Dedicated to Professor Jamshid Moori on the occasion of his 70th birthday

NOTE ON CHARACTER AMENABILITY IN BANACH ALGEBRAS

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We give a survey of results and problems concerning the notion of character amenability in Banach algebras. We also provide different characterizations of this notion of amenability and the relationship that exists between this notion and some important properties of the algebras. Results and problems are surveyed over general Banach algebras and Banach algebras in different classes.

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1. INTRODUCTION

This article is a survey of the character amenability results around the general Banach algebras and Banach algebras in different classes which are known to the author. The notion of amenability of groups was first defined for discrete locally compact groups by Von Neumann [49]. This was later generalized to arbitrary locally compact groups by M. Day [14]. B.E. Johnson in [26] while trying to study the relationship between the group amenability of a locally compact group G and the group algebra $L^1(G)$ came up with the notion of amenability for Banach algebras. He proved that a locally compact group G is amenable as a group if and only if the group algebra $L^1(G)$ is amenable as a Banach algebra. This result of Johnson laid the groundwork for amenability in Banach algebras. Ever since this groundwork, the notion of amenability has become a major issue in Banach algebra theory and in harmonic analysis. For details on amenability in Banach algebras see [36].

After the pioneering work of Johnson in [26], several modifications of the original notion of amenability in Banach algebras are introduced. One of the most important modifications was introduces by A.T. Lau [31] where he introduced the notion of left amenability for a class of F-algebras. This was latter generalized by E. Kaniuth in joint papers with A.T. Lau and J. Pym [29, 30] where they introduced the notion of φ-amenability of Banach algebras. Recently, M.S. Monfared [44], gave an extension of these notions, where he introduced the notion of character amenability. The notion of character amenability as defined in [44] is stronger than left amenability of Lau and also modifies the original definition by Johnson in the sense that it requires continuous derivations from A into dual Banach A-bimodules to be inner, but only those modules are concerned where either of the left or right module action is defined by characters on A. As such character amenability is weaker than the classical amenability introduced by Johnson in [26].

Several authors have studied the notion of character amenability for different classes of Banach algebras, most notably are Alaghmandan, Nasr-Isfahami and Nemati [1], Dashti, Nasr-Isfahami and Renani [12], Essmaili and Filali [18], Hu, Monfared and Traynor [25], Kaniuth, Lau and Pym [29], Mewomo and Okelo [39], Mewomo, Maepa and Uwala [42], Mewomo and Meapa [40, 41], Mewomo and Ogunsola [38], see also [43] and Monfared [44].

The purpose of this note is to give an overview of what has been done so far on character amenability in general Banach algebras and Banach algebras in different classes and raise some problems of interest.

2. PRELIMINARIES

First, we recall some standard notions; for further details, see [8] and [36].

A locally compact group G is amenable if it possesses a translation invariant mean. That is, if there exists a linear functional $\mu : L^{\infty}(G) \to \mathbb{C}$, satisfying

$$\mu(1) = \|\mu\| = 1 \quad and \quad \mu(\delta_x * f) = \mu(f) \quad (x \in G, f \in L^{\infty}(G)).$$

Let A be an algebra. Let X be an A-bimodule. A *derivation* from A to X is a linear map $D : A \to X$ such that

$$D(ab) = Da \cdot b + a \cdot Db \quad (a, b \in A).$$

For example, $\delta_x : a \to a \cdot x - x \cdot a$ is a derivation; derivations of this form are the *inner derivations.*

Let A be a Banach algebra, and let X be an A-bimodule. Then X is a Banach A-bimodule if X is a Banach space and if there is a constant $k > 0$ such that

$$\|a \cdot x\| \le k \|a\| \|x\|, \quad \|x \cdot a\| \le k \|a\| \|x\| \quad (a \in A, x \in X).$$

For example, A itself is Banach A-bimodule, and X', the dual space of a Banach A-bimodule X, is a Banach A-bimodule with respect to the module operations defined by

$$\langle x,\, a\cdot\lambda\rangle = \langle x\cdot a,\, \lambda\rangle, \quad \langle x,\, \lambda\cdot a\rangle = \langle a\cdot x,\, \lambda\rangle \quad (x\in X)$$

for $a\in A$ and $\lambda\in X'$; we say that X' is the *dual module* of X.

Let A be a Banach algebra, and let X be a Banach A-bimodule. Then $\mathcal{Z}^1(A,X)$ is the space of all continuous derivations from A into X, $\mathcal{N}^1(A,X)$ is the space of all inner derivations from A into X, and the first cohomology group of A with coefficients in X is the quotient space

$$\mathcal{H}^1(A,X)=\mathcal{Z}^1(A,X)/\mathcal{N}^1(A,X)\,.$$

The Banach algebra A is *amenable* if $\mathcal{H}^1(A,X')=\{0\}$ for each Banach A-bimodule X. For example, the group algebra $L^1(G)$ of a locally compact group G is amenable if and only if G is amenable [26]. Also, a C^*-algebra is amenable if and only if it is nuclear [7,22].

Let A be a Banach algebra. Then the projective tensor product $A\hat{\otimes}A$ is a Banach A-bimodule where the multiplication is specified by

$$a\cdot(b\otimes c)=ab\otimes c \;\; and \;\; (b\otimes c)\cdot a=b\otimes ca \quad (a,b,c\in A.)$$

Definition 2.1. Let A be a Banach algebra.

1. A bounded approximate diagonal for A is a bounded net (m_α) in $A\hat{\otimes}A$ such that

$$a\cdot m_\alpha - m_\alpha\cdot a\to 0 \;\; and \;\; a\pi_A(m_\alpha)\to a \quad (a\in A).$$

2. A virtual diagonal for A is an element $M\in(A\hat{\otimes}A)''$ such that

$$a\cdot M = M\cdot a \;\; and \;\; \pi_A''(M)a=a \quad (a\in A).$$

Johnson [27] gave the following characterization of amenability:

THEOREM 2.1. *Let A be a Banach algebra. Then the following are equivalent:*

1. *A is amenable*
2. *A has a bounded approximate diagonal*
3. *A has a virtual diagonal*

Definition 2.2. A Banach algebra A is an F-algebra if it is the unique predual of a C^*-algebra $\mathcal{B}$ and the identity element e of $\mathcal{B}$ is a multiplicative linear functional on A.

Example 2.3. The following are examples of F-algebras:

1. The group algebra $L^1(G)$ for any locally compact group G.
2. The semigroup algebra $\ell^1(S)$.

3. The Fourier algebra $A(G)$ for any locally compact group G with pointwise multiplication.

Let A be an F-algebra associated with a C^*-algebra $\mathcal{B}$ and let $P_1(A'') = \{\mu \in \mathcal{B}' : \mu \geq 0, \langle e, \mu \rangle = 1\}$. $P_1(A'') \subset A'' = \mathcal{B}'$. An element $m \in P_1(A'')$ is called a left invariant mean on A' if

$$m(f \cdot a) = \langle a, e \rangle m(f) \quad (a \in A, f \in A').$$

We recall from [31] that the F-algebra A is called left amenable if there is a left invariant mean on A'.

Example 2.4. Lau gave the following examples of left amenable F-algebras.

1. The group algebra $L^1(G)$ for any locally compact group G is left amenable if and only if G is an amenable group.
2. The semigroup algebra $\ell^1(S)$ is left amenable if and only if S is a left amenable semigroup.
3. All commutative F-algebras are left amenable.

We next introduce the notations and basic concepts on character amenability. Let A be a Banach algebra over $\mathbb{C}$ and $\varphi : A \to \mathbb{C}$ be a character on A, that is, an algebra homomorphism from A into $\mathbb{C}$, we let Φ_A denote the character space of A. Also, $\mathcal{M}^A_{\varphi_r}$ denote the class of Banach A- bimodule X for which the right module action of A on X is given by $x \cdot a = \varphi(a)x \quad (a \in A, x \in X, \varphi \in \Phi_A)$, and $\mathcal{M}^A_{\varphi_l}$ denote the class of Banach A- bimodule X for which the left module action of A on X is given by $a \cdot x = \varphi(a)x \quad (a \in A, x \in X, \varphi \in \Phi_A)$. If the right module action of A on X is given by $x \cdot a = \varphi(a)x$, then it is easy to see that the left module action of A on the dual module X' is given by $a \cdot f = \varphi(a)f \quad (a \in A, f \in X', \varphi \in \Phi_A)$. Thus, we note that $X \in \mathcal{M}^A_{\varphi_r}$ (resp. $X \in \mathcal{M}^A_{\varphi_l}$) if and only if $X' \in \mathcal{M}^A_{\varphi_l}$ (resp. $X' \in \mathcal{M}^A_{\varphi_r}$).

Let A be a Banach algebra and let $\varphi \in \Phi_A$, we recall the next definitions from [25] and [44].

Definition 2.5. (i) A is left φ-amenable if every continuous derivation $D : A \to X'$ is inner for every $X \in \mathcal{M}^A_{\varphi_r}$;
(ii) A is right φ-amenable if every continuous derivation $D : A \to X'$ is inner for every $X \in \mathcal{M}^A_{\varphi_l}$;
(iii) A is left character amenable if it is left φ-amenable for every $\varphi \in \Phi_A$;
(iv) A is right character amenable if it is right φ-amenable for every $\varphi \in \Phi_A$;
(v) A is character amenable if it is both left and right character amenable.

3. RESULTS OVER GENERAL BANACH ALGEBRAS

In this section, we surveyed some general theory and results over general Banach algebras. These were applied and used in establishing results for Banach algebras in different classes.

The following results are useful hereditary, stability and general properties of character amenability for Banach algebras.

PROPOSITION 3.1. *Let A be a Banach algebra. Suppose A is character amenable and I is a closed ideal of codimension one in A.*

1. *Then*
   (i) A has a bounded approximate identity and hence factors
   (ii) the unitization algebra $A^\sharp$ is character amenable
   (iii) I has a bounded approximate identity
   (iv) I is character amenable.
2. *Suppose B is another Banach algebra and $\tau : A \to B$ is a continuous homomorphism with $\overline{\tau(A)} = B$.*
   (i) Then B is character amenable
   (ii) In particular, suppose I is a closed ideal of A. Then A/I is character amenable.

Proof.

1. (i) This is [40, Proposition 3.1 (i)]
   (ii) This is [40, Proposition 3.1 (ii)]
   (iii) This is [40, Theorem 3.2 (i)]
   (iv) This is [40, Theorem 3.2 (ii)]
2. (i) This is [42, Proposition 3.3]
   (ii) This is [41, Proposition 3.1]. It follows from the fact that $\tau : A \to A/I$ is a continuous homomorphism with dense range. □

Let $A\hat{\otimes}B$ be the projective tensor product of Banach algebras A and B. For $f \in A', g \in B'$, let $f \otimes g \in (A\hat{\otimes}B)'$ such that

$$(f \otimes g)(a \otimes b) = f(a)g(b) \quad (a \in A, b \in B).$$

Then $\Phi_{A\hat{\otimes}B} = \{\varphi \otimes \psi : \varphi \in \Phi_A, \psi \in \Phi_B\}$. Also, let J be a non-empty set. We denote by $\mathcal{M}_J(A)$ the set of $J \times J$ matrices (a_{ij}) with entries in A such that

$$\|(a_{ij})\| = \sum_{i,j \in J} \|a_{ij}\| < \infty.$$

Then $\mathcal{M}_J(A)$ with the usual matrix multiplication is a Banach algebra. The map

$$\tau : \mathcal{M}_J(A) \to A\hat{\otimes}\mathcal{M}_J(\mathbb{C}) \quad \text{defined by} \quad \tau((a_{ij})) = \sum_{i,j \in J} a_{ij} \otimes E_{ij}$$

$((a_{ij}) \in \mathcal{M}_J(A))$, is an isometric isomorphism of Banach algebras, where (E_{ij}) are the matrix units in $\mathcal{M}_J(\mathbb{C})$. Thus, we have the next results from [29] and [41].

PROPOSITION 3.2. 1. *Let A and B be Banach algebras with $\varphi \in \Phi_A$ and $\psi \in \Phi_B$. Then $A\hat{\otimes}B$ is $(\varphi \otimes \psi)$-amenable if and only if A is φ-amenable and B is ψ-amenable. In particular, $A\hat{\otimes}B$ is left character amenable if and only if A and B are left character amenable.*

2. *Let A be a Banach algebra and J a non-empty set. Then $\mathcal{M}_J(A)$ is left character amenable if and only if A is left character amenable.*

Proof.

1. This is from [29, Theorem 3.3].
2. This is [41, Corollary 3.3]. It clearly follows from Proposition 3.2 (i). □

We recall from [2] that a Banach algebra A is left [right] φ-biflat if there exists a bounded linear operator $\rho : A \to (A\hat{\otimes}A)''$ such that

$$(i)\quad \rho(ab) = \varphi(a)\rho(b) = \rho(a) * b \quad [\rho(ab) = \varphi(b)\rho(a) = a \cdot \rho(b)]$$
$$(ii)\quad (\pi''_A \circ \rho(a))(\varphi) = \varphi(a) \quad (a, b \in A, \varphi \in \Phi_A).$$

Helemskii in [22] showed that a Banach algebra A is amenable if and only if it is biflat and has a bounded approximate identity. We give the character amenability of this result.

The next result is due to [2, Proposition 2.2].

PROPOSITION 3.3. *Let A be a Banach algebra with $\varphi \in \Phi_A$.*
(i) If A is left φ-amenable, then A is left φ-biflat;
(ii If A is left φ-biflat and has a bounded approximate identity, then A is left φ-amenable.

The above definition of φ-biflatness in [2] was generalized in [41] as follows: We say that a Banach algebra A is
(i) left [right] character biflat if it is left [right] φ-biflat for every $\varphi \in \Phi_A$;
(ii) character biflat if it is both left and right character biflat.

THEOREM 3.4. *Let A be a Banach algebra. Then the following are equivalent:*
(i) A is character amenable
(ii) A is character biflat and has a bounded approximate identity.

Proof. This is [41, Theorem 3.6]. This clearly follows from Proposition 3.3 (i) and (ii) and the above definition. □

For a commutative and reflexive Banach algebra, we have the next result due to [25].

THEOREM 3.5. *Let A be a character amenable, reflexive, commutative Banach algebra. Then $A \cong \mathbb{C}^n$ for some $n \in \mathbb{N}$.*

4. SOME CHARACTERIZATIONS AND RESULTS ON SECOND DUAL ALGEBRAS

In this section, we surveyed some characterizations and results over second dual of general Banach algebras. These were applied and used in establishing results for Banach algebras in different classes.

We start by characterizing in terms of bounded approximate identity and certain topological invariant elements in the second dual A'' of the Banach algebra A.

Definition 4.1. Let $\varphi \in \Phi_A$. $\Psi \in A''$ is called

1. φ-topologically left invariant if

$$\langle \Psi,\, a \cdot \lambda \,\rangle = \varphi(a)\langle \Psi,\, \lambda \,\rangle \quad (a \in A, \lambda \in A')\,;$$

2. φ-topologically right invariant if

$$\langle \Psi,\, \lambda \cdot a \,\rangle = \varphi(a)\langle \Psi,\, \lambda \,\rangle \quad (a \in A, \lambda \in A')\,;$$

3. φ-topologically invariant if it it is both left and right.

THEOREM 4.1. *The Banach algebra A is left character amenable if and only if the following two conditions hold:*

1. *A has a bounded approximate identity.*
2. *For every $\varphi \in \Phi_A, \exists$ φ-topologically left invariant element $\Psi \in A''$ such that $\Psi(\varphi) \neq 0$.*

Proof. This is [44, Theorem 4.2]. □

We next give a Johnson-like characterization. Recall from [25] that, for $\varphi \in \Phi_A$, a left (right) φ-virtual diagonal for A is an element M in $(A\hat{\otimes}A)''$ such that

$$(i) \;\; M \cdot a = \varphi(a)M \quad (a \cdot M = \varphi(a)M) \quad (a \in A);$$
$$(ii) \;\; \langle M, \varphi \otimes \varphi\rangle = \pi''(M)(\varphi) = 1.$$

Also, a left [right] φ-approximate diagonal for A is a net (m_α) in $(A\hat{\otimes}A)$, such that

$$(i) \;\; \|m_\alpha \cdot a - \varphi(a)m_\alpha\| \to 0 \quad [\|a \cdot m_\alpha - \varphi(a)m_\alpha\| \to 0] \quad (a \in A);$$

$$(ii) \;\; \langle \varphi \otimes \varphi,\, m_\alpha\rangle \to 1.$$

The following important characterization was shown in [25, Theorem 2.3].

THEOREM 4.2. *Let A be a Banach algebra and $\varphi \in \Phi_A$. The following statements are equivalent:*
(i) A is left [right] φ-amenable
(ii)A has a bounded left [right] φ-approximate diagonal
(iii) A has a left [right] φ-virtual diagonal.

We also give the following characterizations of φ-amenability for Banach algebras. These were shown in [29, Theorem 1.1] and [29, Theorem 1.3].

THEOREM 4.3. *Let A be a Banach algebra and let $\varphi \in \Phi_A$. The following statements are equivalent:*
(i) A is φ-amenable.
(ii) There exists $m \in A''$ such that $m(\varphi) = 1$ and $a \cdot m = \varphi(a)m \quad (a \in A)$.
(iii) There exists a bounded net (u_α) in A such that $\varphi(u_\alpha) = 1$ for all α and

$$\|au_\alpha - \varphi(a)u_\alpha\| \to 0 \quad (a \in A).$$

Let A be a Banach algebra. Then the second dual A'' of A is a Banach A-bimodule for the maps $(a, \Phi) \to a \cdot \Phi$ and $(a, \Phi) \to \Phi \cdot a$ from $A \times A''$ to A'' that extend the product map $A \times A \to A, \quad (a, b) \to ab$ on A. Arens in [3] defined two products, $\Box$ and $\Diamond$, on the second dual A'' of a Banach algebra A; A'' is a Banach algebra with respect to each of these products, and each algebra contains A as a closed subalgebra. The products are called the *first* and *second Arens products* on A'', respectively. For the general theory of Arens products, see [11, 16]. We recall briefly the definitions. For $\Phi \in A''$, we set

$$\langle a, \lambda \cdot \Phi \rangle = \langle \Phi, a \cdot \lambda \rangle, \quad \langle a, \Phi \cdot \lambda \rangle = \langle \Phi, \lambda \cdot a \rangle \quad (a \in A, \lambda \in A'),$$

so that $\lambda \cdot \Phi, \Phi \cdot \lambda \in A'$. Let $\Phi, \Psi \in A''$. Then

$$\langle \Phi \Box \Psi, \lambda \rangle = \langle \Phi, \Psi \cdot \lambda \rangle, \quad \langle \Phi \Diamond \Psi, \lambda \rangle = \langle \Psi, \lambda \cdot \Phi \rangle \quad (\lambda \in A').$$

Suppose that $\Phi, \Psi \in A''$ and that $\Phi = \lim_\alpha a_\alpha$ and $\Psi = \lim_\beta b_\beta$ for nets (a_α) and (b_β) in A. Then

$$\Phi \Box \Psi = \lim_\alpha \lim_\beta a_\alpha b_\beta \quad \textit{and} \quad \Phi \Diamond \Psi = \lim_\beta \lim_\alpha a_\alpha b_\beta,$$

where all limits are taken in the weak-$*$ topology $\sigma(A'', A')$ on A''.

The following result is well known, see [11, Theorem 2.17]:

THEOREM 4.4. *Let A be a Banach algebra. Then both $(A'', \Box)$ and $(A'', \Diamond)$ are Banach algebras containing A as a closed subalgebra.*

The next result is [25, Theorem 3.8], which is the character amenability version of the result on amenability of the second dual.

THEOREM 4.5. *Let A be a Banach algebra. If $(A'', \Box)$ is left [right] character amenable, then A is left [right] character amenable.*

5. RESULTS OVER BANACH ALGEBRAS IN DIFFERENT CLASSES

A fruitful area of research in notions of amenability in Banach algebras has been to describe these notions of amenability in terms of the structures that the algebra sits on. The structures that come to mind are: Banach spaces (algebra of operators), locally compact groups and semigroups (group, measure, Segal, Beurling, Fourier and Fourier-Stieljes algebras), locally compact Hausdorff space (uniform and Lipschitz algebras). In the recent years, various authors have considered character amenability of Banach algebras in terms of these structures that the algebras sit.

In this section, we surveyed results on this relationship for different classes of Banach algebras.

5.1. RESULTS ON CONVOLUTION ALGEBRAS

In this subsection, we surveyed results on group algebras, measure algebras, Fourier algebras and Fourier-Stieltjes algebras.

We recall that a locally compact group is a group G which is also a locally compact Hausdorff space such that the maps

$$G \times G \to G, \quad (g, h) \to gh \quad \textit{and} \quad G \to G, \quad g \to g^{-1}$$

are continuous. Each locally compact group G has a left Haar measure μ. $L^1(G)$, consisting of measurable functions f on G with

$$\|f\|_1 = \int_G |f(t)| \mathrm{d}\mu(t) < \infty,$$

becomes a Banach algebra for the product

$$(f \star g)(t) = \int_G f(s) g(s^{-1}t) \mathrm{d}\mu(s).$$

This is called the group algebra of G. We also denote by $M(G)$, the measure algebra. We recall that the product $\mu, v \in M(G)$ is specified by the formula

$$\langle \mu * v, \lambda \rangle = \int_G \int_G \lambda(st) \mathrm{d}\mu(sdv(t) \quad (\lambda \in C_0(G)),$$

so that $\delta_s * \delta_t = \delta_{st} \quad (s, t \in G)$. It is standard that $M(G)$ is a unital Banach algebra with the convolution product and it is identified with the dual space of all continuous linear functional on the Banach space $C_0(G)$ and that $L^1(G)$ is a closed ideal in $M(G)$. The map $\varphi : \mu \to \mu(G), \quad M(G) \to \mathbb{C}$ is a character on $M(G)$, called the augmented character.

We denote by $A(G)$, the Fourier algebra on G. it is well known that $A(G) \cong L^1(\hat{G})$ when G is abelian and $\hat{G}$ is its dual group. It was shown in [19] that for any locally compact group G, $A(G)$ is a commutative Banach algebra with character space G and in [33] that $A(G)$ has a bounded approximate identity if and only if G is an amenable group.

The Fourier-Stieltjes algebra $B(G)$ is the collection of all coefficient functions of continuous unitary representations of G. It is a Banach algebra containing $A(G)$ as a closed ideal and $B(G) = A(G)$ if and only if G is compact. For details, see [8, 9, 24] and [25].

The character amenability of the group algebra $L^1(G)$ and Fourier algebra $A(G)$ is equivalent to the amenability of the group G. Thus the character amenability of $L^1(G)$ and $A(G)$ is completely determined by the amenability of G. Character amenability of the measure algebra $M(G)$ is equivalent to G being discrete and amenable. Also, for group and measure algebras, $L^1(G)$, $M(G)$, the amenability and character amenability coincide. This follows from Johnson's classical result in [26] and the following result due to [44, Corollary 2.4, Corollary 2.5].

THEOREM 5.1. *Let G be a locally compact.*

1. *Then the following statements are equivalent:*
   (i) $L^1(G)$ is left character amenable.
   (ii) $L^1(G)$ is right character amenable.
   (iii) G is amenable.
2. *The following statements are equivalent:*
   (i) $M(G)$ is character amenable.
   (ii) G is a discrete amenable group.

For Fourier algebra, $A(G)$, the amenability and character amenability do not coincide. This follows from [44, Corollary 2.4] stated below and from Leptin [33] that G is amenable if and only if $A(G)$ has a bounded approximate identity. These then imply that if $A(G)$ is amenable then G must be amenable; the converse to this was shown to be false by Johnson in [28], where he shows that for the compact group $G = SO(3)$, $A(G)$ is not amenable.

THEOREM 5.2. *Let G be a locally compact. Then the following statements are equivalent:*
(i) $A(G)$ is left character amenable.
(ii) $A(G)$ is right character amenable.
(iii) G is amenable.

Proof. This follows [44, Corollary 2.4] with $p = 2$. □

It was shown in [25], that $B(G)$ can be character amenable when G is noncompact. For G compact, $B(G) = A(G)$, and so using Theorem 5.2, we have that $B(G)$ is left or right character amenable if and only if G is amenable.

5.2. RESULTS ON BEURLING AND SEGAL ALGEBRAS

In this subsection, we surveyed results on Beurling and Segal algebras.

Beurling algebras are L^1-algebras associated with locally compact groups G with an extra weight ω on the groups. A weight on a locally group G is a continuous function $\omega : G \to (0, \infty)$ such that

$$\omega(st) \leq \omega(s)\omega(t) \quad (s, t \in G).$$

A weight ω on G is said to be symmetric if $\omega(t^{-1}) = \omega(t) \quad (t \in G)$. For a continuous weight ω on G, we define the weighted spaces

$$L^1(G, \omega) := \{f \textit{ Borel measurable } : \|f\|_{L^1(G,\omega)} = \|\omega f\|_{L^1(G)} < \infty\},$$

and

$$L^\infty(G, \frac{1}{\omega}) := \{f \textit{ Borel measurable } : \|f\|_{L^\infty(G,\frac{1}{\omega})} = \|\frac{f}{\omega}\|_{L^\infty(G)} < \infty\}.$$

$L^1(G, \omega)$ and $L^\infty(G, \frac{1}{\omega})$ are isometric to $L^1(G)$ and $L^\infty(G)$, respectively. Also, $L^\infty(G, \frac{1}{\omega})$ is the dual of $L^1(G, \omega)$ with the duality pairing

$$\langle f, g\rangle = \int_G f(t)g(t)\mathrm{d}\mu(t) \quad (f \in L^1(G, \omega), g \in L^\infty(G, \frac{1}{\omega})),$$

where μ is the left Haar measure on G.

With the multiplication and norm $\|.\|_{L^1(G,\omega)}$, $L^1(G, \omega)$ becomes a Banach algebra and the algebra $L^1(G, \omega)$ is called the Beurling algebra on G. For further details see [10].

Let A be a Banach algebra with norm $\|.\|_A$ and let B be a dense left ideal in A such that
(i) B is a Banach algebra with respect to some norm $\|.\|_B$,
(ii) there is a constant $K > 0$ such that

$$\|b\|_A \leq K\|b\|_B \qquad \text{for all} \quad b \in B,$$

(iii) there is a constant $C > 0$ such that

$$\|ab\|_B \leq C\|a\|_A\|b\|_B \qquad \text{for all} \quad a, b \in B.$$

Then we recall from [32] that B is called an abstract Segal algebra in A.

In the case $A = L^1(G)$, we write $S^1(G)$ instead of B and further insist that $S^1(G)$ is closed under left translation; $L_x f \in S^1(G)$ for all $x \in G$ and $f \in S^1(G)$, where $L_x f(y) = f(x^{-1}y)$ for $y \in G$. By well-known techniques, conditions (i)-(iii) above on $B = S^1(G)$, is equivalent to the map

$$(x, f) \to L_x f : G \times S^1(G) \to S^1(G)$$

is continuous with $\|L_x f\|_{S^1(G)} = \|f\|_{S^1(G)}$ for $f \in S^1(G), x \in G$. Every Segal algebra $S^1(G)$ is an abstract Segal algebra in $L^1(G)$ but not conversely.

For a locally compact group G, we recall from [20] that a Lebesgue-Fourier algebra $\mathcal{L}A(G)$ of a locally compact group G is

$$\mathcal{L}A(G) = L^1(G) \cap A(G),$$

where $|||f||| = \|f\|_1 + \|f\|_{A(G)} \quad (f \in \mathcal{L}A(G))$, and where the product is the convolution product. As shown in [20], pointwise product also provides a Banach algebra structure on $\mathcal{L}A(G)$. It was also shown in [20], that $(\mathcal{L}A(G), |||.|||)$ with convolution product is a Segal algebra in $L^1(G)$ and that $(\mathcal{L}A(G), |||.|||)$ with pointwise multiplication is a commutative abstract Segal algebra in $A(G)$.

The next results give conditions on the weight on G for which;
(i) the amenability and character amenability of $L^1(G, \omega)$ coincide.
(ii) $L^1(G, \omega)$ is not character amenable

PROPOSITION 5.3. *Let ω be a weight on the locally compact group G.*

1. *Suppose ω is symmetric. Then the following statements are equivalent:*
   (i) $L^1(G, \omega)$ is character amenable.
   (ii) $L^1(G, \omega)$ is amenable.
2. *Suppose ω is symmetric and $\lim_{x\to\infty} \omega(x) = \infty$, then $L^1(G, \omega)$ is not character amenable.*

Proof.

1. This is [40, Corollary 4.2].
2. This is [40, Proposition 4.4]. □

It was shown in [46] that $S^1(G)$ can never have a bounded approximate identity unless it coincides with $L^1(G)$. Thus we have that
(i) $S^1(G)$ is not character amenable for all G,
(ii) character amenability and amenability of $S^1(G)$ coincide and this is possible when $S^1(G) = L^1(G)$.

The following results are due to [42, Proposition 4.3, Proposition 4.4]

PROPOSITION 5.4. *Let G be a locally compact group.*

1. *The following statements are equivalent:*
   (i) $\mathcal{L}A(G)$ with convolution product is character amenable
   (ii) G is discrete and amenable.
2. *The following statements are equivalent:*
   (i) $\mathcal{L}A(G)$ with pointwise product is character amenable
   (ii) G is compact, amenable and the Fourier algebra $A(G)$ is character amenable.

5.3. RESULTS ON ALGEBRAS OF FUNCTIONS

In this subsection, we surveyed results on uniform and Lipschitz algebras. We first recall the definition of uniform and Lipschitz algebras. For details, see [8].

Let X be a non-empty compact, Hausdorff topological space. A uniform algebra on X is a closed subalgebra A of $C(X)$ which contains all the constant functions and separate points of X, *i.e.* whenever x and y are in X with $x \neq y$, there exists $f \in A$ with $f(x) \neq f(y)$. Every uniform algebra on X is a commutative, unital Banach algebra with respect to the uniform norm

$$|f|_X = \sup\{|f(x)| : x \in X\}.$$

A function algebra A on X is a subalgebra of $C(X)$ that strongly separates points on X for which the A-topology on X coincides with the given topology. A Banach function algebra on X is a function algebra on X which is a Banach algebra with respect to some norm. A Banach function algebra A is called natural if the character space $\Phi_A = \{\epsilon_x : x \in X\}$, where ϵ_x is the evaluate functional at x given by $\epsilon_x(f) = f(x)$. For a unital Banach function algebra A on X, the Choquet boundary of A, denoted by $\Gamma_0(A)$, consists of all those $x \in X$ for which δ_x is the unique probability measure μ on X such that $f(x) = \int_X f \mathrm{d}\mu$, for every $f \in A$.

Let (X, d) be a compact metric space and $0 < \alpha \leq 1$. The algebra of all complex-valued functions f on X for which

$$p(f) = \sup\{\frac{|f(x) - f(y)|}{d(x,y)^\alpha} : x, y \in X, x \neq y\} < \infty,$$

is denoted by $\mathrm{Lip}(X, \alpha)$ and the subalgebra of those functions f for which

$$\lim_{d(x,y)\to 0} \frac{|f(x) - f(y)|}{d(x,y)^\alpha} = 0,$$

is denoted by $\mathrm{lip}(X, \alpha)$. These are called Lipschitz algebras. It is known that $\mathrm{Lip}(X, \alpha)$ with the norm $\|.\|_\alpha$, given by

$$\|f\|_\alpha = p_\alpha + |f|_X,$$

where $|.|_X$ is the uniform norm, and pointwise product is a Banach algebra. The algebras $\mathrm{Lip}(X, \alpha)$ for $0 < \alpha \leq 1$, and $\mathrm{lip}(X, \alpha)$ for $0 < \alpha < 1$, are Banach function algebras on X under the norm $\|.\|_\alpha$. These algebras were first studied by Sherbert in [48]. Also, they are self-adjoint and separate the points of X, and so they are uniformly dense in $C(X)$, by the Stone-Weierstrass theorem. Thus, they are natural Banach function algebras on X.

The following results show the relationship between the character amenability of the Banach function algebra A and the Choquet boundary of A. It also characterizes the character amenability of the natural uniform unital uniform algebra and uniform algebras.

Theorem 5.5. 1. *Let A be a unital Banach function algebra on a compact space X. Suppose A is character amenable, then $\Gamma_0(A) = X$.*

2. *Let A be a natural unital uniform algebra on a compact space X. A is character amenable if and only if $\Gamma_0(A) = X$.*
3. *Let K be a compact subset of $\mathbb{C}$ and let $P(K)$ be the uniform algebra on K consisting of uniform limits of polynomials. Then $P(K)$ is character amenable if and only if $P(K) = C(K)$.*

Proof.

1. This is [25, Theorem 5.1].
2. This is [25, Corollary 5.2].
3. This is [25, Theorem 5.4]. □

Remark. Theorem 5.5 (3) is the character amenability version of Sheinberg's result on amenability characterizing amenable uniform algebras, see [47].

The character amenability of Lipschitz algebras were studied in [12,25,30]. The choice of $\varphi \in \Phi_{\mathrm{Lip}(X,\alpha)}$, for which $\mathrm{Lip}(X,\alpha)$ is φ- amenable was studied in [30]. In particular, it was shown that $\mathrm{Lip}(X,\alpha)$ is φ_x-amenable if and only if x is an isolated point of X, where $x \to \varphi_x$ is the canonical homeomorphism between X and $\Phi_{\mathrm{Lip}(X,\alpha)}$. This same result is also true for $\mathrm{lip}(X,\alpha)$ for $0 < \alpha < 1$, see also [25]. It was shown in [25] that for infinite compact metric space X with $0 < \alpha < 1$, $\mathrm{Lip}(X,\alpha)$ is not character amenable.

5.4. RESULTS ON SEMIGROUP ALGEBRAS

In this subsection, we surveyed results on semigroup algebras.

We recall that a semigroup is a non-empty set S with an associative binary operation

$$(s,t) \to st; \quad S \times S \to S \quad (s,t \in S).$$

Let S be a semigroup, S is said to be regular if for all $s \in S$, there is $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$. S is an inverse semigroup if such s^* exists and is unique for all $s \in S$. An element $p \in S$ is idempotent if $p^2 = p$. The set of idempotents in S is denoted by $E(S)$. A semigroup S is semilattice if S is commutative and $E(S) = S$.

Let S be a non-empty set. Then

$$\ell^1(S) = \left\{ f \in \mathbb{C}^S : \sum_{s\in S} |f(s)| < \infty \right\},$$

with the norm $\|\cdot\|_1$ given by $\|f\|_1 = \sum_{s\in S} |f(s)|$ for $f \in \ell^1(S)$. We write δ_s for the characteristic function of $\{s\}$ when $s \in S$.

Now suppose that S is a semigroup. For $f, g \in \ell^1(S)$, we set

$$(f \star g)(t) = \left\{ \sum f(r)g(s) : r,s \in S,\, rs = t \right\} \quad (t \in S)$$

so that $f \star g \in \ell^1(S)$. It is standard that $(\ell^1(S), \star)$ is a Banach algebra, called the *semigroup algebra on* S. $\ell^1(S)$ is commutative if and only if S is abelian.

Let S be a semigroup, for each $\varphi \in \Phi_S$, the map $\tilde{\varphi} : \ell^1(S) \to \mathbb{C}$ defined by

$$\tilde{\varphi}(\sum_{s\in S} \alpha_s \delta_s) = \sum_{s\in S} \alpha_s \varphi(s)$$

is a character on $\ell^1(S)$, and every character on $\ell^1(S)$ arises in this way.

For a semigroup S, given $f \in \ell^\infty(S), t \in S$, we define the translation functions ft, tf by

$$ft(s) = f(ts), \quad tf(s) = f(st) \quad (s, t \in S).$$

A continuous linear function μ on $\ell^\infty(S)$ is called
(i) a mean if $\mu(\mathbf{1}) = \|\mu\| = 1$, where $\mathbf{1}$ denotes the constant unit function on S;
(ii) a left invariant if for each $f \in \ell^\infty(S), t \in S$, we have $\mu(ft) = \mu(f)$;
(iii) a right invariant if for each $f \in \ell^\infty(S), t \in S$, we have $\mu(tf) = \mu(f)$.

The semigroup S is left [right] amenable if there exists a left [right] invariant mean on $\ell^\infty(S)$, and it is amenable if it is left and right amenable. For an inverse semigroup, left amenability is equivalent to right amenability, in which case, amenability is equivalent to left or right amenability. For further details, see [8, 11, 17].

One of the results below show that for any semigroup S, if $\ell^1(S)$ is character amenable, then S is amenable and regular.

THEOREM 5.6. *Let S be a semigroup.*

1. *Suppose $\varphi_S \in \Phi_{\ell^1(S)}$.*
   (i) If $\ell^1(S)$ is left [right] φ_S-amenable, then S is left [right] amenable.
   (ii) If $\ell^1(S)$ is character amenable, then S is amenable and regular.
2. *Suppose S is inverse and $E(S)$ finite. Then $\ell^1(S)$ is left character amenable if each maximal subgroup of S is amenable.*
3. *Suppose $E(S)$ finite. If $\ell^1(S)$ is character amenable, then it has an identity.*
4. *Suppose S is uniformly locally finite semilattice. Then $\ell^1(S)$ is character amenable if and only if S is finite.*

Proof.

1. This is [41, Proposition 4.1].
2. This is [41, Theorem 4.5].
3. This is [41, Corollary 4.2].
4. This is [18, Corollary 2.8]. □

Remark. Theorem 5.6 (1) is true for any semigroup, but the converse is false. The partial converse is given in Theorem 5.6 (2).

The character amenability of $\ell^1(S)$ for certain semigroups S such Brandt semigroup, Rees semigroup, semillatice, inverse semigroup with uniformly locally finite idempotent set were studied in [41] and [18]. It was shown in [41] that for a Brandt semigroup S over a group G with index set I, the left character amenability of $\ell^1(S)$ is equivalent to the amenability of G and I being finite. The authors in [18] also characterize the character amenability of Brandt semigroup algebras. It was also shown in [41] that for a Rees semigroup S with a zero over a group G, the left character amenability of $\ell^1(S)$ is equivalent to its amenability, this is in turn equivalent to G being amenable.

5.5. PROBLEMS ON ALGEBRAS OF OPERATORS ON BANACH SPACE

A class of algebras where the notion of character amenability is yet to be explored are the algebras of operators on a Banach space (*e.g.* $B(E)$-the algebra of bounded operators on a Banach space E, and the closed ideals of $B(E)$). This is because, the character amenability of algebras of operators on a Banach space depend extensively on the geometry of the Banach space E and the fact that the structure of closed ideals of $B(E)$ appears to be little understood.

For a Banach space E, we denote by $B(E)$ the algebra of bounded operators on E. $B(E)$ is a Banach algebra with the composition product and the operator norm

$$\|T\| = \sup\{\|T(x)\| : x \in E, \|x\| \leq 1\}.$$

We also denote by $F(E)$, the set of finite rank operators, $N(E)$, the set of nuclear operators, $A(E)$, the set of approximable operators, and $K(E)$, the set of compact operators. These are all two-sided operator ideals of $B(E)$. They are closed ideals, hence Banach algebras in their natural norms except $F(E)$. $F(E)$ is the smallest non-zero ideal in $B(E)$, while its closure $A(E)$ is the smallest non-zero closed ideal in $B(E)$.

$F(E)$ can be written in terms of the tensor product between E and the dual space E' called the algebraic tensor product as follows:

$$F(E) \cong E \otimes E' = span\{x \otimes \lambda \in B(E) : x \in E, \lambda \in E', (x \otimes \lambda)(y) = \lambda(y)x \quad y \in E\}.$$

The completion of $F(E)$ in the operator norm is $A(E)$. This norm is called the injective norm and the completion $A(E)$ corresponds to the injective tensor product denoted by $X \check{\otimes} X^*$, which carries this norm. There is another natural norm for $T \in F(E)$, the projective tensor norm

$$\|T\|_p = \inf\{\sum_{k=1}^{n} \|x_k\|\|\omega_k\| : T = \sum_{k=1}^{n} x_k \otimes \omega_k\};$$

where the infimum extends over all positive integer n; $x_k \in E$, $\omega_k \in E'$. By completing $E \otimes E' \cong F(E)$ in the projective tensor norm, instead of the operator norm, we get the projective tensor product denoted by $E\hat{\otimes}E'$ called the tensor algebra of E. $E\hat{\otimes}E'$ is a Banach algebra. See [8] and [23] for further details.

We also recall that a Banach space E is said to have the approximation property (A.P), if for every compact set $K \subset E$ and every $\epsilon > 0$, there is an operator $T \in F(E)$ such that

$$\|T(x) - x\| \leq \epsilon \quad for\ every\ x \in K.$$

If in addition, the operator T can always be chosen with $\|T\| \leq C$ for some $C \geq 1$ fixed, then X is said to have the C-approximation property (C-A.P). A Banach space X is said to have the bounded approximation property (B.A.P) if it has the C-A.P for some C, and X has the metric approximation property (M.A.P) if it has the 1-A.P. For details on this and bounded compact approximation property see [45].

It was shown in [26] that $K(E)$ is amenable for $E = \ell^p$, $(1 < p < \infty)$ and $E = C[0, 1]$. Thus $K(E)$ is character amenable for these Banach spaces.

The amenability and character amenability of the group algebra $L^1(G)$ coincide and are clearly related to the group G. Thus the character amenability is an important property of groups which has many characterizations and also many cohomological characterizations of $L^1(G)$. It will therefore be good to study the characterizations of the character amenability of the closed operator ideals of $B(E)$. Thus, we raised the following questions:

Question 5.1.

1. It was shown in [13] that $B(\ell^p)$ is not amenable for $1 \leq p \leq \infty$, since character amenability is weaker than amenability, then we asked: Is $B(\ell^p)$ character amenable for any or all $p \in [1, \infty]$?
2. Also, Mewomo in [35], showed that $A(T^2)$ is weakly amenable, where T^2 is the 2-convexified Tsirelson's space. Is $A(T^2)$ character amenable?
3. In general, what are the infinite-dimensional Banach spaces E for which $B(E)$ or any of its closed ideals are character amenable?

Question 5.2.

1. Can a technique be developed to determine whether or not for a given Banach space E, any of the closed ideal of $B(E)$ is character amenable?
2. What are the conditions on the Banach space E that will ensure the character amenability of any of the closed ideals of $B(E)$?

Question 5.3. Since character amenable Banach algebras have bounded approximate identities, then by [15, Theorem 2.6], we have the following:

1. If $K(E)$ is character amenable, then E has the bounded compact approximation property.
2. If $A(E)$ is character amenable, then E has the bounded approximation property.

Thus, we ask the following questions: Is character amenability of $A(E)$ or $K(E)$ equivalent to E or the dual space E' have the bounded approximation property?

Question 5.4.

1. What are the necessary and sufficient condition for the character amenability of any of the closed ideals of $B(E)$?
2. What are the intrinsic properties of the Banach space E which is equivalent to the character amenability of any of the closed ideals of $B(E)$?

6. CONCLUSION

The above survey of results and problems to some extent aims at producing a catalogue listing of general results on character amenability for general Banach algebras and Banach algebras in different classes. It also provides different characterizations and hereditary properties of character amenable Banach algebras and the relationship that exits between important properties of the algebra and character amenability. This survey serves as a reference point for future research on character amenability in Banach algebras.

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