EXTENSIONS, DILATIONS AND SPECTRAL ANALYSIS OF SINGULAR STURM-LIOUVILLE OPERATORS

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A space of boundary values is constructed for minimal symmetric Sturm-Liouville operator acting in $L_\varrho^2(a,b)$ with defect index $(1,1)$ (in limit-circle case at a (b) and limit-point case at b (a)). All maximal dissipative, maximal accumulative and self-adjoint extensions of such a symmetric operator are described in terms of boundary conditions at a (b). In each case, we construct a self-adjoint dilation of the dissipative operator and its incoming and outgoing spectral representations, which allows us to determine the scattering matrix. We establish a functional model of the dissipative operator and construct its characteristic function in terms of the Weyl-Titchmarsh function on the self-adjoint operator. We also prove the completeness of the root functions of the dissipative Sturm-Liouville operators.

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1. INTRODUCTION

The theory of extensions of symmetric operators is one of the basic directions of operator theory. The first fundamental results in this theory were obtained by von Neumann [21], although the apparent origins can be found in the well-known works of Weyl (see [28]). Results obtained for the representation of linear relations turned out to be very useful to describe several classes of extensions of symmetric operators. The first result of this type is due to Rofe-Beketov [25]. Kochubei [16] and Bruk [7] introduced independently the term 'space of boundary values' and described all maximal dissipative, maximal accumulative, self-adjoint, etc. extensions of symmetric operators by means of this term (see [12] and see also the survey article [11]). However, irrespective of the general scheme, the problem of the description of the maximal dissipative (accumulative), self-adjoint and other extensions of a given symmetric operator in terms of the boundary conditions is considerably interesting, particularly in the case of singular differential operators as usual boundary conditions are, in general, meaningless at the singular ends of the considered interval.

We know [19, 22–24] that the theory of dilations with application of operator models provides an adequate approach to the spectral theory of dissipative (contractive) operators. Characteristic function is one of the central parts of this theory as it carries the complete information on the spectral properties of the dissipative operator. Thus, the dissipative operator becomes the model in the incoming spectral representation of the dilation. The problem of the completeness of the system of eigenvectors and associated vectors is solved by means of the factorization of the characteristic function. The computation of the characteristic functions of dissipative operators is preceded by the construction and investigation of a self-adjoint dilation and of the corresponding scattering problem, in which the characteristic function is realized as the scattering matrix. Efficiency of this approach for dissipative Schrödinger and Sturm-Liouville operators has been demonstrated in [1–5, 22–24].

The present paper considers the minimal symmetric Sturm-Liouville operator acting in the space $\mathcal{L}^2_\varrho(a,b)$ with defect index $(1,1)$ (in Weyl's limit-circle case at a (b) and limit-point case at b (a)). A space of boundary values is constructed and all maximal dissipative, maximal accumulative and self-adjoint extensions are described by using the boundary conditions at a (b). For each case, we define a self-adjoint dilation of the maximal dissipative operator and its incoming and outgoing spectral representations so that we can determine the scattering matrix with respect to the scheme of Lax and Phillips [17]. Incoming spectral representation is used to construct a functional model of the maximal dissipative operator and its characteristic function in terms of the Weyl-Titchmarsh function of the self-adjoint operator. Finally, we prove the completeness of the system of eigenfuntions and associated functions (or root functions) of maximal dissipative Sturm-Liouville operators by utilizing the results obtained in the theory of characteristic function.

2. MAXIMAL DISSIPATIVE, SELF-ADJOINT EXTENSIONS OF SYMMETRIC OPERATORS AND SELF-ADJOINT DILATIONS OF THE DISSIPATIVE OPERATORS

We consider Sturm-Liouville differential expression with two singular end points a and b:

(2.1)

$$M(f):=\frac{1}{\varrho(t)}[-(p(t)f(t)')'+q(t)f(t)]\ (t\in\Omega=(a,b),-\infty\le a<b\le+\infty),$$

where p, q and ϱ are real valued, Lebesgue measurable functions on Ω, and $\frac{1}{p}, q, \varrho \in \mathcal{L}^1_{loc}(\Omega)$, $\varrho > 0$ almost everywhere on Ω.

In order to pass from the differential expression to operators, we introduce the Hilbert space $\mathcal{L}^2_\varrho(\Omega)$ consisting of all functions f such that $\int_a^b \varrho(t)\,|f(t)|^2\,\mathrm{d}t < +\infty$ with the inner product $(f, g) = \int_a^b \varrho(t) f(t)\overline{g(t)}\mathrm{d}t$.

Let $\mathcal{D}_{\max}$ denote the linear set of all functions $f \in \mathcal{L}^2_\varrho(\Omega)$ such that f and pf' are locally absolute continuous functions on Ω and $M(f) \in \mathcal{L}^2_\varrho(\Omega)$. We define the *maximal operator* $\mathcal{M}_{\max}$ on $\mathcal{D}_{\max}$ as $\mathcal{M}_{\max} f = M(f)$.

For any functions $g, h \in \mathcal{D}_{\max}$, we have Green's formula

$$(\mathcal{M}_{\max} g, h) - (g, \mathcal{M}_{\max} h) = [g, h](b) - [g, h](a), \tag{2.2}$$

where

$$[g, h](t) := W_t(g, \overline{h}) := g(t)(p\overline{h}')(t) - (pg')(t)\overline{h(t)}, t \in \Omega,$$
$$[g, h](a) := \lim_{t \to a^+} [g, h](t), [g, h](b) := \lim_{t \to b^-} [g, h](t).$$

In $\mathcal{L}^2_\varrho(\Omega)$, consider the dense linear set $\mathcal{D}'$ which is composed of smooth, compactly supported functions. Denote by $\mathcal{M}'$ the restriction of the operator $\mathcal{M}_{\max}$ to $\mathcal{D}'$. It follows from (2.2) that $\mathcal{M}'$ is symmetric. Consequently, it admits closure denoted by $\mathcal{M}_{\min}$. The *minimal operator* $\mathcal{M}_{\min}$ is a symmetric operator with defect index $(0, 0)$, $(1, 1)$ or $(2, 2)$ and $\mathcal{M}^*_{\min} = \mathcal{M}_{\max}$ (see [6, 8–10, 13, 20, 26–29]). When the defect index is $(0, 0)$, $\mathcal{M}_{\min}$ becomes self-adjoint, *i.e.*, $\mathcal{M}^*_{\min} = \mathcal{M}_{\min} = \mathcal{M}_{\max}$.

We assume that $\mathcal{M}_{\min}$ has defect index $(1, 1)$. Let the minimal (maximal) operators generated by the expression M on the intervals $(a, c]$ and $[c, b)$ $(c \in \Omega)$ be denoted by $\mathcal{M}^-_{\min}$ $(\mathcal{M}^-_{\max})$ and $\mathcal{M}^+_{\min}$ $(\mathcal{M}^+_{\max})$ respectively, and let $\mathcal{D}^\mp_{\min}$ $(\mathcal{D}^\mp_{\max})$ be the domain of $\mathcal{M}^\mp_{\min}$ $(\mathcal{M}^\mp_{\max})$. We know [20, 27, 29] that the defect number $def\mathcal{M}_{\min}$ of the operator $\mathcal{M}_{\min}$ is obtained by the formula $def\mathcal{M}_{\min} = def\mathcal{M}^+_{\min} + def\mathcal{M}^-_{\min} - 2$. Then, we have $def\mathcal{M}^+_{\min} + def\mathcal{M}^-_{\min} = 3$. If we set $k_+ = def\mathcal{M}^+_{\min}$, $k_- = def\mathcal{M}^-_{\min}$, then we get $1 \le k_\pm \le 2$.

(**a**) Assume that $\mathcal{M}^-_{\min}$ and $\mathcal{M}^+_{\min}$ have defect indices $(1, 1)$ and $(2, 2)$, respectively (see [6, 8–10, 13, 20, 26–29]). Since the operator $\mathcal{M}_{\min}$ has defect index $(1, 1)$, we have $[g, h](a) = 0$ for all $g, h \in \mathcal{D}_{\max}$ and

$$(\mathcal{M}_{\max} g, h) - (g, \mathcal{M}_{\max} h) = [g, h](b), \ \forall g, h \in \mathcal{D}_{\max}. \tag{2.3}$$

The domain $\mathcal{D}_{\min}$ of $\mathcal{M}_{\min}$ consists of precisely those functions $g \in \mathcal{D}_{\max}$ satisfying the condition

$$[g, h](b) = 0, \ \forall h \in \mathcal{D}_{\max}. \tag{2.4}$$

Let $\sigma(t)$ and $\omega(t)$ be the solutions of the equation

$$M(g) = 0 \ (t \in \Omega_+ := [c, b)\,, \ c \in \Omega) \tag{2.5}$$

with the initial conditions

$$\sigma(c) = 1,\ (p\sigma')(c) = 0,\ \omega(c) = 0,\ (p\omega')(c) = 1. \tag{2.6}$$

The Wronskian of two solutions of (2.5) does not depend on t, and they are linearly independent if and only if their Wronskian is non-zero. Constancy of the Wronskian together with conditions (2.6) imply that

$$[\sigma,\omega](t) = [\sigma,\omega](c) = 1\ (c < t \le b). \tag{2.7}$$

Thus, σ and ω form a fundamental system of solutions of (2.5). Since $\mathcal{M}^{+}_{\min}$ has defect index $(2,2)$, $\sigma,\omega \in \mathcal{L}^2_{\varrho}(\Omega_+)$ and moreover $\sigma,\omega \in \mathcal{D}^{+}_{\max}$.

Lemma 2.1. *For arbitrary functions $g,h \in \mathcal{D}^{+}_{\max}$, we have the equality*

$$[g,h](t) = [g,\sigma](t)[\overline{h},\omega](t) - [g,\omega](t)[\overline{h},\sigma](t),\ \ c \le t \le b. \tag{2.8}$$

Proof. Since the functions σ and ω are real valued and $[\sigma,\omega](t) = 1$ $(c \le t \le b)$, we have

$$\begin{gathered}
[g,\sigma](t)[\overline{h},\omega](t) - [g,\omega](t)[\overline{h},\sigma](t) = \left(gp\sigma' - pg'\sigma\right)(t)(\overline{h}p\omega' - p\overline{h}'\omega)(t) \\
- \left(gp\omega' - pg'\omega\right)(t)(\overline{h}p\sigma' - p\overline{h}'\sigma)(t) = (gp\sigma'\overline{h}p\omega' - gp\sigma'p\overline{h}'\omega - pg'\sigma\overline{h}p\omega' \\
+pg'\sigma p\overline{h}'\omega - gp\omega'\overline{h}p\sigma' + gp\omega'p\overline{h}'\sigma + pg'\omega\overline{h}p\sigma' - pg'\omega p\overline{h}'\sigma)(t) \\
= (-gp\overline{h}' + pg'\overline{h})(t)\left(p\sigma'\omega - \sigma p\omega'\right)(t) = [g,h](t),
\end{gathered}$$

which proves the lemma. □

Lemma 2.2. *The domain $\mathcal{D}_{\min}$ of the operator $\mathcal{M}_{\min}$ consist of precisely those functions $g \in \mathcal{D}_{\max}$ satisfying the following boundary conditions*

$$[g,\sigma](b) = [g,\omega](b) = 0. \tag{2.9}$$

Proof. We know that the domain $\mathcal{D}_{\min}$ of $\mathcal{M}_{\min}$ coincides with the set of all functions $g \in \mathcal{D}_{\max}$ satisfying the condition (2.4). Due to Lemma 2.1, (2.4) is equivalent to

$$[g,\sigma](b)[\overline{h},\omega](b) - [g,\omega](b)[\overline{h},\sigma](b) = 0. \tag{2.10}$$

Since $[\overline{h},\omega](b)$ and $[\overline{h},\sigma](b)$ $(g \in \mathcal{D}_{\max})$ can be taken arbitrarily, equality (2.10) holds for all $g \in \mathcal{D}_{\max}$ if and only if the conditions (2.9) are fulfilled. The lemma is proved. □

The concept of the space of boundary values of the operator plays an important role in the theory of extensions. The triplet $(\mathfrak{H}, G_1, G_2)$, where $\mathfrak{H}$ is a Hilbert space, G_1 and G_2 are linear mappings from $\mathcal{D}(S^*)$ into $\mathfrak{H}$, is called (see [7, 12, p. 155, 16]) a *space of boundary values* of a closed symmetric operator S acting in a Hilbert space H with equal (finite or infinite) defect index if

(i) $(S^*f,g)_H - (f,S^*g)_H = (G_1f,G_2g)_{\mathfrak{H}} - (G_2f,G_1g)_{\mathfrak{H}}$, $\forall f,g \in \mathcal{D}(S^*)$, and

(ii) for every $y_1, y_2 \in \mathfrak{H}$ there exists a vector $f \in \mathcal{D}(S^*)$ such that $G_1f = y_1$ and $G_2f = y_2$.

We consider the following two linear maps from $\mathcal{D}_{\max}$ into $\mathbb{C}$

$$G_1g = [g,\sigma](b),\ G_2g = [g,\omega](b). \tag{2.11}$$

So, we can state the next result.

Theorem 2.3. *The triplet $(\mathbb{C}, G_1, G_2)$ defined according to (2.11) is a space of boundary values of the operator $\mathcal{M}_{\min}$.*

Proof. The first requirement of the definition of a space of boundary values is fulfilled because of (2.3) and Lemma 2.1:

$$(G_1g,G_2h)_{\mathbb{C}} - (G_2g,G_1h)_{\mathbb{C}} = [g,\sigma](b)[\overline{h},\omega](b) - [g,\omega](b)[\overline{h},\sigma](b)$$
$$= [g,h](b) = (\mathcal{M}_{\max}g,h) - (g,\mathcal{M}_{\max}h)\ (\forall g,h \in \mathcal{D}_{\max}). \qquad \square$$

Lemma 2.4. *For any complex numbers $\delta_0, \delta_1, \epsilon_0$ and ϵ_1 there is a function $u \in \mathcal{D}^+_{\max}$ satisfying the boundary conditions*

$$u(c) = \delta_0,\ (pu')(c) = \delta_1,\ [u,\sigma](b) = \epsilon_0,\ [u,\omega](b) = \epsilon_1. \tag{2.12}$$

Proof. Let f be an arbitrary function in $\mathcal{L}^2_\varrho(\Omega_+)$ satisfying

$$(f,\sigma) = \epsilon_0 + \delta_1,\ (f,\omega) = \epsilon_1 - \delta_0. \tag{2.13}$$

There exists such a function f, even among the linear combination of σ and ω. In fact, if we set $f = c_1\sigma + c_2\omega$, then conditions (2.13) give us a system of equations in the constants c_1 and c_2 whose determinant is the Gram determinant of the linearly independent functions σ and ω and is therefore non-zero.

Let $u(t)$ denote the solution of the equation $M(u) = f(t)$ $(c \le t < b)$ satisfying the initial conditions $u(c) = \delta_0$, $(pu')(c) = \delta_1$. We claim that u is the desired function. First, we observe that $u(t)$ can be expressed by

$$u(t) = \delta_0\sigma(t) + \delta_1\omega(t) + \int_c^t \{\sigma(t)\omega(\xi) - \sigma(\xi)\omega(t)\}\,\varrho(\xi)f(\xi)\mathrm{d}\xi.$$

As $\sigma, \omega \in \mathcal{L}^2_\varrho(\Omega_+)$, we have $u \in \mathcal{L}^2_\varrho(\Omega_+)$ and moreover $u \in \mathcal{D}^+_{\max}$.

Next, we apply Green's formula to u and σ and obtain $(f,\sigma) = (M(u),\sigma)$ $= [u,\sigma](b) - [u,\sigma](c) + (u, M(\sigma))$. Since $M(\sigma) = 0$, we have $(u, M(\sigma)) = 0$. Moreover, conditions $u(c) = \delta_0$, $(pu')(c) = \delta_1$ imply that $[u,\sigma](c) = u(c)(p\sigma')(c) - (pu')(c)\sigma(c) = -\delta_1$. Therefore,

$$(f,\sigma) = [u,\sigma](b) + \delta_1. \tag{2.14}$$

Then, we conclude from (2.13) and (2.14) that $[u,\sigma](b) = \epsilon_0$.

For ω, we can similarly show that

$$(f,\omega) = (M(u),\omega) = [u,\omega](b) - [u,\omega](c) + (u, M(\omega)) = [u,\omega](b) - \delta_0. \tag{2.15}$$

Then from (2.13) and (2.15) we have $[u,\omega](b) = \epsilon_1$. Lemma 2.4 is proved. □

The next lemma proves the second requirement of the definition of a space of boundary values.

Lemma 2.5. *For any complex numbers δ_0, δ_1 there is a function $u \in \mathcal{D}_{\max}$ satisfying*

$$[u,\sigma](b) = \delta_0,\ [u,\omega](b) = \delta_1. \tag{2.16}$$

Proof. The operator $\mathcal{M}^+_{\min}$ has defect index $(2,2)$. By Lemma 2.4, we can find a function $u_+ \in \mathcal{D}^+_{\max}$ satisfying the conditions

$$u_+(c) = 0,\ (pu'_+)(c) = 0,\ [u_+,\sigma](b) = \delta_0,\ [u_+,\omega](b) = \delta_1. \tag{2.17}$$

Now we let

$$u(t) = \begin{cases} 0, & a < t \le c, \\ u_+(t), & c \le t < b. \end{cases}$$

Then, $u \in \mathcal{D}_{\max}$ and $[u,\sigma](b) = \delta_0$, $[u,\omega](b) = \delta_1$. The Lemma 2.5 and Theorem 2.3 are verified. □

Recall that a linear operator T (with domain $\mathcal{D}(T)$) acting on some Hilbert space H is called *dissipative* (*accumulative*) if $\Im(Tf,f) \ge 0$ ($\Im(Tf,f) \le 0$) for all $f \in \mathcal{D}(T)$ and *maximal dissipative* (*maximal accumulative*) if it does not have a proper dissipative (accumulative) extensions (see [12], p. 149).

Using Theorem 2.3 and [7, 12, Theorem 1.6, p. 156, 16] we can state the following result.

Theorem 2.6. *Every maximal dissipative (accumulative) extensions $\mathcal{T}^+_\eta$ of $\mathcal{M}_{\min}$ is determined by the equality $\mathcal{T}^+_\eta u = \mathcal{M}_{\max} u$ of the functions u in $\mathcal{D}_{\max}$ satisfying the boundary condition*

$$[u,\sigma](b) - \eta[u,\omega](b) = 0, \tag{2.18}$$

where $\Im\eta \ge 0$ or $\eta = \infty$ ($\Im\eta \le 0$ or $\eta = \infty$). Conversely, for an arbitrary number η with $\Im\eta \ge 0$ or $\eta = \infty$ ($\Im\eta \le 0$ or $\eta = \infty$) condition (2.18) determines a maximal dissipative (accumulative) extension of $\mathcal{M}_{\min}$. The self-adjoint extension of $\mathcal{M}_{\min}$ are obtained precisely when η is a real number or infinity. Here for $\eta = \infty$ condition (2.18) should be replaced by $[u,\omega](b) = 0$.

(**b**) Now, let $\mathcal{M}^-_{\min}$ have defect index $(2,2)$ and $\mathcal{M}^+_{\min}$ have defect index $(1,1)$ (see [6, 8-10, 13, 20, 26-29]). The operator $\mathcal{M}_{\min}$ has defect index $(1,1)$. Then $[u,v](b) = 0$ for all $u, v \in \mathcal{D}_{\max}$ and

$$(\mathcal{M}_{\max}u, v) - (u, \mathcal{M}_{\max}v) = -[u,v](a),\ \forall u, v \in \mathcal{D}_{\max}. \tag{2.19}$$

LEMMA 2.7. *The domain $\mathcal{D}_{\min}$ of the operator $\mathcal{M}_{\min}$ consist of precisely those functions $u \in \mathcal{D}_{\max}$ satisfying the following boundary conditions*

$$[u,\sigma](a) = [u,\omega](a) = 0. \tag{2.20}$$

We consider the following linear maps from $\mathcal{D}_{\max}$ into $\mathbb{C}$

$$F_1 u = [u,\omega](a),\ \ F_2 u = [u,\sigma](a). \tag{2.21}$$

Then, the next two theorems can be proved in the same way as in the case (**a**).

THEOREM 2.8. *The triplet $(\mathbb{C}, F_1, F_2)$ defined according to (2.21) is a space of boundary values of the operator $\mathcal{M}_{\min}$.*

THEOREM 2.9. *Every maximal dissipative (accumulative) extension $\mathcal{T}_\gamma^-$ of $\mathcal{M}_{\min}$ is determined by the equality $\mathcal{T}_\gamma^- u = \mathcal{M}_{\max} u$ on the functions u in $\mathcal{D}_{\max}$ satisfying the boundary condition*

$$[u,\omega](a) - \gamma[u,\sigma](a) = 0, \tag{2.22}$$

where $\Im\gamma \geq 0$ or $\gamma = \infty$ ($\Im\gamma \leq 0$ or $\gamma = \infty$). Conversely, for an arbitrary number γ with $\Im\gamma \geq 0$ or $\gamma = \infty$ ($\Im\gamma \leq 0$ or $\gamma = \infty$) condition (2.22) determines a maximal dissipative (accumulative) extension of $\mathcal{M}_{\min}$. The self-adjoint extension of $\mathcal{M}_{\min}$ are obtained precisely when γ is a real number or infinity. Here for $\gamma = \infty$ condition (2.22) should be replaced by $[u,\sigma](a) = 0$.

In this section, we shall study the maximal dissipative operators $\mathcal{T}_\gamma^-$ ($\Im\gamma > 0$) and $\mathcal{T}_\eta^+$ ($\Im\eta > 0$) generated by the differential expression M and boundary conditions (2.18) and (2.22), respectively.

We add the 'incoming' and 'outgoing' channels $\mathcal{L}^2(-\infty, 0)$ and $\mathcal{L}^2(0,\infty)$ to the space $\mathcal{H} := \mathcal{L}^2_\varrho(\Omega)$ and then form the *main Hilbert space of the dilation* $\mathbb{H} := \mathcal{L}^2(-\infty,0) \oplus \mathcal{H} \oplus \mathcal{L}^2(0,\infty)$ in which we consider the operator $\mathbb{T}_\gamma^-$ generated by the expression

$$\mathbb{T}\langle v_-, y, v_+\rangle = \langle i\frac{\mathrm{d}v_-}{\mathrm{d}\xi}, M(y), i\frac{\mathrm{d}v_+}{\mathrm{d}\varsigma}\rangle \tag{2.23}$$

on the set $\mathcal{D}(\mathbb{T}_\gamma^-)$ of vectors $\langle v_-, y, v_+\rangle$ satisfying the conditions: $v_- \in \mathcal{W}_2^1(-\infty, 0)$, $v_+ \in \mathcal{W}_2^1(0,\infty)$, $y \in \mathcal{D}_{\max}$,

$$[y,\omega](a) - \gamma[y,\sigma](a) = \delta v_-(0),\ \ [y,\omega](a) - \overline{\gamma}[y,\sigma](a) = \delta v_+(0), \tag{2.24}$$

where $\delta^2 := 2\Im\gamma$, $\delta > 0$ and $\mathcal{W}_2^1$ is the Sobolev space. Then we assert the following theorem.

THEOREM 2.10. *The operator $\mathbb{T}_\gamma^-$ is self-adjoint in $\mathbb{H}$ and is a self-adjoint dilation of the maximal dissipative operator $\mathcal{T}_\gamma^-$.*

Proof. Suppose that $Y, Z \in \mathcal{D}(\mathbb{T}_\gamma^-)$, $Y = \langle v_-, y, v_+\rangle$ and $Z = \langle \psi_-, z, \psi_+\rangle$. If we use integration by parts and (2.19), we get

$$(2.25)\quad \left(\mathbb{T}_\gamma^- Y, Z\right)_{\mathbb{H}} = \int_{-\infty}^{0} iv_-'\bar{\psi}_- \mathrm{d}\xi + (\mathcal{M}_{\max} y, z)_{\mathcal{H}} + \int_0^\infty iv_+'\bar{\psi}_+ \mathrm{d}\xi$$
$$= iv_-(0)\overline{\psi_-(0)} - iv_+(0)\overline{\psi_+(0)} - [y,z](a) + \left(Y, \mathbb{T}_\gamma^- Z\right)_{\mathbb{H}}.$$

Moreover, using the boundary conditions (2.24) for the components of the vectors y, z and the relation

$$(2.26)\qquad [y,z](t) = [y,\sigma](t)\,[\overline{z},\omega](t) - [y,\omega](t)\,[\overline{z},\sigma](t)\ (a \le t \le c),$$

one can see by direct computation that $iv_-(0)\overline{\psi_-(0)} - iv_+(0)\overline{\psi_+(0)} - [y,z](a) = 0$. Thus, $\mathbb{T}_\gamma^-$ is symmetric. In order to prove that $\mathbb{T}_\gamma^-$ is self-adjoint, we need to show that $(\mathbb{T}_\gamma^-)^* \subseteq \mathbb{T}_\gamma^-$.

Take $Z = \langle \psi_-, z, \psi_+\rangle \in \mathcal{D}((\mathbb{T}_\gamma^-)^*)$. Let $(\mathbb{T}_\gamma^-)^* Z = Z^* = \langle \psi_-^*, z^*, \psi_+^*\rangle \in \mathbb{H}$ so that

$$(2.27)\qquad (\mathbb{T}_\gamma^- Y, Z)_{\mathbb{H}} = (Y, Z^*)_{\mathbb{H}},\ \forall Y \in \mathcal{D}(\mathbb{T}_\gamma^-).$$

Here we can choose the vectors with suitable components as $Y \in \mathcal{D}(\mathbb{T}_\gamma^-)$ to show that $\psi_- \in \mathcal{W}_2^1(-\infty,0)$, $\psi_+ \in \mathcal{W}_2^1(0,\infty)$, $z \in \mathcal{D}_{\max}$ and $Z^* = \mathbb{T}Z$, where the expression $\mathbb{T}$ is given by (2.23). Hence, (2.27) can be written as $(\mathbb{T}Y, Z)_{\mathbb{H}} = (Y, \mathbb{T}Z)_{\mathbb{H}}$, $\forall Y \in \mathcal{D}(\mathbb{T}_\gamma^-)$. Therefore, the sum of the integral terms in the bilinear form $(\mathbb{T}Y, Z)_{\mathbb{H}}$ must be zero:

$$(2.28)\qquad iv_-(0)\overline{\psi_-(0)} - iv_+(0)\overline{\psi_+(0)} - [y,z](a) = 0$$

for all $Y = \langle v_-, y, v_+\rangle \in \mathcal{D}(\mathbb{T}_\gamma^-)$. Further, solving the boundary conditions (2.24) for $[y,\sigma](a)$ and $[y,\omega](a)$ we find that $[y,\sigma](a) = \frac{i}{\delta}\left(v_+(0) - v_-(0)\right)$, $[y,\omega](a) = \delta v_-(0) + \frac{i\gamma}{\delta}\left(v_+(0) - v_-(0)\right)$. Then, we use (2.26) and see that (2.28) is equivalent to the equality $iv_-(0)\overline{\psi_-(0)} - iv_+(0)\overline{\psi_+(0)} = -[y,z](a) = \frac{i}{\delta}\left(v_+(0) - v_-(0)\right)[\overline{z},\omega](a) - [\delta v_-(0) + \frac{i\gamma}{\delta}\left(v_+(0) - v_-(0)\right)][\overline{z},\sigma](a)$. Since the values $v_\pm(0)$ may be any complex numbers, a comparison of the coefficients of $v_\pm(0)$ on the left and on the right of the last equality shows that the vector $Z = \langle \psi_-, z, \psi_+\rangle$ satisfies the following boundary conditions $[z,\omega](a) - \gamma[z,\sigma](a) = \delta\psi_-(0)$, $[z,\omega](a) - \overline{\gamma}[z,\sigma](a) = \delta\psi_+(0)$. As a result, we obtain the inclusion $(\mathbb{T}_\gamma^-)^* \subseteq \mathbb{T}_\gamma^-$ implying that $\mathbb{T}_\gamma^- = (\mathbb{T}_\gamma^-)^*$.

The self-adjoint operator $\mathbb{T}_\gamma^-$ generates in $\mathbb{H}$ a unitary group $\mathbb{U}^-(s) := \exp[i\mathbb{T}_\gamma^- s]$ $(s \in \mathbb{R} := (-\infty,\infty))$. Let $\mathcal{P} : \mathbb{H} \to \mathcal{H}$ and $\mathcal{P}_1 : \mathcal{H} \to \mathbb{H}$ be the mappings acting according to the formulas $\mathcal{P} : \langle v_-, x, v_+\rangle \to x$ and $\mathcal{P}_1 : x \to \langle 0, x, 0\rangle$. Let $\mathcal{Z}(s) = \mathcal{P}\mathbb{U}^-(s)\mathcal{P}_1$ $(s \ge 0)$. The family $\{\mathcal{Z}(s)\}$ $(s \ge 0)$ of operators is a strongly continuous semi-group of completely non-unitary contractions on $\mathcal{H}$. Denote by $\mathcal{A}$ the generator of this semi-group: $\mathcal{A}x =$

$\lim_{s\to+0}[(is)^{-1}(\mathcal{Z}(s)x-x)]$. The domain of $\mathcal{A}$ consists of all vectors for which the above limit exists. The operator $\mathcal{A}$ is maximal dissipative and the operator $\mathbb{T}_\gamma^-$ is called the self-adjoint dilation of $\mathcal{A}$ [19, 22-23]. We shall show that $\mathcal{A}=\mathcal{T}_\gamma^-$, and hence $\mathbb{T}_\gamma^-$ is a self-adjoint dilation of $\mathcal{T}_\gamma^-$. For this purpose, we first confirm the equality [19, 22, 23]

$$\mathcal{P}(\mathbb{T}_\gamma^- - \lambda I)^{-1}\mathcal{P}_1 x = (\mathcal{T}_\gamma^- - \lambda I)^{-1}x,\ x\in\mathcal{H},\ \Im\lambda<0. \tag{2.29}$$

If we set $(\mathbb{T}_\gamma^- - \lambda I)^{-1}\mathcal{P}_1x = Z = \langle\psi_-, z, \psi_+\rangle$, then we have $(\mathbb{T}_\gamma^- - \lambda I)Z = \mathcal{P}_1 x$, and in turn $\mathcal{M}_{\max}z-\lambda z = x$, $\psi_-(\xi)=\psi_-(0)e^{-i\lambda\xi}$ and $\psi_+(\varsigma)=\psi_+(0)e^{-i\lambda\varsigma}$. Since $Z\in\mathcal{D}(\mathbb{T}_\gamma^-)$ and hence $\psi_-\in\mathcal{L}^2(-\infty,0)$, it follows that $\psi_-(0)=0$. Thus, z satisfies the boundary condition $[z,\omega](a)-\gamma[z,\sigma](a)=0$. Therefore, $z\in\mathcal{D}(\mathcal{T}_\gamma^-)$ and in fact, $z=(\mathcal{T}_\gamma^- - \lambda I)^{-1}x$ since a point λ with $\Im\lambda<0$ cannot be an eigenvalue of a dissipative operator. We note that $\psi_+(0)$ is given by $\psi_+(0)=\delta^{-1}([z,\omega](a)-\overline{\gamma}[z,\sigma](a))$. Thus,

$$(\mathbb{T}_\gamma^- - \lambda I)^{-1}\mathcal{P}_1 x = \left\langle 0, ((\mathbb{T}_\gamma^- - \lambda I)^{-1}x, \delta^{-1}([z,\omega](a)-\overline{\gamma}[z,\sigma](a))e^{-i\lambda\varsigma}\right\rangle$$

for $x\in\mathcal{H}$ and $\Im\lambda<0$. Applying the mapping $\mathcal{P}$, we obtain (2.29).

Now, we can easily show that $\mathcal{A}=\mathcal{T}_\gamma^-$. Indeed, we have by (2.29) that

$$\begin{aligned}(\mathcal{T}_\gamma^- - \lambda I)^{-1} &= \mathcal{P}(\mathbb{T}_\gamma^- - \lambda I)^{-1}\mathcal{P}_1 = -i\mathcal{P}\int_0^\infty \mathbb{U}^-(s)e^{-i\lambda s}\mathrm{d}s\mathcal{P}_1\\ &= -i\int_0^\infty \mathcal{Z}(s)e^{-i\lambda s}\mathrm{d}s = (\mathcal{A}-\lambda I)^{-1},\ \Im\lambda<0,\end{aligned}$$

from which $\mathcal{T}_\gamma^- = \mathcal{A}$ follows directly. Hence, Theorem 2.10. is proved. □

To construct a self-adjoint dilation of the dissipative operator $\mathcal{T}_\eta^+$ ($\Im\eta>0$) in the space $\mathbb{H}$ we deal with the operator $\mathbb{T}_\eta^+$ generated by (2.23) on the set $\mathcal{D}(\mathbb{T}_\eta^+)$ of vectors $\langle v_-, y, v_+\rangle$ satisfying the conditions: $v_-\in\mathcal{W}_2^1(-\infty,0)$, $v_+\in\mathcal{W}_2^1(0,\infty)$, $y\in\mathcal{D}_{\max}$, $[y,\sigma](b)-\eta[y,\omega](b)=\beta v_-(0)$, $[y,\sigma](b)-\overline{\eta}[y,\omega](b)=\beta v_+(0)$, $\beta^2:=2\Im\eta$, $\beta>0$.

We may prove the following assertion analogously.

Theorem 2.11. *The operator $\mathbb{T}_\eta^+$ is self-adjoint in $\mathbb{H}$ and is a self-adjoint dilation of the dissipative operator $\mathcal{T}_\eta^+$.*

3. SCATTERING THEORY OF DILATIONS, FUNCTIONAL MODELS AND COMPLETENESS OF THE SYSTEM OF ROOT FUNCTIONS OF THE DISSIPATIVE OPERATORS

Let $\varphi(t,\lambda)$ and $\psi(t,\lambda)$ denote the solutions of the equation $M(y)=\lambda y$ ($t\in\Omega$) satisfying the conditions $[\varphi,\sigma](a)=-1$, $[\varphi,\omega](a)=0$, $[\psi,\sigma](a)=$

0, $[\psi,\omega](a) = 1$. The Weyl-Titchmarsh function $m_\infty^-(\lambda)$ of the self-adjoint operator $\mathcal{T}_\infty^-$ generated by the boundary condition $[y,\omega](a) = 0$ is uniquely determined due to the condition $\psi(t,\lambda) + m_\infty^-(\lambda)\varphi(t,\lambda) \in \mathcal{L}_\varrho^2(\Omega)$, $\Im\lambda \neq 0$. In this case $m_\infty^-(\lambda)$ is in general not a meromorphic function on $\mathbb{C}$, but is a holomorphic function with $\Im\lambda \neq 0$, $\Im\lambda\Im m_\infty^-(\lambda) > 0$ and $\overline{m_\infty^-(\lambda)} = m_\infty^-(\bar{\lambda})$ ($\Im\lambda \neq 0$). In this section, we assume that the function $m_\infty^-(\lambda)$ is meromorphic in $\mathbb{C}$, which implies, equivalently, that any self-adjoint extension of the operator $\mathcal{M}_{\min}$ has a purely discrete spectrum (see [6, 8, 9, 14, 15, 18, 20, 26, 27]).

An important property of the unitary group $\mathbb{U}^-(s) = \exp[i\mathbb{T}_\gamma^- s]$ ($\mathbb{U}^+(s) = \exp[i\mathbb{T}_\eta^+ s]$) ($s \in \Omega$) makes it possible to apply to it the Lax-Phillips scheme [17]. Namely, it has incoming and outgoing subspaces $\mathbb{D}^- := \langle \mathcal{L}^2(-\infty,0), 0, 0\rangle$ and $\mathbb{D}^+ := \langle 0,0,\mathcal{L}^2(0,\infty)\rangle$ possessing the following properties

(**1**) $\mathbb{U}^\pm(s)\mathbb{D}^- \subset \mathbb{D}^-$, $s \leq 0$ and $\mathbb{U}^\pm(s)\mathbb{D}^+ \subset \mathbb{D}^+$, $s \geq 0$;

(**2**) $\cap_{s\leq 0}\mathbb{U}^\pm(s)\mathbb{D}^- = \cap_{s\geq 0}\mathbb{U}^\pm(s)\mathbb{D}^+ = \{0\}$;

(**3**) $\overline{\cup_{s\geq 0}\mathbb{U}^\pm(s)\mathbb{D}^-} = \overline{\cup_{s\leq 0}\mathbb{U}^\pm(s)\mathbb{D}^+} = \mathbb{H}$;

(**4**) $\mathbb{D}^- \perp \mathbb{D}^+$.

Property (**4**) is obvious. To prove property (**1**) for $\mathbb{D}^+$ (the proof for $\mathbb{D}^-$ is similar), we set $R_\lambda^\pm := (\mathbb{T}_\gamma^\pm - \lambda I)^{-1}$, for all λ with $\Im\lambda < 0$ and for any $Y = \langle 0,0,v_+\rangle \in \mathbb{D}^+$ we have

$$R_\lambda^\pm Y = \langle 0,0,-ie^{-i\lambda\xi}\int_0^\xi e^{-i\lambda s}v_+(s)\mathrm{d}s\rangle$$

which implies that $R_\lambda^\pm Y \in \mathbb{D}^+$, therefore, if $Z \perp \mathbb{D}^+$, then

$$0 = (R_\lambda^\pm Y, Z)_{\mathbb{H}} = -i\int_0^\infty e^{-i\lambda s}(\mathbb{U}^\pm(s)Y,Z)_{\mathbb{H}}\mathrm{d}\lambda,\ \Im\lambda < 0.$$

Hence, we have $(\mathbb{U}^\pm(s)Y,Z)_{\mathbb{H}} = 0$ for all $s \geq 0$ and in turn $\mathbb{U}^\pm(s)\mathbb{D}^+ \subset \mathbb{D}^+$ for $s \geq 0$. Property (**1**) is proved.

To prove property (**2**) for $\mathbb{D}^+$ (the proof for $\mathbb{D}^-$ is similar), we use the mappings $\mathcal{P}^+ : \mathbb{H} \to \mathcal{L}^2(0,\infty)$ and $\mathcal{P}_1^+ : \mathcal{L}^2(0,\infty) \to \mathbb{D}^+$ acting according to the formulas $\mathcal{P}^+ : \langle v_-, u, v_+\rangle \to v_+$ and $\mathcal{P}_1^+ : v \to \langle 0,0,v\rangle$, respectively. The semi-group of isometries $\mathcal{V}(s) = \mathcal{P}^+\mathbb{U}^-(s)\mathcal{P}_1^+$ ($s \geq 0$) is a one-sided shift in $\mathcal{L}^2(0,\infty)$ and, in fact, the generator of the semi-group of the one-sided shift $\mathcal{Y}(s)$ in $\mathcal{L}^2(0,\infty)$ is the differential operator $i\frac{d}{d\zeta}$ with boundary condition $v(0) = 0$. On the other side, the generator $\mathcal{B}$ of the semi-group of isometries $\mathcal{V}(s)$ ($s \geq 0$) is the operator $\mathcal{B}v = \mathcal{P}^+\mathbb{T}_\gamma^-\mathcal{P}_1^+Y = \mathcal{P}^+\mathbb{T}_\gamma^-\langle 0,0,v\rangle = \mathcal{P}^+\langle 0,0,i\frac{\mathrm{d}v}{d\zeta}\rangle = i\frac{\mathrm{d}v}{d\zeta}$, where $v \in \mathcal{W}_2^1(0,\infty)$ and $v(0) = 0$. We know that a semi-group is uniquely determined by its generator, hence $\mathcal{V}(s) = \mathcal{Y}(s)$, and

$$\cap_{s\geq 0}\mathbb{U}^-(s)\mathbb{D}^+ = \langle 0,0,\cap_{s\geq 0}\mathcal{Y}(s)\mathcal{L}^2(0,\infty)\rangle = \{0\}$$

(the proof for $\mathbb{U}^+(s)$ is similar) proving the property (**2**).

The scattering matrix is defined in terms of the theory of spectral representations in the scheme of the Lax-Phillips scattering theory. Now, we proceed to construct them. During this process, we also prove property (**3**) of the incoming and outgoing subspaces.

We recall that the linear operator $\mathbf{A}$ (with domain $\mathcal{D}(\mathbf{A})$) acting in the Hilbert space $\mathbf{H}$ is called *completely non-self-adjoint* (or *pure*) if the invariant subspace $\mathbf{K} \subseteq \mathcal{D}(\mathbf{A})$ ($\mathbf{K} \neq \{0\}$) of the operator $\mathbf{A}$ whose restriction to $\mathbf{K}$ is self-adjoint, does not exist.

LEMMA 3.1. *The operator $\mathcal{T}_\gamma^-$ ($\mathcal{T}_\eta^+$) is completely non-self-adjoint (pure).*

Proof. Let $\mathcal{H}' \subset \mathcal{H}$ be a non-trivial subspace in which $\mathcal{T}_\gamma^-$ (the proof for $\mathcal{T}_\eta^+$ is similar) induces a self-adjoint operator $\mathcal{T}_\gamma'$ with domain $\mathcal{D}(\mathcal{T}_\gamma') = \mathcal{H}' \cap \mathcal{D}(\mathcal{T}_\gamma^-)$. If $Y \in \mathcal{D}(\mathcal{T}_\gamma')$, then $f \in \mathcal{D}(\mathcal{T}_\gamma'^*)$, and $[f,\omega](a) - \gamma[f,\sigma](a) = 0$, $[f,\omega](a) - \overline{\gamma}[f,\sigma](a) = 0$. For the eigenfunctions y of the operator $\mathcal{T}_\gamma^-$ lying in $\mathcal{H}'$ that are eigenfunctions of $\mathcal{T}_\gamma'$, we have $[y,\sigma](a) = 0$. From the boundary condition $[y,\omega](a) - \gamma[y,\sigma](a) = 0$, we obtain $[y,\omega](a) = 0$, and $y(t,\lambda) \equiv 0$. Since $m_\infty^-(\lambda)$ is a meromorphic function in $\mathbb{C}$, the resolvent $R_\lambda(\mathcal{T}_\gamma^-)$ of the operator $\mathcal{T}_\gamma^-$ is a compact operator, and hence the spectrum of $\mathcal{T}_\gamma'$ is purely discrete. If we use the theorem on expansion in eigenfunctions of the self-adjoint operator $\mathcal{T}_\gamma'$, we find $\mathcal{H}' = \{0\}$, *i.e.* the operator $\mathcal{T}_\gamma^-$ is pure. The lemma is proved. □

We set

$$\mathbb{H}_-^\pm = \overline{\cup_{s\geq 0}\mathbb{U}^\pm(s)\mathbb{D}^-},\ \ \mathbb{H}_+^\pm = \overline{\cup_{s\leq 0}\mathbb{U}^\pm(s)\mathbb{D}^+}.$$

LEMMA 3.2. $\mathbb{H}_-^\pm + \mathbb{H}_+^\pm = \mathbb{H}$.

Proof. Taking property (**1**) of the subspace $\mathbb{D}^\pm$ into consideration, it becomes easy to show that the subspace $\mathbb{H}'_\pm = \mathbb{H} \ominus \left(\mathbb{H}_-^\pm + \mathbb{H}_+^\pm\right)$ is invariant relative to the group $\{\mathbb{U}^\pm(s)\}$ and has the form $\mathbb{H}'_\pm = \langle 0, \mathcal{H}'_\pm, 0\rangle$, where $\mathcal{H}'_\pm$ is a subspace in $\mathcal{H}$. Therefore, if the subspace $\mathbb{H}'_\pm$ (and hence also $\mathcal{H}'_\pm$) were non-trivial, then the unitary group $\{\mathbb{U}^\pm(s)\}$, restricted to this subspace, would be a unitary part of the group $\{\mathbb{U}^\pm(s)\}$, and hence the restriction $\mathcal{T}_\gamma^{-\prime}$ ($\mathcal{T}_\eta^{+\prime}$) of $\mathcal{T}_\gamma^-$ ($\mathcal{T}_\eta^+$) to $\mathcal{H}'_-$ ($\mathcal{H}'_+$) would be a self-adjoint operator in $\mathcal{H}'_-$ ($\mathcal{H}'_+$). Purity of the operator $\mathcal{T}_\gamma^-$ ($\mathcal{T}_\eta^+$) leads to $\mathcal{H}'_\pm = \{0\}$, *i.e.* $\mathbb{H}'_\pm = \{0\}$. The lemma is proved. □

Let us use the following notation $\theta(t,\lambda) = \psi(t,\lambda) + m_\infty^-(\lambda)\varphi(t,\lambda)$,

$$\Theta_\gamma^-(\lambda) = \frac{m_\infty^-(\lambda) - \gamma}{m_\infty^-(\lambda) - \overline{\gamma}}. \tag{3.1}$$

Let $\Upsilon_\lambda^-(t,\xi,\varsigma) = \langle e^{-i\lambda\xi}, (m_\infty^-(\lambda) - \gamma)^{-1}\delta\theta(t,\lambda), \overline{\Theta}_\gamma^-(\lambda)e^{-i\lambda\varsigma}\rangle$. We note that the vectors $\Upsilon_\lambda^-(t,\xi,\varsigma)$ for real λ do not belong to the space $\mathbb{H}$. However, $\Upsilon_\lambda^-(t,\xi)$ satisfy the equation $\mathbb{T}\Upsilon=\lambda\Upsilon$ ($\lambda\in\mathbb{R}$), and the corresponding boundary conditions for the operator $\mathbb{T}_\gamma^-$.

With the help of the vector $\Upsilon_\lambda^-(t,\xi)$, we define the transformation $\mathcal{F}_-: Y\to\tilde{Y}_-(\lambda)$ by $(\mathcal{F}_-Y)(\lambda):=\tilde{Y}_-(\lambda):=\frac{1}{\sqrt{2\pi}}(Y,\Upsilon_\lambda^-)_{\mathbb{H}}$ on the vector $Y=\langle v_-,y,v_+\rangle$ in which v_-,v_+ and y are smooth, compactly supported functions.

Lemma 3.3. *The transformation $\mathcal{F}_-$ maps $\mathbb{H}_-^-$ isometrically onto $\mathcal{L}^2(\mathbb{R})$. For all vectors $Y,Z\in\mathbb{H}_-^-$ the Parseval equality and the inversion formula hold:*

$$\begin{aligned}(Y,Z)_{\mathbb{H}} &= (\tilde{Y}_-,\tilde{Z}_-)_{\mathcal{L}^2}=\int_{-\infty}^{\infty}\tilde{Y}_-(\lambda)\overline{\tilde{Z}_-(\lambda)}\mathrm{d}\lambda,\\ Y &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{Y}_-(\lambda)\Upsilon_\lambda^-\mathrm{d}\lambda,\end{aligned}$$

where $\tilde{Y}_-(\lambda):=(\mathcal{F}_-Y)(\lambda)$ and $\tilde{Z}_-(\lambda):=(\mathcal{F}_-Z)(\lambda)$.

Proof. For $Y,Z\in\mathbb{D}^-$, $Y=\langle v_-,0,0\rangle$, $Z=\langle\psi_-,0,0\rangle$ we have

$$\tilde{Y}_-(\lambda):=\frac{1}{\sqrt{2\pi}}(Y,\Upsilon_\lambda^-)_{\mathbb{H}}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}v_-(\xi)e^{i\lambda\xi}\mathrm{d}\xi\in\mathcal{H}_-^2$$

and, in view of the usual Parseval equality for Fourier integrals,

$$(Y,Z)_{\mathbb{H}}=\int_{-\infty}^{0}v_-(\xi)\overline{\psi_-(\xi)}\mathrm{d}\xi=\int_{-\infty}^{\infty}\tilde{Y}_-(\lambda)\overline{\tilde{Z}_-(\lambda)}\mathrm{d}\lambda=(\mathcal{F}_-Y,\mathcal{F}_-Z)_{\mathcal{L}^2}.$$

Here and below, let $\mathcal{H}_\pm^2$ denote the Hardy classes in $\mathcal{L}^2(\mathbb{R})$ consisting of the functions analytically extendable to the upper and lower half-planes, respectively.

In order to extend the Parseval equality to the whole $\mathbb{H}_-^-$, we consider in $\mathbb{H}_-^-$ the dense set $\mathbb{H}_-'$ of vectors obtained from the smooth, compactly supported functions in $\mathbb{D}^-$ as follows: $Y\in\mathbb{H}_-'$ if $Y=\mathbb{U}^-(s)Y_0$, $Y_0=\langle v_-,0,0\rangle$, $v_-\in C_0^\infty(-\infty,0)$, where $s=s_Y$ is a non-negative number depending on Y. In this case, if $Y,Z\in\mathbb{H}_-'$, then we have $\mathbb{U}_{-s}^-Y,\mathbb{U}_{-s}^-Z\in\mathbb{D}^-$ for $s>s_Y$ and $s>s_Z$ and, moreover, the first components of these vectors lie in the space $C_0^\infty(-\infty,0)$. Therefore, being the operators $\mathbb{U}^-(s)$ ($s\in\mathbb{R}$) unitary, the equality $\mathcal{F}_-\mathbb{U}^-(s)Y=(\mathbb{U}^-(s)Y,\mathbb{U}_\lambda^-)_{\mathbb{H}}=e^{i\lambda s}(Y,\mathbb{U}_\lambda^-)_{\mathbb{H}}=e^{i\lambda s}\mathcal{F}_-Y$ leads us to

$$\begin{aligned}(3.2)\quad (Y,Z)_{\mathbb{H}}&=(\mathbb{U}_s^-Y,\mathbb{U}_{-s}^-Z)_{\mathbb{H}}=(\mathcal{F}_-\mathbb{U}_{-s}^-Y,\mathcal{F}_-\mathbb{U}_{-s}^-Z)_{\mathcal{L}^2}\\ &=(e^{-i\lambda s}\mathcal{F}_-Y,e^{-i\lambda s}\mathcal{F}_-Z)_{\mathcal{L}^2}=(\mathcal{F}_-Y,\mathcal{F}_-Z)_{\mathcal{L}^2}.\end{aligned}$$

By closure in (3.2), we have the Parseval equality for the whole space $\mathbb{H}_-^-$. The inversion formula follows from the Parseval equality if all integrals in it are understood as limits in the mean of integrals over finite intervals. Finally,

$$\mathcal{F}_{-}\mathbb{H}_{-}^{-} = \overline{\cup_{s\geq 0}\mathcal{F}_{-}\mathbb{U}^{-}(s)\mathbb{D}^{-}} = \overline{\cup_{s\geq 0}e^{-i\lambda s}\mathcal{H}_{-}^{2}} = \mathcal{L}^{2}(\mathbb{R}),$$

i.e. $\mathcal{F}_{-}$ maps $\mathbb{H}_{-}^{-}$ onto the whole $\mathcal{L}^2(\mathbb{R})$, which completes the proof. □

Let $\Upsilon_\lambda^+(t,\xi,\varsigma) = \langle \Theta_\gamma^-(\lambda)e^{-i\lambda\xi}, (m_\infty^-(\lambda)-\overline{\gamma})^{-1}\delta\theta(t,\lambda), e^{-i\lambda\varsigma}\rangle$. It is clear that the vectors $\Upsilon_\lambda^+(t,\xi,\varsigma)$ do not lie in the space $\mathbb{H}$ for real values of λ. Nevertheless, $\Upsilon_\lambda^+(t,\xi)$ satisfies $\mathbb{T}\Upsilon = \lambda\Upsilon$ ($\lambda\in\mathbb{R}$), and the corresponding boundary conditions for the operator $\mathbb{T}_\gamma^-$. Using the vector $\Upsilon_\lambda^+(t,\xi,\varsigma)$, we can define the transformation $\mathcal{F}_+ : Y \to \tilde{Y}_+(\lambda)$ as

$$(\mathcal{F}_+Y)(\lambda) := \tilde{Y}_+(\lambda) := \frac{1}{\sqrt{2\pi}}(Y,\Upsilon_\lambda^+)_{\mathbb{H}},$$

on vectors $Y = \langle v_-, y, v_+\rangle$ in which v_-, v_+ and y are smooth, compactly supported functions. The proof of the next result is analogous to that of Lemma 3.3.

LEMMA 3.4. *The transformation $\mathcal{F}_+$ maps $\mathbb{H}_+^-$ isometrically onto $\mathcal{L}^2(\mathbb{R})$ and for all vectors $Y, Z \in \mathbb{H}_+^-$, the Parseval equality and the inversion formula hold:*

$$\begin{aligned}(Y,Z)_{\mathbb{H}} &= (\tilde{Y}_+,\tilde{Z}_+)_{\mathcal{L}^2} = \int_{-\infty}^{\infty}\tilde{Y}_+(\lambda)\overline{\tilde{Z}_+(\lambda)}\mathrm{d}\lambda,\\ Y &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{Y}_+(\lambda)\Upsilon_\lambda^+\mathrm{d}\lambda,\end{aligned}$$

where $\tilde{Y}_+(\lambda) := (\mathcal{F}_+Y)(\lambda)$ and $\tilde{Z}_+(\lambda) := (\mathcal{F}_+Z)(\lambda)$.

According to (3.1), the function $\Theta_\gamma^-(\lambda)$ satisfies the equality $\left|\Theta_\gamma^-(\lambda)\right| = 1$ for $\lambda\in\mathbb{R}$. Then, the explicit formula given for the vectors Υ_λ^+ and Υ_λ^- implies that

$$\Upsilon_\lambda^- = \overline{\Theta_\gamma^-}(\lambda)\Upsilon_\lambda^+ \ (\lambda\in\mathbb{R}). \tag{3.3}$$

Hence, the equality $\mathbb{H}_-^- = \mathbb{H}_+^-$ holds true due to Lemmas 3.3 and 3.4. Combining this result by Lemma 3.2, we obtain $\mathbb{H} = \mathbb{H}_-^- = \mathbb{H}_+^-$, and, thus property (**3**) for $\mathbb{U}^-(s)$ above has been established for the incoming and outgoing subspaces. Thus, the transformation $\mathcal{F}_-$ maps $\mathbb{H}$ isometrically onto $\mathcal{L}^2(\mathbb{R})$ with the subspace $\mathbb{D}^-$ mapped onto $\mathcal{H}_-^2$ and the operators $\mathbb{U}^-(s)$ passing into the operators of multiplication by $e^{i\lambda s}$. In other words, $\mathcal{F}_-$ is the incoming spectral representation for the group $\{\mathbb{U}^-(s)\}$. Similarly, $\mathcal{F}_+$ is the outgoing spectral representation for $\{\mathbb{U}^-(s)\}$. It follows from (3.3) that the passage from the $\mathcal{F}_+$-representation of a vector $f\in\mathbb{H}$ to its $\mathcal{F}_-$-representation is realized by multiplication by the function $\Theta_\gamma^-(\lambda) : \tilde{Y}_-(\lambda) = \Theta_\gamma^-(\lambda)\tilde{Y}_+(\lambda)$. It is given in [17] that the scattering matrix (function) of the group $\{\mathbb{U}^-(s)\}$ with respect to the subspaces $\mathbb{D}^-$ and $\mathbb{D}^+$ is the coefficient by which the $\mathcal{F}_-$-representation of a vector $Y \in \mathbb{H}$ must be multiplied to obtain the corresponding $\mathcal{F}_+$-representation: $\tilde{Y}_+(\lambda) = \overline{\Theta}_\gamma^-(\lambda)\tilde{Y}_-(\lambda)$. Hence, we have proved the following theorem.

Theorem 3.5. *The function $\overline{\Theta}_\gamma^-(\lambda)$ is the scattering function (matrix) of the unitary group $\{\mathbb{U}^-(s)\}$ (of the self-adjoint operator $\mathbb{T}_\gamma^-$).*

Let $\Theta(\lambda)$ be an arbitrary inner function on the upper half-plane [19]. If we set $\mathcal{N} = \mathcal{H}_+^2 \ominus \Theta\mathcal{H}_+^2$, then $\mathcal{N} \neq \{0\}$ becomes a subspace of the Hilbert space $\mathcal{H}_+^2$. Let us now consider the semi-group of the operators $\mathcal{Y}(s)$ $(s \geq 0)$ acting in $\mathcal{N}$ according to the formula $\mathcal{Y}(s)\varphi = \mathcal{P}\left[e^{i\lambda s}\varphi\right]$, $\varphi := \varphi(\lambda) \in \mathcal{N}$, where $\mathcal{P}$ is the orthogonal projection from $\mathcal{H}_+^2$ onto $\mathcal{N}$. The generator of the semi-group $\{\mathcal{Y}(s)\}$ is described by $\mathcal{S} : \mathcal{S}\varphi = \lim_{s\to+0}[(is)^{-1}(\mathcal{Y}(s)\varphi - \varphi)]$, being a dissipative operator acting in $\mathcal{N}$ with domain $\mathcal{D}(\mathcal{S})$ composed of all functions $\varphi \in \mathcal{N}$ for which the above limit exists. The operator $\mathcal{S}$ is called a *model dissipative operator* (This model dissipative operator associated with the names of Lax and Phillips [17] is a special case of a more general model dissipative operator constructed by Sz.-Nagy and Foiaş [19]). The basic assertion is that $\Theta(\lambda)$ is the characteristic function of the operator $\mathcal{S}$.

Let $\mathbb{K} = \langle 0, \mathcal{H}, 0\rangle$ so that $\mathbb{H} = \mathbb{D}^- \oplus \mathbb{K} \oplus \mathbb{D}^+$. We can see from the explicit form of the unitary transformation $\mathcal{F}_-$ that under the mapping $\mathcal{F}_-$,

$$\begin{aligned}(3.4)\quad &\mathbb{H} \to \mathcal{L}^2(\mathbb{R}),\ Y \to \tilde{Y}_-(\lambda) = (\mathcal{F}_- Y)(\lambda),\ \mathbb{D}^- \to \mathcal{H}_-^2,\ \mathbb{D}^+ \to \Theta_\gamma^-\mathcal{H}_+^2,\\ &\mathbb{K} \to \mathcal{H}_+^2 \ominus \Theta_\gamma^-\mathcal{H}_+^2,\ \mathbb{U}^-(s)Y \to (\mathcal{F}_-\mathbb{U}^-(s)\mathcal{F}_-^{-1}\tilde{Y}_-)(\lambda) = e^{i\lambda s}\tilde{Y}_-(\lambda).\end{aligned}$$

The formulas (3.4) show that the operator $\mathcal{T}_\gamma^-$ is unitary equivalent to the model dissipative operator with characteristic function $\Theta_\gamma^-(\lambda)$. Since the characteristic functions of unitary equivalent dissipative operators coincide with [19, 22–24], we have proved the next result.

Theorem 3.6. *The characteristic function of the maximal dissipative operator $\mathcal{T}_\gamma^-$ coincides with the function $\Theta_\gamma^-(\lambda)$ defined in (3.1).*

We denote by $\phi(t,\lambda)$ and $\chi(t,\lambda)$ the solutions of the equation $M(y) = \lambda y$ $(t \in \Omega)$ with conditions $[\phi,\sigma](b) = 0$, $[\phi,\omega](b) = -1$, $[\chi,\sigma](b) = 1$, $[\chi,\omega](b) = 0$. Through the condition $\chi(t,\lambda) + m_\infty^+(\lambda)\phi(t,\lambda) \in \mathcal{L}_\varrho^2(\Omega)$, $\Im\lambda \neq 0$, Weyl-Titchmarsh function $m_\infty^+(\lambda)$ of the self-adjoint operator $\mathcal{T}_\infty^+$ generated by the boundary condition $[y,\omega](b) = 0$ is uniquely determined. Then, $m_\infty^+(\lambda)$ is not a meromorphic function on $\mathbb{C}$ in general, but it is a holomorphic function with $\Im\lambda \neq 0$, $\Im\lambda\Im m_\infty^+(\lambda) > 0$ and $\overline{m_\infty^+(\lambda)} = m_\infty^+(\bar{\lambda})$ $(\Im\lambda \neq 0)$. In this section, we assume that the function $m_\infty^+(\lambda)$ is meromorphic in $\mathbb{C}$. Then this condition becomes equivalent to the fact that any self-adjoint extension of the operator $\mathcal{M}_{\min}$ has a purely discrete spectrum (see [6, 8, 9, 14, 15, 18, 20, 26, 27]).

We adopt the following notation $\vartheta(t,\lambda)=\chi(t,\lambda)+m_\infty^+(\lambda)\phi(t,\lambda)$,

$$\Theta_\eta^+(\lambda)=\frac{m_\infty^+(\lambda)-\eta}{m_\infty^+(\lambda)-\overline{\eta}}. \tag{3.5}$$

Set $\Psi_\lambda^-(t,\xi,\varsigma)=\langle e^{-i\lambda\xi},(m_\infty^+(\lambda)+\eta)^{-1}\beta\vartheta(t,\lambda),\overline{\Theta}_\eta^+(\lambda)e^{-i\lambda\varsigma}\rangle$. Note that the vector $\Psi_\lambda^-(t,\xi,\varsigma)$ does not belong to $\mathbb{H}$ for $\lambda\in\mathbb{R}$. However, we can apply the expression $\mathbb{T}$ to them. It is easy to see that Ψ_λ^- satisfy the equation $\mathbb{T}\Psi=\lambda\Psi$ $(\lambda\in\mathbb{R})$ and boundary conditions (2.25).

Using Ψ_λ^-, we define the transformation $\Phi_-:Y\to\tilde{Y}_-(\lambda)$ by $(\Phi_-Y)(\lambda):=\tilde{Y}_-(\lambda):=\frac{1}{\sqrt{2\pi}}(Y,\Psi_\lambda^-)_{\mathbb{H}}$ on the vector $Y=\langle v_-,y,v_+\rangle$ in which v_-,v_+ and y are smooth, compactly supported functions. The proof of the next result is similar to that of Lemma 3.3.

Lemma 3.7. *The transformation Φ_- maps $\mathbb{H}_-^+$ isometrically onto $\mathcal{L}^2(\mathbb{R})$. For all vectors $Y,Z\in\mathbb{H}_-^+$ the Parseval equality and the inversion formula hold:*

$$\begin{aligned}(Y,Z)_{\mathbb{H}} &= (\tilde{Y}_-,\tilde{Z}_-)_{\mathcal{L}^2}=\int_{-\infty}^{\infty}\tilde{Y}_-(\lambda)\overline{\tilde{Z}_-(\lambda)}\mathrm{d}\lambda,\\ Y &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{Y}_-(\lambda)\Psi_\lambda^-\mathrm{d}\lambda,\end{aligned}$$

where $\tilde{Y}_-(\lambda):=(\Phi_-Y)(\lambda)$ and $\tilde{Z}_-(\lambda):=(\Phi_-Z)(\lambda)$.

We set $\Psi_\lambda^+(t,\xi,\varsigma)=\langle\Theta_\eta^+(\lambda)e^{-i\lambda\xi},(m_\infty^+(\lambda)-\overline{\eta})^{-1}\beta\vartheta(t,\lambda),e^{-i\lambda\varsigma}\rangle$. Although the vector $\Psi_\lambda^+(t,\xi,\varsigma)$ does not belong to $\mathbb{H}$ for $\lambda\in\mathbb{R}$, the expression $\mathbb{T}$ can be applied to it and we can see that Ψ_λ^+ satisfies the equation $\mathbb{T}\Psi=\lambda\Psi$ $(\lambda\in\mathbb{R})$ and boundary conditions (2.25).

With the help of vector $\Psi_\lambda^+(t,\xi,\varsigma)$, we define the transformation $\Phi_+:Y\to\tilde{Y}_+(\lambda)$ on vectors $Y=\langle v_-,y,v_+\rangle$, in which v_-,v_+ and y are smooth, compactly supported functions by setting

$$(\Phi_+Y)(\lambda):=\tilde{Y}_+(\lambda):=\frac{1}{\sqrt{2\pi}}(Y,\Psi_\lambda^+)_{\mathbb{H}}.$$

Lemma 3.8. *The transformation Φ_+ maps $\mathbb{H}_+^+$ isometrically onto $\mathcal{L}^2(\mathbb{R})$, and for all vectors $Y,Z\in\mathbb{H}_+^+$, the Parseval equality and the inversion formula hold:*

$$\begin{aligned}(Y,Z)_{\mathbb{H}} &= (\tilde{Y}_+,\tilde{Z}_+)_{\mathcal{L}^2}=\int_{-\infty}^{\infty}\tilde{Y}_-(\lambda)\overline{\tilde{Z}_-(\lambda)}\mathrm{d}\lambda,\\ Y &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{Y}_+(\lambda)\Psi_\lambda^+\mathrm{d}\lambda,\end{aligned}$$

where $\tilde{Y}_+(\lambda):=(\Phi_+Y)(\lambda)$ and $\tilde{Z}_+(\lambda):=(\Phi_+Z)(\lambda)$.

According to (3.5), the function $\Theta_\eta^+(\lambda)$ satisfies $\left|\Theta_\eta^+(\lambda)\right| = 1$ for $\lambda \in \mathbb{R}$. Therefore, it is seen from the explicit formula for the vectors Ψ_λ^+ and Ψ_λ^- that

$$\Psi_\lambda^- = \overline{\Theta}_\eta^+(\lambda)\Psi_\lambda^+, \ \lambda \in \mathbb{R}. \tag{3.6}$$

Therefore, Lemma 3.7 and 3.8 imply that $\mathbb{H}_-^+ = \mathbb{H}_+^+$. Together with Lemma 3.2, this shows that $\mathbb{H} = \mathbb{H}_-^+ = \mathbb{H}_+^+$. From the formula (3.6) it follows that passage from the Φ_--representation of an vector $Y \in \mathbb{H}$ to its Φ_+-representation is accomplished as follows: $\tilde{Y}_+(\lambda) = \overline{\Theta}_\eta^+(\lambda)\tilde{Y}_-(\lambda)$. According to [17], we have now reached the following conclusion.

Theorem 3.9. *The function $\overline{\Theta}_\eta^+(\lambda)$ is the scattering function (matrix) of the unitary group $\{\mathbb{U}^+(s)\}$ (of the self-adjoint operator $\mathbb{T}_\eta^+$).*

From the explicit form of unitary transformation Φ_-, we have

$$\begin{aligned} &\mathbb{H} \to \mathcal{L}^2(\mathbb{R}),\ Y \to \tilde{Y}_-(\lambda) = (\Phi_- Y)(\lambda),\ \mathbb{D}^- \to \mathcal{H}_-^2,\ \mathbb{D}^+ \to \Theta_\eta^+\mathcal{H}_+^2, \\ &\mathbb{K} \to \mathcal{H}_+^2 \ominus \Theta_\eta^+\mathcal{H}_+^2,\ \Psi^+(s)Y \to (\Phi_-\Psi^+(s)\Phi_-^{-1}\tilde{Y}_-)(\lambda) = e^{i\lambda s}\tilde{Y}_-(\lambda). \end{aligned} \tag{3.7}$$

Formulas (3.7) show that the operator $\mathcal{T}_\eta^+$ is unitarily equivalent to the model dissipative operator with characteristic function $\Theta_\eta^+(\lambda)$. We have thus proved the following assertion.

Theorem 3.10. *The characteristic function of the maximal dissipative operator $\mathcal{T}_\eta^+$ coincides with the function $\Theta_\eta^+(\lambda)$ defined by (3.5).*

Let $\mathfrak{B}$ denote the linear operator in the Hilbert space $\mathfrak{H}$ with the domain $\mathcal{D}(\mathfrak{B})$. The complex number λ_0 is called an *eigenvalue* of the operator $\mathfrak{B}$ if there exists a non-zero vector $y_0 \in \mathcal{D}(\mathfrak{B})$ such that $\mathfrak{B}y_0 = \lambda_0 y_0$. Such vector y_0 is called the *eigenvector* of the operator $\mathfrak{B}$ corresponding to the eigenvalue λ_0. The vectors $y_1, y_2, ..., y_k$ are called the *associated vectors* of the eigenvector y_0 if they belong to $\mathcal{D}(\mathfrak{B})$ and $\mathfrak{B}y_j = \lambda_0 y_j + y_{j-1},\ j = 1, 2, ..., k$. The vector $y \in \mathcal{D}(\mathfrak{B})$, $y \neq 0$ is called a *root vector* of the operator $\mathfrak{B}$ corresponding to the eigenvalue λ_0, if all powers of $\mathfrak{B}$ are defined on this vector and $(\mathfrak{B} - \lambda_0 I)^n y = 0$ for some integer n. The set of all root vectors of $\mathfrak{B}$ corresponding to the same eigenvalue λ_0 with the vector $y = 0$ forms a linear set $\mathfrak{N}_{\lambda_0}$ and is called the root lineal. The dimension of the lineal $\mathfrak{N}_{\lambda_0}$ is called the *algebraic multiplicity* of the eigenvalue λ_0. The root lineal $\mathfrak{N}_{\lambda_0}$ coincides with the linear span of all eigenvectors and associated vectors of $\mathfrak{B}$ corresponding to the eigenvalue λ_0. Consequently, the completeness of the system of all eigenvectors and associated vectors of $\mathfrak{B}$ is equivalent to the completeness of the system of all root vectors of this operator.

We know that the characteristic function of a dissipative operator $\mathcal{T}_\gamma^-$ ($\mathcal{T}_\eta^+$) carries full information about the spectral properties of this operator [19, 22–24]. For example, the absence of a singular factor $s(\lambda)$ of the characteristic function $\Theta_\gamma^-(\lambda)$ in the factorization $\Theta_\gamma^-(\lambda) = s(\lambda)\mathcal{B}(\lambda)$ (where $\mathcal{B}(\lambda)$ is a Blaschke product) guarantees the completeness of the system of eigenfunctions and associated functions (or root functions) of the dissipative Sturm-Liouville operator $\mathcal{T}_\gamma^-$.

THEOREM 3.11. *Let $m_\infty^-(\lambda)$ be a meromorphic function in $\mathbb{C}$. Then for all values of γ with $\Im\gamma > 0$, except possibly for a single value $\gamma = \gamma^0$, the characteristic function $\Theta_\gamma^-(\lambda)$ of the maximal dissipative operator $\mathcal{T}_\gamma^-$ is a Blaschke product and the spectrum of $\mathcal{T}_\gamma^-$ is purely discrete and belongs to the open upper half plane. The operator $\mathcal{T}_\gamma^-$ $(\gamma \neq \gamma^0)$ has an infinite number of isolated eigenvalues with finite multiplicity and limit points at infinity, and the system of all eigenfunctions and associated functions (or all root functions) of this operator is complete in the space $\mathcal{L}_\varrho^2(\Omega)$.*

Proof. It is clear from (3.1) and (3.5) that $\Theta_\gamma^-(\lambda)$ is an inner function in the upper half-plane and, moreover, it is meromorphic in the whole λ-plane. Therefore, it can be factored as follows $\Theta_\gamma^-(\lambda) = e^{i\lambda d}\mathcal{B}_\gamma(\lambda)$, $d := d(\gamma) > 0$, where $\mathcal{B}_\gamma(\lambda)$ is a Blaschke product. Further, the last equality results in

$$\left|\Theta_\gamma^-(\lambda)\right| \leq e^{-d(\gamma)\Im\lambda},\ \Im\lambda \geq 0. \tag{3.8}$$

If we express $m_\infty^-(\lambda)$ in terms of $\Theta_\gamma^-(\lambda)$, we find from (3.1) and (3.5) that

$$m_\infty^-(\lambda) = \frac{\overline{\gamma}\Theta_\gamma^-(\lambda) - \gamma}{\Theta_\gamma^-(\lambda) - 1}. \tag{3.9}$$

If $d(\gamma) > 0$ for a given value γ $(\Im\gamma > 0)$, then (3.8) shows that $\lim_{s\to+\infty}\Theta_\gamma^-(is) = 0$. Moreover, (3.9) gives us that $\lim_{s\to+\infty} m_\infty^-(is) = \gamma$. Since $m_\infty^-(\lambda)$ is independent of γ, $d(\gamma)$ can be non-zero at not more than a single point $\gamma = \gamma^0$ (and, further, $\gamma^0 = \lim_{s\to+\infty} m_\gamma^\pm(is)$). The theorem is proved. □

The proof of the next result is similar to that of Theorem 3.11.

THEOREM 3.12. *Let $m_\infty^+(\lambda)$ be a meromorphic function in $\mathbb{C}$. Then for all values of η with $\Im\eta > 0$, except possibly for a single value $\eta = \eta^0$, the characteristic function $\Theta_\eta^+(\lambda)$ of the maximal dissipative operator $\mathcal{T}_\eta^+$ is a Blaschke product and the spectrum of $\mathcal{T}_\eta^+$ is purely discrete and belongs to the open upper half plane. The operator $\mathcal{T}_\eta^+$ $(\eta \neq \eta^0)$ has an infinite number of isolated eigenvalues with finite multiplicity and limit points at infinity, and the system of all eigenfunctions and associated functions (or all root functions) of this operator is complete in the space $\mathcal{L}_\varrho^2(\Omega)$.*

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