

EXTENSION OF SQUARE STABLE EXCHANGE RINGS

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An ideal I of a ring R is square stable if $aR + bR = R$ with $a \in I, b \in R$ implies that $a^2 + by \in R$ is invertible for a $y \in R$. We prove that an exchange ideal I of a ring R is square stable if and only if for any $a \in I$, $a^2 \in J(R)$ implies that $a \in J(R)$ if and only if every regular element in I is strongly regular. This extends main characterization of exchange rings of square stable range one. Further, we prove that a regular ideal I of a ring R is square stable if and only if eRe is strongly regular for all idempotents $e \in I$ if and only if $aR + bR = R$ with $a \in 1 + I, b \in R$ implies that $a^2 + by \in U(R)$ for a $y \in R$. As a consequence, we prove that every square stable regular ideal is strongly separative.

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1. INTRODUCTION

Let R be a not necessarily unitary ring. Then there is a canonical unitization $\overline{R} = R \oplus \mathbb{Z}$, with the multiplication $(a, m)(b, n) = (ab + mb + na, mn)$ for $a, b \in R$ and $m, n \in \mathbb{Z}$. Evidently, \overline{R} contains R as an ideal. In [6], Khurana, Lam and Wang defined an element of a ring R to have square stable range one if $aR + bR = R$ (for any $b \in R$) implies that $a^2 + bx$ is a unit for some $x \in R$. A ring R is said to have square stable range one if its every element has the stable range one. The motivation of this article is to explore the square stable range one for rings without units. We shall therefore seek a definition which is intrinsic for the non-unital case.

Let I be an ideal of a ring R . We say that I is square stable if every element in I has stable range one in R . From this, we see that every ideal of a ring having square stable range one is square stable. Also a ring R has square stable range one if and only if it is square stable as an ideal of itself. Let I be an ideal of a commutative ring R . As in square stable range one, we see that I is square stable if and only if whenever $aR + bR = R$ with $a \in I, b \in R$, there exists $Y \in M_2(R)$ such that $aI_2 + bY \in GL_2(R)$. As many known results on

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stable range one cannot be extended to square stable ideals, we focus on those special only for such ideals.

An ideal I of a ring R is an exchange ideal if for any $a \in I$ there exists an idempotent $e \in I$ and $x, y \in I$ such that $e = ax = a + y - ay$. An ideal I of R is an exchange ideal if and only if for any $a \in I$ there exists an idempotent $e \in R$ such that $e \in aR$ and $1 - e \in (1 - a)R$ [1]. A ring R is an exchange ring if it is exchange as an ideal of itself. Recall that a ring R has stable range one if $aR + bR = R$ with $a, b \in R$ implies that $a + by \in R$ is a unit for a $y \in R$. Camillo and Yu proved that an exchange ring has stable range one if and only if every regular element in R is unit-regular [3, Lemma 1.3.1]. Here, an element $a \in R$ is (unit) regular if there exists a (unit) $x \in R$ such that $a = axa$. An element $a \in R$ is called strongly regular if $a \in a^2R \cap Ra^2$. Obviously, $\{\text{strongly regular elements}\} \subsetneq \{\text{unit-regular elements}\} \subsetneq \{\text{regular elements}\}$ in a ring R . In [6, Theorem 5.8], it is proved that an exchange ring R has square range one if and only if every regular element in R is strongly regular. The aim of the present paper is to extend this result to exchange ideals of R . We shall prove, in Section 3, that an exchange ideal I of a ring R is square stable if and only if for any $a \in I$, $a^2 \in J(R)$ implies that $a \in J(R)$ if and only if every regular element in I is strongly regular. This extends the main characterization of exchange rings of square stable range one.

An ideal I of a ring R is regular if every element in I is regular. Clearly, every regular ideals of a ring is an exchange ideals. Recall that I has stable range one if $aR + bR = R$ with $a \in 1 + I, b \in R$ implies that $a + by \in R$ is invertible. In Section 4, we observe that square stable regular ideals possess a similar characterization. We shall prove that a regular ideal I of a ring R is square stable if and only if eRe is strongly regular for all idempotents $e \in I$ if and only if $aR + bR = R$ with $a \in 1 + I, b \in R$ implies that $a^2 + by \in U(R)$ for a $y \in R$. As a consequence, we prove that every square stable regular is strongly separative.

Throughout, all rings are associative with an identity, and all ideals of a ring are two-sided ideals. $J(R)$ and $U(R)$ will denote the Jacobson radical and the set of all units of a ring R , respectively. We use $M_2(R)$ to stand for the ring of all 2×2 matrices over a ring R , and $GL_2(R) = U(M_2(R))$, *i.e.*, the 2-dimensional general linear group of R .

2. EXCHANGE IDEALS

Let I be an ideal of a ring R . If $I \subseteq J(R)$, we easily check that I is a square stable exchange ideals. Thus, the Jacobson radical and prime radical of any ring are both square stable exchange ideals [1]. Furthermore, every nil

ideal is a square stable exchange ideal. One easily checks that an ideal I of a ring R is square stable if and only if for any $a \in I, r \in R$ there exists $x \in R$ such that $a^2 + (1 - ar)x \in U(R)$. From this, we claim that the ring \mathbb{Z} of all integers has no any non-zero square stable ideal. Let $I = n\mathbb{Z}$ ($n \neq 0, 1$) be a non-zero square stable ideal of \mathbb{Z} . Choose $a = n, r = 2n$. Then we have some $x \in \mathbb{Z}$ such that $a^2 + (1 - ar)x \in U(\mathbb{Z})$. This implies that $(2x - 1)n^2 = x \pm 1$. This gives a contradiction as there is no any integer $n(\neq 0, 1)$ satisfying these equations. Here are some pertinent examples.

Example 2.1. Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$, and let $I = \begin{pmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{pmatrix}$. Then I is a square stable exchange ideal of R , while R has not square range one.

Proof. Clearly, $I \subseteq J(R)$. Thus, I is a square stable exchange ideal. As \mathbb{Z} has no square stable range one [6, Proposition 2.1], we easily check that R has no square stable range one. \square

Example 2.2. Let \mathbb{Z} be the ring of all integers. Then $\mathbb{Z}[[x]]$ has no square stable one, while $x\mathbb{Z}[[x]]$ is a square stable exchange ideal of $\mathbb{Z}[[x]]$.

Proof. Assume that $\mathbb{Z}[[x]]$ has stable range one. Since $3 \times 4 - 11 = 1$, we can find a $y(x) \in \mathbb{Z}[[x]]$ such that $9 - 11y(x) \in U(\mathbb{Z}[[x]])$. We note that $u(x) \in U(\mathbb{Z}[[x]])$ if and only if $u(0) = \pm 1$. Hence, $9 - 11y(0) = \pm 1$ where $y(0) \in \mathbb{Z}$. But there is no any integer y as the root of the equations $9 - 11y = \pm 1$. This gives a contradiction. Therefore $\mathbb{Z}[[x]]$ has no square stable range one. As $x\mathbb{Z}[[x]] \in J(\mathbb{Z}[[x]])$, $x\mathbb{Z}[[x]]$ is a stable range exchange ideal of $\mathbb{Z}[[x]]$, and we are done. \square

LEMMA 2.3. *Let I be a square ideal of a ring R , and let $a \in I$. If $a^2 \in J(R)$, then $a \in J(R)$.*

Proof. Suppose that $a^2 \in J(R)$. For any $x \in R$, we have $aR + (1 - ax)R = R$ with $a \in I$. Since I is square stable, we can find some $y \in R$ such that $u := a^2 + (1 - ax)y \in U(R)$. It follows by $a^2 \in J(R)$ that $1 - ax \in R$ is right invertible. This implies that $a \in J(R)$, as desired. \square

LEMMA 2.4. *Let R be a ring and $ax + b = 1$ with $a, x, b \in R$. If $a \in R$ is unit-regular, then $a + by \in U(R)$ for a $y \in R$.*

Proof. This is obvious as in the proof of [3, Lemma 1.3.1]. \square

THEOREM 2.5. *Let I be an exchange ideal of a ring R . Then the following are equivalent:*

- (1) *I is square stable.*
- (2) *For any $a \in I$, $a^2 \in J(R) \implies a \in J(R)$.*

Proof. (1) \Rightarrow (2) This is obvious by Lemma 2.3.

(2) \Rightarrow (1) Let $f \in R$ be an idempotent. Then $(fa(1-f))^2 = 0 \in J(R)$. By hypothesis, $fa(1-f) \in J(R)$. Hence, $\overline{fa} = \overline{faf}$. Likewise, $\overline{af} = \overline{faf}$. Thus, $\overline{fa} = \overline{af}$ in $R/J(R)$.

Now let $ax + b = 1$ for $a \in I, x, b \in R$ then $b = 1 - ax \in 1 + I$. As I is an exchange ideal, there exists an idempotent $e \in R$ such that $e = bs$ and $1 - e = (1 - b)t$ for some $s, t \in R$. This implies that $axt + e = 1$, and so $(1 - e)axt(1 - e)a = (1 - e)a$. Thus, $\overline{(1 - e)a(1 - e)xt + e} = \overline{1}$ in $R/J(R)$, and then

$$\overline{((1 - e)a)^2(xt)^2 + e} = \overline{1}.$$

This shows that

$$\overline{((1 - e)a)^2} = \overline{((1 - e)a)^2(xt)^2((1 - e)a)^2}.$$

Set $c = (xt)^2((1 - e)a)^2(xt)^2$. Then

$$\overline{((1 - e)a)^2} = \overline{((1 - e)a)^2 c ((1 - e)a)^2} \text{ and } \overline{c} = \overline{c((1 - e)a)^2 c}.$$

Accordingly, $\overline{((1 - e)a)^2} = \overline{((1 - e)a)^2 cd}$, where $d = 1 - ((1 - e)a)^2 c + ((1 - e)a)^2$. One easily checks that $\overline{(d)}^{-1} = 1 - ((1 - e)a)^2 c + c \in R/J(R)$. Hence, $\overline{((1 - e)a)^2} \in R/J(R)$ is unit-regular. By virtue of Lemma 2.4, there exists a $y \in R$ such that $\overline{(1 - e)a^2 + ey} \in U(R/J(R))$. As every unit lifts modulo $J(R)$, we have a $u \in U(R)$ such that $(1 - e)a^2 + ey = u + r$ for a $r \in J(R)$. Therefore $a^2 + bs(y - a^2) = a^2 + e(y - a^2) \in U(R)$, as desired. \square

As an immediate consequence of Theorem 2.5, we see that an exchange ring R has square stable range one if and only if $R/J(R)$ is reduced if and only if $R/J(R)$ is abelian [6, Theorem 4.4].

COROLLARY 2.6. *Let I be an exchange ideal of a ring R . Then the following are equivalent:*

- (1) I is square stable.
- (2) For any $a \in I$ and idempotent $e \in R$, $ae - ea \in J(R)$.

Proof. (1) \Rightarrow (2) For any $a \in I$ and idempotent $e \in R$, we see that $(ea(1 - e))^2 = 0 \in J(R)$. It follows by Theorem 2.5 that $ea - eae \in J(R)$. Likewise, $eae - ae \in J(R)$. Hence, $ae - ea = (ae - eae) + (eae - ea) \in J(R)$.

(2) \Rightarrow (1) Given $ax + b = 1$ with $a \in I, x, b \in R$, there exists an idempotent $e \in R$ such that $e = bs$ and $1 - e = (1 - b)t$ for some $s, t \in R$. Hence, $(1 - e)axt + e = 1$. By hypothesis, we have some $r \in J(R)$ such that $(1 - e)a = (1 - e)a(1 - e) + r$. Hence, $(1 - e)a^2(xt)^2 + e = 1 - rxt \in U(R)$. Hence, $(1 - e)a^2(xt)^2(1 - rxt)^{-1} + e(1 - rxt)^{-1} = 1$. This shows that $e(1 - rxt)^{-1} = e$,

and so $(1-e)a^2(xt)^2(1-rxt)^{-1} + e = 1$. It follows that $(1-e)a^2(xt)^2(1-rxt)^{-1}(1-e)a^2 = (1-e)a^2$. As $(1-e)a^2(xt)^2(1-rxt)^{-1} \in R$ is an idempotent, we see that

$$\overline{(1-e)a^2} \in (\overline{(1-e)a^2})^2(R/J(R)) \cap (R/J(R))\overline{(1-e)a^2}^2,$$

and so $\overline{(1-e)a^2} \in R/J(R)$ is strongly regular. Hence, it is unit-regular [7]. As I is an exchange ideal, we have an idempotent $f \in R$ and a unit $u \in R$ such that $\overline{(1-e)a^2} = \overline{fu}$. Thus, we can find some $s \in J(R)$ such that $fu(xt)^2(1-rxt)^{-1} + e = 1 - s(xt)^2(1-rxt)^{-1}$. It follows that

$$fu(xt)^2(1-rxt)^{-1}(1-s(xt)^2(1-rxt)^{-1})^{-1} + e(1-s(xt)^2(1-rxt)^{-1})^{-1} = 1.$$

As $fu \in R$ is unit-regular, it follows by Lemma 2.4 that $fu + e(1-s(xt)^2(1-rxt)^{-1})^{-1}z \in U(R)$. Therefore $(1-e)a^2 + e(1-s(xt)^2(1-rxt)^{-1})^{-1}z \in U(R)$. Consequently,

$$a^2 + bs((1-s(xt)^2(1-rxt)^{-1})^{-1}z - a^2) \in U(R),$$

hence the result. \square

The following result will play an important role in the proof of the main result in this section.

THEOREM 2.7. *Let I be an exchange ideal of a ring R . Then the following are equivalent:*

- (1) *I is square stable.*
- (2) *For any regular $a \in I$, $\bar{a} \in R/J(R)$ is strongly regular.*

Proof. (1) \Rightarrow (2) Let $a \in R$ be regular. Write $a = axa$ for some $x \in R$. Since $ax + (1-ax) = 1$, we can find a $y \in R$ such that $a^2 + (1-ax)y = u \in U(R)$. Hence, $a^2 = axa^2 = ax(a^2 + (1-ax)y) = axu$. Thus, $ax = a^2u$, and so $a = (ax)a = a^2ua$. Therefore $a \in a^2R$, and so $\bar{a} \in (\bar{a})^2(R/J(R))$. Write $a = a^2x$ for some $x \in R$. Hence, $a^2(a-xa^2) = 0$. This shows that $(a-xa^2)^3 = a(a-xa^2)(a-xa^2) = a^2(a-xa^2) = 0$. Hence, $(a-xa^2)^4 = 0 \in J(R)$. By using Theorem 2.5, $(a-xa^2)^2 \in J(R)$, and so $a-xa^2 \in J(R)$. This shows that

$$\bar{a} \in \overline{a^2}(R/J(R)) \cap (R/J(R))\overline{a^2}.$$

That is, $\bar{a} \in R/J(R)$ is strongly regular.

(2) \Rightarrow (1) Given $bc + d = 1$ with $b \in I, c, d \in R$, we see that $d \in 1 + I$, and so we can find an idempotent $e \in R$ such that $e = ds$ and $1-e = (1-d)t$ for some $s, t \in R$. Hence, $bct + e = 1$. Let $p = b(ct)b$. Then $p(ct)p = b(ct)b(ct)b(ct)b = b(ct)b = \bar{p}$. That is, $p \in I$ is regular. By hypothesis, we have some $s \in R$ such that $\bar{p} = \overline{p^2s}$ in $R/J(R)$. Hence,

$$\overline{p^2(sct) + e} = \overline{p(ct) + e} = \overline{b(ct)b(ct) + e} = \overline{(1-e) + e} = \bar{1}.$$

Since $p^2 = (bctb)p = (1 - e)(bp) = bp - ebp \in b^2R + eR$, we see that

$$\overline{b^2}(R/J(R)) + \overline{e}(R/J(R)) = R/J(R).$$

Write $\overline{b^2y + ez} = \overline{1}$ for some $y, z \in R$. Thus, we can find a $v \in R$ such that $b^2yv + ezv = 1$ in R . Since $ezv \in 1 + I$, we have an idempotent f such that $f = ezvk$ and $1 - f = (1 - ezv)l$ for some $k, l \in R$. It follows that $b^2yvl + f = 1$, and so $(1 - f)b^2yvl + f = 1$. We infer that $(1 - f)b^2 \in I$ is regular.

By hypothesis, $\overline{(1 - f)b^2} \in R/J(R)$ is strongly regular. Thus, $\overline{(1 - f)b^2} \in R/J(R)$ is unit-regular [7]. Since $\overline{(1 - f)b^2yl + f} = \overline{1}$, by virtue of Lemma 2.4, there exists some $t \in R$ such that $\overline{(1 - f)b^2 + ft} \in U(R/J(R))$. That is, $\overline{b^2 + f(t - b^2)} \in U(R/J(R))$. As every unit lifts modulo $J(R)$, we see that $b^2 + f(t - b^2) \in U(R)$, and so $b^2 + ezk(t - b^2) \in U(R)$. Therefore $b^2 + dszk(t - b^2) \in U(R)$, as required. \square

In view of Theorem 2.7, we show that every exchange ideal of an abelian ring is square stable. Furthermore, we can enhance Khurana-Lam-Wang's theorem [6, Theorem 5.8] as follows:

COROLLARY 2.8. *Let R be an exchange ring. Then the following are equivalent:*

- (1) R has square stable range one.
- (2) For any regular $a \in R$, $\overline{a} \in R/J(R)$ is strongly regular.

We have at our disposal all the information necessary to prove the following result.

THEOREM 2.9. *Let I be an exchange ideal of a ring R . Then the following are equivalent:*

- (1) I is square stable.
- (2) For any regular $a \in I$, $a \in a^2R$ and aR is Dedekind-finite.
- (3) Every regular element in I is strongly regular.

Proof. (1) \Rightarrow (2) Suppose I is square stable. For any regular $a \in R$, $a = aca$ for a $c \in R$. As $aR + (1 - ac)R = R$, we have a $y \in R$ such that $u := a^2 + (1 - ac)y \in U(R)$. This shows that $acu = ac(a^2 + (1 - ac)y) = aca^2 = a^2$. Hence, $ac = a^2u^{-1}$, and so $a = a^2u^{-1}a \in a^2R$.

Set $e = ac$. Then $aR = eR$, and so it will suffice to show that eRe is Dedekind-finite. Given $xy = e$ in eRe , then $\overline{x} \in R/J(R)$ is regular. In light of Theorem 3.5, $\overline{x} \in R/J(R)$ is strongly regular. Write $\overline{x} = \overline{tx^2}$ for some $t \in R$. We may assume that $t \in eRe$. As $\overline{xy} = \overline{e}$, we get $\overline{tx^2y} = \overline{tx}$. Hence, $\overline{tx} = \overline{e}$. This shows that $e - tx \in J(R)$, and so $e - tx \in J(eRe)$. It follows that $tx = e - (e - tx) \in U(eRe)$. Thus, $x \in eRe$ is left invertible. Clearly, $x \in eRe$

is right invertible. This implies that $x \in U(eRe)$, and then $yx = e$. Therefore eRe is Dedekind-finite.

(2) \Rightarrow (3) Let $a \in I$ be regular. By hypothesis, $aR = a^2R$ and aR Dedekind-finite. Construct a map $\varphi : Ra \rightarrow Ra^2, ra \mapsto ra^2$. Then φ is an R -epimorphism. If $ra^2 = 0$, then $ra = 0$, and so φ is an R -monomorphism. This implies that $Ra \cong Ra^2$. As $a \in R$ is regular, so is $a^2 \in R$. This, Ra^2 is a direct summand of R . Write $Ra^2 \oplus D = R$. Then

$$Ra = Ra \bigcap (Ra^2 \oplus D) = Ra^2 \oplus Ra \bigcap D.$$

In view of [6, Lemma 5.1], Ra is Dedekind-finite. Hence, $D = 0$. Therefore, $Ra = Ra^2$, and then $a \in a^2R \bigcap Ra^2$, as required.

(3) \Rightarrow (1) For any regular $a \in R$, $a \in R$ is strongly regular. Hence, $\bar{a} \in R/J(R)$ is strongly regular. In light of Theorem 2.7, we complete the proof. \square

As an immediate consequence, we derive that an exchange ring R has square stable range one if and only if every regular element in R is strongly regular [6, Theorem 5.8].

3. REGULAR IDEALS

In this section, we explore more explicit characterization of square stable regular ideals. The following example provides a large class of such regular ideals.

Example 3.1. If $n = \prod_{i=1}^m p_i^{k_i}$ is the prime power decomposition of the positive integer n , and p_j is an odd prime and $k_j = 1$ for at least one $j \in \{1, \dots, m\}$, then $\mathbb{Z}_n[i] = \{a + bi \mid a, b \in \mathbb{Z}_n, i^2 = -1\}$ has a nonzero square stable regular ideal. This is obvious by [8, Corollary 3.12].

Our starting point is the following technical lemma.

LEMMA 3.2 ([3], Lemma 13.1.19). *Let I be a regular ideal of a ring R and $x_1, x_2, \dots, x_m \in I$. Then there exists an idempotent $e \in I$ such that $x_i \in eRe$ for all $i = 1, 2, \dots, m$.*

Recall that an ideal I of a ring R has stable range one provided that $aR + bR = R$ with $a \in 1 + I, b \in R \implies a + by \in U(R)$. It is known that a regular ideal I has stable range one if and only if eRe is unit-regular for all idempotents $e \in I$. Surprisingly, square stable ideals possess a similar characterization.

THEOREM 3.3. *Let I be a regular ideal of a ring R . Then the following are equivalent:*

- (1) I is square stable.
- (2) eRe is strongly regular for all idempotents $e \in I$.

Proof. (1) \Rightarrow (2) Let $e \in I$ be an idempotent. Then eRe is regular. If $x^2 = 0$ in eRe , then $x \in J(R)$ by Theorem 3.3. As $x \in I$ is regular, we have a $y \in I$ such that $x = xyx$, and then $x(1 - yx) = 0$. This implies that $x = 0$. Hence, eRe is reduced. As eRe is regular, eRe is strongly regular.

(2) \Rightarrow (1) Suppose that $ax + b = 1$ with $a \in I, x, b \in R$. Then $b \in 1 + I$, and so $a, 1 - b \in I$. By view of Lemma 3.2, there exists an idempotent $e \in I$ such that $a, 1 - b \in eRe$. Write $a = ea'e$ and $1 - b = eb'e$. Then $ea'ex + b = 1$, and so $(ea'e)(exe) + ebe = e$. Since eRe is strongly regular, by virtue of [6, Theorem 5.2], eRe has square stable range one. Hence, there exists a $y \in R$ such that $(ea'e)^2 + ebeye \in U(eRe)$. Thus, we have a $u \in R$ such that

$$((ea'e)^2 + ebeye)(eue) = (eue)((ea'e)^2 + ebeye) = e.$$

This shows that

$$\begin{aligned} ((ea'e)^2 + ebeye + 1 - e)(eue + 1 - e) &= (eue + 1 - e)((ea'e)^2 + ebeye + 1 - e) \\ &= e + (1 - e) \\ &= 1. \end{aligned}$$

Clearly, $b(1 - e) = 1 - e$ and $be = e - ea'exe = ebe$, and then

$$\begin{aligned} (a^2 + b(eye + 1 - e))(eue + 1 - e) &= (eue + 1 - e)(a^2 + b(eye + 1 - e)) \\ &= 1. \end{aligned}$$

Therefore $a^2 + b(eye + 1 - e) \in U(R)$, and the result follows. \square

Example 3.4. Let I be regular ideal of a ring R . Then I is square stable if and only if I is reduced.

Proof. Suppose that I is square stable. If $x^2 = 0$ with $x \in I$, then there exists some idempotent $e \in I$ such that $x \in eRe$, by Lemma 3.2. In view of Theorem 2.5, eRe is strongly regular, hence, it is reduced. This implies that $x = 0$, as desired.

Conversely, assume that I is reduced. Then eRe is reduced for all idempotent $e \in I$. Hence, eRe is strongly regular. Therefore I is square stable, in terms of Theorem 3.3. \square

We now characterize strongly regular rings in terms of square stable ideals.

COROLLARY 3.5. *Let R be a regular ring. Then R is strongly regular if and only if*

- (1) I is square stable;
- (2) R/I is strongly regular;
- (3) Every units of R/I lifts to a unit of R .

Proof. Suppose that R is strongly regular. Then for any idempotent $e \in I$, eRe is strongly regular. In view of Theorem 3.3, I is square stable. (2) is obvious. Clearly, R is unit-regular. If $\overline{xy} = \overline{1}$. Then $x = xux$ for a $u \in U(R)$. Hence, $\overline{x} = \overline{u}^{-1}$. (3) holds.

Conversely, assume that (1) – (3) hold. Given $ax + b = 1$ with $a, x, b \in R$, then $\overline{ax + b} = \overline{1}$ in R/I . By (2), R/I has square stable range one, and then so does R/I . Thus, there exists a $y \in R$ such that $\overline{a^2 + by} \in U(R/I)$. By (3), we have a $u \in U(R)$ such that $\overline{a^2 + by} = \overline{u}$. Hence, $(a^2 + by)u - 1 \in I$. This shows that

$$((a^2 + by)u)(u^{-1}x) + b(1 - yx) = 1.$$

Since I is square stable, eRe is strongly regular for all idempotent $e \in I$. Hence, I has stable range one. Thus, we can find a $z \in R$ such that $(a^2 + by)u + b(1 - yx)z \in U(R)$; that is, $a^2 + b((1 - yx)z + yu) \in U(R)$. Therefore R has square range one. In light of [6, Theorem 5.4], R is strongly regular. \square

We now come to the main result of this section.

THEOREM 3.6. *Let I be a regular ideal of a ring R . Then the following are equivalent:*

- (1) I is square stable;
- (2) $aR + bR = R$ with $a \in 1 + I, b \in R \implies a^2 + by \in U(R)$ for a $y \in R$.

Proof. (1) \implies (2) Given $aR + bR = R$ with $a \in 1 + I, b \in R$, then we have $x, y \in R$ such that $ax + by = 1 - a$. As $a - 1 \in I$, there exists an idempotent $e \in I$ such that $1 - a = (1 - a)e$. Hence, $e - ae = axe + bye$. Clearly, $a(1 - e) = 1 - e$, and so one easily checks that

$$\begin{aligned} (eae)(e + exe) + ebye &= eae(1 + x)e + ebye \\ &= ea(e + (1 - e))(1 + x)e + ebye \\ &= e(a(1 + x) + by)e \\ &= e. \end{aligned}$$

Since I is square stable, by virtue of Theorem 3.3, eRe is strongly regular. Thus, we have a $z \in eRe$ such that $v := (eae)^2 + ebye z \in U(eRe)$. Let $w = (1 - e)a^2e + (1 - e)byz$. Obviously, $(eae)^2 = eae = e(ae + a(1 - e))ae = ea^2e$, and that $a^2(1 - e) = a(1 - e) = 1 - e$, and so $v = e(a^2 + byz)e$. Hence, $v + w = (a^2 + byz)e$ and $1 - e = (a^2 + byz)(1 - e)$. This shows that $a^2 + byz = v + w + 1 - e$. Clearly, $vw = v^{-1}w = w^2 = w(1 - e) = 0$ and $(1 - e)w = w$.

Hence, we check that

$$(v + w + 1 - e)^{-1} = v^{-1} - wv^{-1} + 1 - e,$$

and then

$$(a^2 + byz)^{-1} = v^{-1} - wv^{-1} + 1 - e.$$

Therefore, $a^2 + byz \in U(R)$, as required.

(2) \Rightarrow (1) Let $e \in I$ be an idempotent. Given $ax + b = e$ with $a, x, b \in eRe$, then $(a + 1 - e)(x + 1 - e) + b = 1$ with $a + 1 - e \in 1 + I$. By hypothesis, we can find a $y \in R$ such that $u := (a + 1 - e)^2 + by \in U(R)$. This shows that

$$u^{-1}((a + 1 - e)^2 + by) = ((a + 1 - e)^2 + by)u^{-1} = 1.$$

As $(a + 1 - e)^2 = a^2 + 1 - e$, we see that $(1 - e)u^{-1} = 1 - e$, and so $u^{-1}e = eu^{-1}e$. Therefore we have

$$(eu^{-1}e)(a^2 + b(eye)) = (a^2 + b(eye))(eu^{-1}e) = e.$$

Accordingly, $a^2 + b(eye) \in U(eRe)$. That is, eRe is square stable. In light of Theorem 3.3, I is square stable, hence the result. \square

COROLLARY 3.7. *Let I be a regular ideal of a ring R . Then the following are equivalent:*

- (1) I is square stable;
- (2) Every element in I is strongly regular.
- (3) Every element in $1 + I$ is strongly regular.

Proof. (1) \Leftrightarrow (2) As every regular ideal is an exchange ideal, this is obvious by Theorem 2.9.

(1) \Rightarrow (3) Let $a \in 1 + I$. Then $a - a^2 \in I$. Since I is regular, we see that $a - a^2 \in I$ is regular. Clearly, I is an exchange ideal of R . In view of Theorem 2.7, $a - a^2 \in R$ is strongly regular. So $a - a^2 = (a - a^2)^2x = y(a - a^2)^2$ for some $x, y \in R$, and then $a = a^2(1 + (1 - a)^2x) = (y(1 - a)^2 + 1)a^2$. Therefore $a \in R$ is strongly regular.

(3) \Rightarrow (1) Suppose that $ax + b$ with $a \in 1 + I, x, b \in R$. Then $a - a^2 \in I$ is regular. Thus, we can find some $x \in R$ such that $a - a^2 = (a - a^2)z(a - a^2)$. Hence, $a = a(a + (1 - a)z(1 - a))a$, i.e., $a \in R$ is regular. By hypothesis, $a \in R$ is strongly regular. In view of [6, Theorem 5.2], there exists a $y \in R$ such that $a^2 + by \in U(R)$. This completes the proof, by Theorem 3.6. \square

An ideal I of a ring R is strongly separative provided that for all finitely generated right R -modules A and B with $A = AI, B = BI, A \oplus A \cong A \oplus B \Rightarrow A \cong B$ (cf. [2, 5, 9]). As an immediate consequence we have

COROLLARY 3.8. *Every square stable regular ideal of a ring is strongly separative.*

Proof. Let I be a square regular ideal of a ring R . Let $a \in 1 + I$. Then $a \in R$ is strongly regular. Hence, $a \in 1 + I$ is unit-regular. In light of [4, Theorem 4], I is strongly separative, as asserted. \square

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