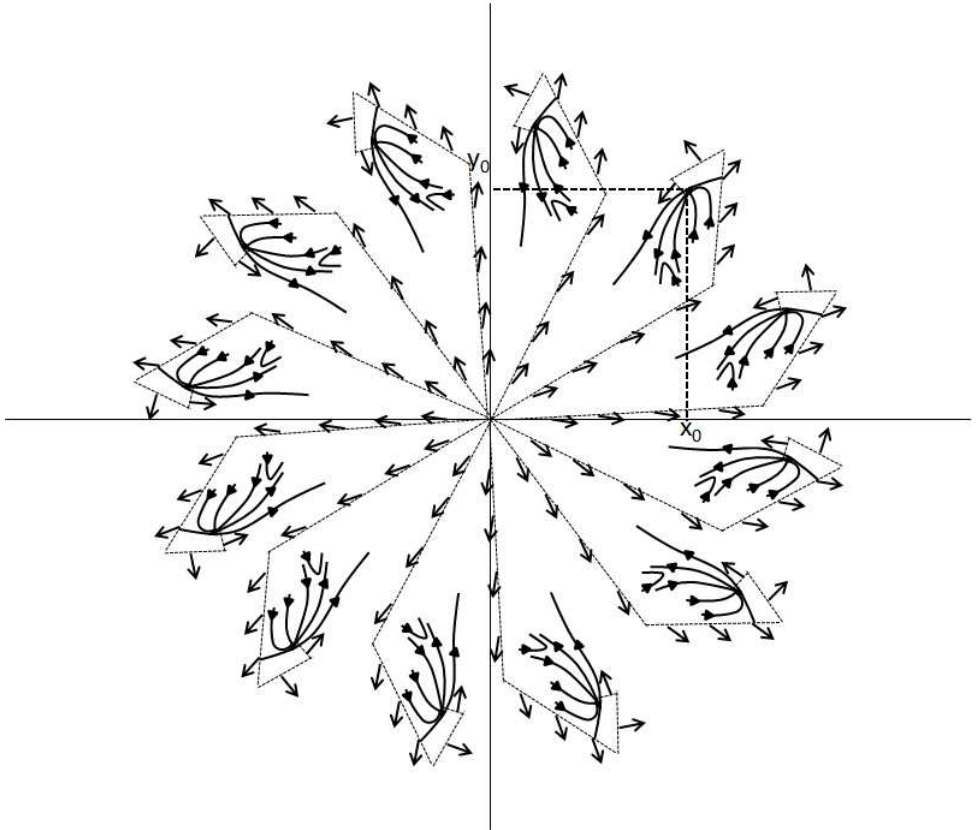


Fig. 1 – The polygonal curve with no contact with the flow of the differential equation and the separatrices of the saddle-nodes of system (10).

Again, as in many occasions occurred throughout this paper we use the \mathbb{Z}_{12} -equivariance of system (1). It gives us the possibility of studying the flow in a just $\frac{1}{12}$ of the phase space.

We build the polygonal line by working in cartesian coordinates. We define the following polygonal line.

$$(x(t), y(t)) = \begin{cases} (t, t) & \text{if } 0 \leq x \leq \sqrt{\frac{1}{2.4}}, \\ \left(t, 1.5 \left(t - \sqrt{\frac{1}{2.4}} \right) + \sqrt{\frac{1}{2.4}} \right) & \text{if } \sqrt{\frac{1}{2.4}} \leq t < 1.4, \\ (-0.1(t - 1.4) + 1.4, 1.8) & \text{if } 1.4 \leq t < 2, \\ \begin{cases} ((t - 3)(x_0 - 1) + x_0, \\ (t - 3)(y_0 - 1.8) + y_0) \end{cases} & \text{if } 2 \leq t < 3. \end{cases}$$

The flow associated to the differential equation is transversal to the polygonal line. We get that when evaluated on the corresponding segments of the polygonal line, the scalar product between the normal to each segment and the flow of this equation is negative. To show the calculation, we will exemplify it for the first segment.

Let's take the segment $L \equiv \{y = x\}$ and when substituting it into the system in cartesian coordinates (8) we get

$$\begin{cases} \dot{x} = 16p_1x^9 - 16p_2x^9 + 32x^{11} + 32s_1x^{11} - 32s_2x^{11} \\ \dot{y} = 16p_1x^9 + 16p_2x^9 + 32x^{11} + 32s_1x^{11} + 32s_2x^{11}. \end{cases}$$

A normal vector to the line L is $(-1, 1)$ and the scalar product of (\dot{x}, \dot{y}) with $(-1, 1)$ yields $f(x) = 32p_2x^9 + 64s_2x^{11}$. Solving this last equation leads to $f(x) < 0$ for

$$-\sqrt{\frac{-p_2}{2s_2}} \leq x \leq \sqrt{\frac{-p_2}{2s_2}}.$$

So we choose $0 \leq x \leq \sqrt{\frac{-p_2}{2s_2}}$, and we get that this scalar product is negative in the region where the polygonal line is defined as (t, t) . The remaining segments of the polygonal line are calculated similarly. We denote $(x_0, y_0) = (1.19, 1.63)$ the coordinates point of the saddle node represented in Figure 1.

Next we combine the fact that infinity is repeller with the Poincaré-Bendixson Theorem to conclude that the only possible ω -limit for the unstable separatrix of the saddle-node is a periodic orbit; moreover, this has to surround the twelve saddle-nodes, see again Figure 1.

For $p_1 = \Sigma_A^+$ then $A(\theta)$ keeps its sign. Using similar arguments as in part (a) of the proof enables us to conclude that the limit cycle is hyperbolic.

If we slightly change p_1 towards zero but still very close to Σ_A^+ , it follows that $B(\theta)$ is positive because $\Sigma_A^+ > \Sigma_B^+$; therefore, there are 25 critical points as shown in Lemma 7: the origin (which is a focus), twelve saddles and twelve critical points of index +1 on Θ . Next we apply one more time part (a) of the proof of this theorem, we know that the upper bound for the limit cycles surrounding the origin is one. For $p_1 = \Sigma_A^+$ the limit cycle is hyperbolic and still. Hence we get that the vector field corresponding to $B(\theta)$ of constant sign, has 25 non-zero critical points and a limit cycle which surrounds them together with the origin. \square

Acknowledgments. I would like to thank my parents for supporting me during several months in 2015, when I was unemployed. I also thank Professor Bard Ermentrout for his kind help with the XPPAUT software.

REFERENCES

- [1] M.J. Álvarez, A. Gasull and R. Prohens, *Limit cycles for cubic systems with a symmetry of order 4 and without infinite critical points*. Proc. Amer. Math. Soc. **136** (2008), 1035–1043.
- [2] A.A. Andronov, E.A. Leontovich, I.I. Gordon and A.G. Maier, *Qualitative Theory of Second-Order Dynamic Systems*. John Wiley and Sons, New-York (1973).
- [3] V. Arnold, *Chapitres Supplémentaires de la Théorie des Équations Différentielles Ordinaires*. Éditions MIR, Moscou, 1978.
- [4] L.A. Cherkas, *On the number of limit cycles of an autonomous second-order system*. Differ. Equ. **5** (1976), 666–668.
- [5] S.N Chow, C. Li and D. Wang, *Normal Forms and Bifurcation of Planar Vector Fields*. Cambridge Univ. Press, 1994.
- [6] B. Coll, A. Gasull and R. Prohens, *Differential equations defined by the sum of two quasi-homogeneous vector fields*. Canad. J. Math. **49** (1997), 212–231.
- [7] A. Gasull and J. Llibre, *Limit cycles for a class of Abel equation*. SIAM J. Math. Anal. **21** (1990), 1235–1244.
- [8] M. Golubitsky and D.G. Schaeffer, *Singularities and Groups in Bifurcation Theory I*. Appl. Math. Sci. **51** Springer-Verlag, 1985.
- [9] M. Golubitsky, I. Stewart and D.G. Schaeffer, *Singularities and Groups in Bifurcation Theory II*. Appl. Math. Sci. **69** Springer-Verlag, 1988.
- [10] N.G. Lloyd, *A note on the number of limit cycles in certain two-dimensional systems*. J. Lond. Math. Soc. **20** (1979), 277–286.
- [11] A. Zegeeling, *Equivariant unfoldings in the case of symmetry of order 4*. Bulg. Math. Publ. **19** (1993), 71–79.

Received 20 October 2015

13 Grof Miko Imre Street,
Sfântu-Gheorghe, 520003,
Romania
adrian_murza@hotmail.com