HARNACK INEQUALITIES FOR WEIGHTED SUBELLIPTIC P-LAPLACE EQUATIONS CONSTRUCTED BY HÖRMANDER VECTOR FIELDS

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This paper deals with the following weighted subelliptic p-Laplace equation

$$L_p u := div_X \left(\langle A(x)Xu(x), Xu(x) \rangle^{\frac{p-2}{2}} A(x)Xu(x) \right) = g(x),$$

where the system $X = (X_1, ..., X_m)$ satisfies Hörmander's condition, $u \in W^{1,p}(\Omega, w)$, $1 < p < Q$, $A(x)$ is a bounded measurable and $m \times m$ symmetric matrix satisfying

$$\lambda^{-1} w(x)^{2/p} |\xi|^2 \leq \langle A(x)\xi, \xi \rangle \leq \lambda w(x)^{2/p} |\xi|^2, for\ \xi \in \mathbb{R}^m,$$

with $w = w(x)$ being an A_p function. We first establish a maximum principle for weak solutions to the equation $L_p u = g$ with the weighted Sobolev inequality and the extension of Moser iteration technique to the weight case. Next, the local boundedness and the Harnack inequality for nonnegative weak solutions to $L_p u = g$ are proved. As an application, the Hölder continuity for nonnegative weak solutions is given. Unlike in many papers, we do not impose any restriction in advance for measures of metric balls.

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1. INTRODUCTION

In this paper, we consider the following weighted subelliptic p-Laplace equation

$$L_p u := div_X \left(\langle A(x)Xu(x), Xu(x) \rangle^{\frac{p-2}{2}} A(x)Xu(x) \right) = g(x), \tag{1.1}$$

where $X = (X_1, ..., X_m)$, $X_k (k = 1, ..., m)$ are smooth vector fields satisfying Hörmander's condition. Let X_k^* be the adjoint of X_k, $div_X v = - \sum_{k=1}^{m} X_k^* v_k$, $v =$

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$(v_1, ..., v_m)$, $1 < p < Q$. We suppose that the coefficient matrix $A = (a_{ij}(x))$ is a bounded measurable and $m \times m$ symetric matrix satisfying

$$\lambda^{-1} w(x)^{2/p} |\xi|^2 \leq a_{ij}\xi_i\xi_j \leq \lambda w^{2/p}(x)|\xi|^2, \ x \in \Omega, \ \xi \in \mathbb{R}^m, \tag{1.2}$$

where $\lambda \geq 1$ is a constant, $\Omega \subset \mathbb{R}^N$ is a bounded connected domain and w is a Muckenhoupt weight with respect to the Carnot-Carathéodory metric.

Since Hörmander's fundamental work [16], subelliptic equations constructed by vector fields have received a strong impulse and today's literature on the subject is quite numerous (see Bony [1], Fefferman and Phong [10], Nagel, Stein and Wainger [22], etc.). It is generally known that Harnack inequalities are an important tool for regularity research (such as Hölder continuity and continuity) of elliptic equations, see Gilbarg and Trudinger [14], Ladyžhenskaya [17], Chen and Wu [5], etc. As well Harnack inequalities of weak solutions to subelliptic equations have received much concern. Capogna, Danielli and Garofalo in [4] showed the Harnack inequality for quasilinear subelliptic equations, also refer to [18]. Lu in [20] proved the Harnack inequality for second order weighted subelliptic homogeneous equations

$$Lu := -\sum_{i,j=1}^{m} X_i^* \left(a_{ij}\left(x\right) X_j u\right) = 0, \tag{1.3}$$

where $A = (a_{ij}(x))$ satisfies

$$c^{-1} w(x)|\xi|^2 \leq \langle A\xi, \xi\rangle \leq c w(x)|\xi|^2, \ \xi \in \mathbb{R}^m.$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^m$ and $w \in A_2$. Recently, the Harnack inequality of the homogeneous subelliptic p-Laplace equations ($p \geq 2$) with two weights was given by Ferrari [11] and the Hölder continuity for the subelliptic p-Laplace equations with one weight based on [11] was proved by Cruz-Uribe, Moen and Naibo [6]. Related papers see [7, 21] for Harnack inequalities and [8, 9, 26] for Hölder continuities.

In this paper, we first prove the maximum principle for weak solutions to (1.1) by virtue of the weighted Sobolev inequality in [20] and the extension of Moser iteration technique to the weight case. Next the local boundedness and Harnack inequality for the nonnegative weak solutions to (1.1) are derived. The method we used here is inspired by Serrin [24], Capogna, Danielli, and Garofalo [4] and the weighted John-Nirenberg inequality is useful. Finally, we establish the Hölder continuity for weak solutions to (1.1) provided that the right hand term satisfies a condition related to the weighted Morrey space.

The Sobolev inequality for Hörmander vector fields without weight is well known [13, 19]:

$$(1.4)\qquad \left(\frac{1}{|B_r|}\int_{B_r}|f|^q\mathrm{d}x\right)^{\frac{1}{q}}\le Cr\left(\frac{1}{|B_r|}\int_{B_r}|Xf|^p\mathrm{d}x\right)^{\frac{1}{p}},$$

where $f\in C_0^\infty(\bar{B}), 1<p<Q$, the exponent $q\in[p,Qp/(Q-p)]$. Since (1.1) is weighted, we actually need the weighted Sobolev inequality which reads [20]

$$(1.5)\qquad \left(\frac{1}{w(B_r)}\int_{B_r}|f|^q w\mathrm{d}x\right)^{\frac{1}{q}}\le Cr\left(\frac{1}{w(B_r)}\int_{B_r}|Xf|^p w\mathrm{d}x\right)^{\frac{1}{p}},$$

where $q\in[p,Qp/(Q-1)+\delta_p)$, $\delta_p>0$ is a constant depending on p. The difference of q in (1.4) and (1.5) leads to disparity of results and proofs between equations without weight and with weight.

For any $g(x)\in[W^{1,p}(\Omega,w)]^{-1}$, the dual space of the weighted Sobolev space $W^{1,p}(\Omega,w)$, we say that $u\in W^{1,p}(\Omega,w)$ is a weak solution to (1.1) if

$$(1.6)\qquad \int_\Omega\langle A(x)Xu,Xu\rangle^{\frac{p-2}{2}}\langle A(x)Xu,X\varphi(x)\rangle\,\mathrm{d}x=\int_\Omega g(x)\varphi(x)\mathrm{d}x$$

for any $\varphi\in W_0^{1,p}(\Omega,w)$.

Now we are ready to summarize the main results in this paper.

Theorem 1.1 (Maximum principle). *Let $u\in W^{1,p}(B_r,w)$ be a weak solution to (1.1) and $g/w\in L^q(B_r,w)$ $(q>Q)$, then there exists a positive constant C depending only on λ,p,Q,Ω and w, such that*

$$(1.7)\qquad \sup_{B_r}u^+\le\sup_{\partial B_r}u^+ + Cr^{\frac{p}{p-1}}w(B_r)^{-\frac{1}{q(p-1)}}\left\|\frac{g}{w}\right\|_{L^q(B_r,w)}^{\frac{1}{p-1}}.$$

Theorem 1.2 (Harnack inequality). *Let $u\in W^{1,p}(\Omega,w)$ be a nonnegative weak solution to (1.1) and $g/w\in L^q(\Omega,w)(q>Q)$, then there exist positive constants C and R_0, such that for any $0<R\le R_0$, $B_R=B(x,R)$, $B_{4R}\subset\Omega$, we have*

$$(1.8)\qquad \sup_{B_R}u\le C(\inf_{B_R}u+K(R)),$$

and

$$(1.9)\qquad K(R)=\left(R^{pQ/q}w(B_{2R})^{-\frac{1}{q}}\left\|\frac{g}{w}\right\|_{L^q(B_{2R},w)}\right)^{\frac{1}{p-1}}.$$

Specially, if $g=0$, then

$$\sup_{B_R}u\le C\inf_{B_R}u.$$

As an application of Theorem 1.2, we have

THEOREM 1.3 (Hölder continuity). *Let* $u \in W^{1,p}(\Omega, w)$ *be a weak solution to (1.1),* $\sup_{\Omega} |u| = M < \infty$. *Assume* $g/w \in L^{q, pQ-\alpha q(p-1)}(\Omega, w)$, $0 < \alpha < \min\left\{1, \frac{pQ}{q(p-1)}\right\}$, *then u is locally Hölder continuous in* Ω, *that is for any* $\Omega' \subset\subset \Omega$, *it follows*

$$\sup_{x,y\in\Omega'} \frac{|u(x)-u(y)|}{d(x,y)^{\alpha}} \le C. \tag{1.10}$$

Note that (1.1) becomes (1.3) when $p = 2$. Moreover, if $w = 1$, then L_p is the usual subelliptic p-Laplacian and so the same conclusions for corresponding equations can directly be obtained. Since deeper understandings for the measure of metric ball B_R induced from vector fields appeared (see Bramanti [2]), it reminds one that any restriction on the measure of metric ball should be cautious. In this paper there is no restriction in advance for the measures of metric balls.

The article is organized as follows: Section 2 collects some facts for Hörmander's vector fields, A_p weights, weighted Sobolev spaces and their properties. The proof of Theorem 1.1 is given in Section 3 by using the weighted Sobolev inequality and the extension of Moser iteration technique. In Section 4, we prove the local boundedness for nonnegative weak solutions to $L_p u = g$. Section 5 is devoted to the proof of Theorem 1.2 by virtue of the weighted John-Nirenberg inequality and the conclusion in Section 4. The proof of Theorem 1.3 is in Section 6.

2. PRELIMINARIES

Let $X_1, ..., X_m$ $(m \le N)$ be a system of C^∞ vector fields satisfying Hörmander's condition on a neighborhood of $\bar{\Omega}$, *i.e.*, there is a positive integer s such that all commutators of $X_1, ..., X_m$ up to order s span the tangent space of $\mathbb{R}^N$ at every point of Ω (see [16, 22, 23]).

Definition 2.1 (Carnot-Carathéodory distance). An absolutely continuous curve $\gamma : [0, T] \to \Omega$ is called sub-unit with respect to the system $X = (X_1, ..., X_m)$, if whenever $\gamma'(t)$ exists one has that for all $\xi \in \mathbb{R}^N$,

$$<\gamma'(t), \xi>^2 \le \sum_{j=1}^{m} <X_j(\gamma(t)), \xi>^2, \text{ a.e. } t \in [0, T].$$

The length of γ is defined by $l_S(\gamma) = T$. Given $x, y \in \Omega$, we denote by $\Phi(x, y)$ the collection of all sub-unit curves connecting x to y. For any $x, y \in \Omega$, the Carnot-Carathéodory distance generated by X is defined by

$$d(x, y) = \inf\{l_S(\gamma) : \gamma \in \Phi(x, y)\}.$$

We will denote by

$$B_R(x_0) = B(x_0, R) = \{x \in \Omega : d(x_0, x) < R\}$$

the metric ball centered at x_0 of radius R and write simply B_R whenever x_0 is not stressed. The following doubling property holds true (see [22]): there exist constants $c_D, R_D > 0$ such that for any $x_0 \in \Omega$, $0 < 2R < R_D$, $B(x_0, 2R) \subset \Omega$,

$$|B(x_0, 2R)| \leq c_D \, |B(x_0, R)| \, ,$$

It easily implies that for any $R \leq R_D$ and $t \in (0, 1)$,

$$|B_{tR}| \geq c_D^{-1} t^Q \, |B_R|$$

where $Q = \log_2 c_D \geq N$ is called the local homogeneous dimension of Ω with respect to $X_1, ..., X_m$.

Next we describe A_p weights depending on the Carnot-Carathéodory distance here.

Definition 2.2 ([11,20]). Let w be a nonnegative integrable function on Ω and $1 < p < \infty$, we say w is an A_p weight, denoted by $w \in A_p$, if

$$[w]_{A_p} := \sup_{r>0} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} w(x) \mathrm{d}x \right) \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} w(x)^{\frac{1}{1-p}} \mathrm{d}x \right)^{p-1} = C_w < \infty,$$

where C_w is called the A_p constant of w.

Functions in A_p enjoy

$$w(B(x, 2r)) \leq C w(B(x, r)), \;\; x \in \Omega, \;\; r > 0, \tag{2.1}$$

which implies that any $w \in A_p$ defines a measure in $\mathbb{R}^N$.

The weighted L^p space is the set

$$L^p(\Omega, w) = \{u : \Omega \to R : \int_\Omega |u|^p w \mathrm{d}x < +\infty\}$$

with the norm

$$\|u\|_{L^p(\Omega,w)} = \left(\int_\Omega |u|^p w \mathrm{d}x \right)^{1/p}$$

One evidently has the weighted interpolation inequality (its proof is similar to one without weight, see Gilbarg and Trudinger [14])

$$\|h\|_{L^s(\Omega,w)} \leq \varepsilon \|h\|_{L^r(\Omega,w)} + \varepsilon^{-\mu_1} \|h\|_{L^p(\Omega,w)}, \tag{2.2}$$

where $1 < p \le s \le r$, $\mu_1 = \left(\frac{1}{p}-\frac{1}{s}\right)\left(\frac{1}{s}-\frac{1}{r}\right)^{-1}$, and the weighted Hölder inequality

$$\int_\Omega |f(x)g(x)|\, w(x)\mathrm{d}x \le \|f\|_{L^p(\Omega,w)}\|g\|_{L^q(\Omega,w)}, \tag{2.3}$$

where $p>1, q>1, \frac{1}{p}+\frac{1}{q}=1$.

The weighted Sobolev spaces is

$$W^{1,p}(\Omega,w)=\{u\in L^p(\Omega,w): X_j u\in L^p(\Omega,w), j=1,...,m\},$$

with the norm

$$\|u\|_{W^{1,p}(\Omega,w)}=\|u\|_{L^p(\Omega,w)}+\|Xu\|_{L^p(\Omega,w)}.$$

In addition, we denote by $W_0^{1,p}(\Omega,w)$ the closure of smoothly and compactly supported functions in $W^{1,p}(\Omega,w)$ with respect to the norm $\|u\|_{W^{1,p}(\Omega,w)}$.

We recall the following weighted Sobolev inequalities.

Lemma 2.3 ([20], Theorem B). *(1) Let $w\in A_p$, $p>1$ and $E\subset\subset\Omega$, then there exist constants $r_0>0$ and $C>0$, such that for any metric ball $B=B(x,r)\subset\Omega$, $x\in E$, and any $f\in W^{1,p}(\bar{B},w)$, the following inequality holds*

$$\left(\frac{1}{w(B)}\int_B |f-f_B|^q w\mathrm{d}x\right)^{\frac{1}{q}} \le Cr\left(\frac{1}{w(B)}\int_B |Xf|^p w\mathrm{d}x\right)^{\frac{1}{p}}, \tag{2.4}$$

provided $0<r<r_0$, $Q\ge 2$, $p\le q<p\left(Q/(Q-1)+\delta_p\right)$, $\delta_p>0$ only depends on p and C_w, where r_0 and C depend only on C_w, E, Ω and δ_p, f_B means $\frac{1}{w(B)}\int_B fw\mathrm{d}x$;

(2) Let $f\in W_0^{1,p}(\bar{B},w)$, then

$$\left(\frac{1}{w(B)}\int_B |f|^q w\mathrm{d}x\right)^{\frac{1}{q}} \le Cr\left(\frac{1}{w(B)}\int_B |Xf|^p w\mathrm{d}x\right)^{\frac{1}{p}}. \tag{2.5}$$

It is known from (2.5) that that if $1<p<Q$ and $p\le q<p\left(Q/(Q-1)+\delta_p\right)$, then $W_0^{1,p}(B_r,w)$ embeds in $L^q(B_r,w)$.

Now we introduce weighted Morrey spaces.

Definition 2.4. For $1\le p<\infty$, $\vartheta>0$, we say that u is in a weighted Morrey space $L^{p,\vartheta}(\Omega,w)$, if $u\in L^p(\Omega,w)$ and

$$\|u\|_{L^{p,\vartheta}(\Omega,w)} := \sup_{\substack{x\in\Omega\\ 0<r<d_0}} \left(\frac{r^\vartheta}{w(\Omega\cap B(x,r))}\int_{\Omega\cap B(x,r)} |u(y)|^p w(y)\mathrm{d}y\right)^{\frac{1}{p}} < +\infty,$$

where $d_0 = diam(\Omega)$.

3. PROOF OF THEOREM 1.1

We will investigate two cases $g = 0$ and $g \neq 0$, and use (2.5), (2.2) and the Moser iteration technique with weight.

Proof of Theorem 1.1. If $g = 0$, then u is a weak solution to $L_p u = 0$, and u^+ is also. Denoting

$$l = \sup_{\partial B_r} u^+,$$

and picking $\varphi = (u^+ - k)^+ \in W_0^{1,p}(B_r, w)$ (for $k > l$) as a test function to $L_p u = 0$, we get from (1.2) and (2.5) that

$$\begin{aligned} 0 &= \int_{B_r} \langle AXu^+, Xu^+ \rangle^{\frac{p-2}{2}} \left\langle AXu^+, X(u^+ - k)^+ \right\rangle \mathrm{d}x \\ &\geq \lambda^{-\frac{p}{2}} \int_{B_r} \left| X(u^+ - k)^+ \right|^p w \mathrm{d}x \\ &\geq \lambda^{-\frac{p}{2}} C r^{-p} \int_{B_r} \left((u^+ - k)^+ \right)^p w \mathrm{d}x, \end{aligned}$$

which implies $u^+ \leq k$ a.e. in B_r. By the arbitrariness of k,

$$\sup_{B_r} u^+ \leq \sup_{\partial B_r} u^+,$$

so (1.7) is true.

If $g \neq 0$, we prove (1.7) with two steps.

Step 1. We first prove that if $u \in W^{1,p}(B_r, w)$ satisfies

$$\int_{B_r} \langle AXu, Xu \rangle^{\frac{p-2}{2}} \langle AXu, X\varphi \rangle \, \mathrm{d}x \leq \int_{B_r} g\varphi \, \mathrm{d}x \tag{3.1}$$

for any $\varphi \in W_0^{1,p}(B_r, w)$ and $u \leq 0$ on ∂B_r, then there exists a positive constant C such that

(3.2)

$$\sup_{B_r} u^+ \leq C w(B_r)^{-\frac{1}{p}} \left(\left\| u^+ \right\|_{L^p(B_r, w)} + r^{\frac{p}{p-1}} w(B_r)^{\frac{1}{p} - \frac{1}{q(p-1)}} \left\| g/w \right\|_{L^q(B_r, w)}^{\frac{1}{p-1}} \right).$$

In fact, define

$$H(z) = \begin{cases} z^\beta - k^\beta, & z \in [k, M]; \\ \beta M^{\beta-1}(z - M) + M^\beta - k^\beta, & z \in [M, +\infty), \end{cases} \tag{3.3}$$

where $\beta > 1$ and $k > 0$ is to be chosen later, and denote

$$G(t) = \begin{cases} \int_k^t (H'(s))^p ds, & t \in [k, +\infty); \\ (H'(k))^p (t - k), & t \in (-\infty, k), \end{cases}$$

$$h(x) = u^+(x) + k,$$

$$\varphi(x)=G(h(x))=\int_k^{h(x)}\big(H'(s)\big)^p\mathrm{d}s.$$

Then $G'(h)=(H'(h))^p>0$, $\varphi\geq 0$. Using $u\leq 0$ on ∂B_r, it follows that $h(x)=k$ and $\varphi(x)=0$ on ∂B_r, and then $\varphi\in W_0^{1,p}(B_r,w)$. Taking $\varphi(x)$ into (3.1) and using (1.2), we have

$$\begin{aligned}(3.4)\quad \lambda^{-\frac{p}{2}}\int_{B_r}|Xh|^pG'(h)w\mathrm{d}x &\leq \int_{B_r}\langle AXu,Xu\rangle^{\frac{p-2}{2}}\big\langle AXu,G'(h)Xh\big\rangle\,\mathrm{d}x\\ &= \int_{B_r}\langle AXu,Xu\rangle^{\frac{p-2}{2}}\langle AXu,X\left(G(h)\right)\rangle\,\mathrm{d}x\\ &\leq \int_{B_r} g\,G(h)\mathrm{d}x.\end{aligned}$$

Noting

$$H''(z)=\begin{cases}\beta(\beta-1)z^{\beta-2}, & z\in[k,M];\\ 0, & z\in[M,+\infty),\end{cases}$$

we know that $H''(z)\geq 0$ and so $H'(z)$ is increasing, thus

$$G(h)\leq(h-k)\big(H'(h)\big)^p\leq hG'(h). \tag{3.5}$$

Taking (3.5) into (3.4) and using $h(x)\geq k$, it follows

$$\lambda^{-\frac{p}{2}}\int_{B_r}|Xh|^pG'(h)w\mathrm{d}x\leq\int_{B_r}|g|\,hG'(h)\,\mathrm{d}x\leq\frac{1}{k^{p-1}}\int_{B_r}|g|\,\big|hH'(h)\big|^p\mathrm{d}x,$$

therefore

(3.6)

$$\int_{B_r}|XH(h)|^pw\mathrm{d}x=\int_{B_r}|Xh|^pG'(h)w\mathrm{d}x\leq\lambda^{\frac{p}{2}}\frac{1}{k^{p-1}}\int_{B_r}|g|\,\big|hH'(h)\big|^p\mathrm{d}x.$$

In terms of $u\leq 0$ on ∂B_r and $H(k)=0$, we get $H(h)\in W_0^{1,p}(B_r,w)$. Using (2.5), (3.6) and (2.3) shows

$$\begin{aligned}\left(\int_{B_r}|H(h)|^{\frac{Qp}{Q-1}}w\mathrm{d}x\right)^{\frac{Q-1}{Q}} &\leq Cr^pw(B_r)^{-\frac{1}{Q}}\int_{B_r}|XH(h)|^pw\mathrm{d}x\\ &\leq Cr^pw(B_r)^{-\frac{1}{Q}}\lambda^{\frac{p}{2}}\frac{1}{k^{p-1}}\int_{B_r}|g|\,\big|hH'(h)\big|^p\mathrm{d}x\\ &\leq C\lambda^{\frac{p}{2}}\frac{1}{k^{p-1}}r^pw(B_r)^{-\frac{1}{Q}}\left(\int_{B_r}|g/w|^q w\mathrm{d}x\right)^{\frac{1}{q}}\left(\int_{B_r}\big|hH'(h)\big|^{pq'}w\mathrm{d}x\right)^{\frac{1}{q'}},\end{aligned}$$

where $q'=\frac{q}{q-1}$. Taking in (3.3)

$$k=r^{\frac{p}{p-1}}w(B_r)^{-\frac{1}{q(p-1)}}\,\|g/w\|_{L^q(B_r,w)}^{\frac{1}{p-1}}, \tag{3.7}$$

then

$$\|H(h)\|_{L^{\frac{Qp}{Q-1}}(B_r,w)} \le Cw(B_r)^{\frac{1}{pq}-\frac{1}{pQ}} \left\|hH'(h)\right\|_{L^{pq'}(B_r,w)}, \tag{3.8}$$

where $C = C(\lambda, p, Q, w, \Omega)$. Letting $M \to \infty$ in (3.3) and substituting $H(h) = h^\beta - k^\beta$ and $H'(h) = \beta h^{\beta-1}$ into (3.8), we obtain

$$\left\|h^\beta - k^\beta\right\|_{L^{\frac{Qp}{Q-1}}(B_r,w)} \le C\beta w(B_r)^{\frac{1}{pq}-\frac{1}{pQ}} \|h\|^\beta_{L^{\beta pq'}(B_r,w)}. \tag{3.9}$$

Noting $k \le h(x)$, it yields from (3.9) that

$$\begin{aligned}\left(\int_{B_r} h^{\beta\frac{Qp}{Q-1}} w\mathrm{d}x\right)^{\frac{Q-1}{Qp}} &\le \left(\int_{B_r} \left|h^\beta - k^\beta\right|^{\frac{Qp}{Q-1}} w\mathrm{d}x\right)^{\frac{Q-1}{Qp}} + \left(\int_{B_r} k^{\beta\frac{Qp}{Q-1}} w\mathrm{d}x\right)^{\frac{Q-1}{Qp}}\\ &\le C\beta w(B_r)^{\frac{1}{pq}-\frac{1}{pQ}} \|h\|^\beta_{L^{\beta pq'}(B_r,w)} + w(B_r)^{\frac{1}{pq}-\frac{1}{pQ}} \left(\int_{B_r} k^{\beta pq'} w\mathrm{d}x\right)^{\frac{1}{pq'}}\\ &\le C\beta w(B_r)^{\frac{1}{pq}-\frac{1}{pQ}} \|h\|^\beta_{L^{\beta pq'}(B_r,w)},\end{aligned}$$

that is

$$\|h\|_{L^{\beta\frac{Qp}{Q-1}}(B_r,w)} \le (C\beta)^{\frac{1}{\beta}} w(B_r)^{\frac{1}{\beta}(\frac{1}{pq}-\frac{1}{pQ})} \|h\|_{L^{\beta pq'}(B_r,w)}.$$

Letting $\mu = \frac{Qp/(Q-1)}{pq'} > 1$, the inequality above becomes

$$\|h\|_{L^{\beta pq'\mu}(B_r,w)} \le (C\beta)^{\frac{1}{\beta}} w(B_r)^{\frac{1}{\beta}(\frac{1}{pq}-\frac{1}{pQ})} \|h\|_{L^{\beta pq'}(B_r,w)}. \tag{3.10}$$

Now setting $\beta = \mu^j$ $(j = 0, 1, ...)$ in (3.10) leads to

$$\|h\|_{L^{pq'\mu^{j+1}}(B_r,w)} \le (C\mu^j)^{\mu^{-j}} w(B_r)^{\mu^{-j}(\frac{1}{pq}-\frac{1}{pQ})} \|h\|_{L^{\mu^j pq'}(B_r,w)},$$

and iterating these inequalities gives

$$\begin{aligned}\|h\|_{L^{pq'\mu^{m+1}}(B_r,w)} &\le \prod_{j=0}^m (C\mu^j)^{\mu^{-j}} \prod_{j=0}^m w(B_r)^{\mu^{-j}(\frac{1}{pq}-\frac{1}{pQ})} \|h\|_{L^{pq'}(B_r,w)}\\ &= C^{\sum_{j=0}^m \mu^{-j}} \mu^{\sum_{j=0}^m j\mu^{-j}} w(B_r)^{\sum_{j=0}^m \mu^{-j}(\frac{1}{pq}-\frac{1}{pQ})} \|h\|_{L^{pq'}(B_r,w)}.\end{aligned}$$

Letting $m \to \infty$, we have

$$\begin{aligned}\|h\|_{L^\infty(B_r,w)} &\le C^{\sum_{j=0}^\infty \mu^{-j}} \mu^{\sum_{j=0}^\infty j\mu^{-j}} w(B_r)^{\sum_{j=0}^\infty \mu^{-j}(\frac{1}{pq}-\frac{1}{pQ})} \|h\|_{L^{pq'}(B_r,w)}\\ &\le C^\sigma \mu^\tau w(B_r)^{\sigma(\frac{1}{pq}-\frac{1}{pQ})} \|h\|_{L^{pq'}(B_r,w)},\end{aligned} \tag{3.11}$$

where $\sigma = 1/(1-\mu^{-1})$, $\tau = \mu/(\mu-1)^2$. Using (2.2) with $s = pq'$, $p = p$ and $r = \infty$,

$$\|h\|_{L^{pq'}(B_r,w)} \le \varepsilon\|h\|_{L^\infty(B_r,w)} + \varepsilon^{1-1/q'} \|h\|_{L^p(B_r,w)},$$

and substituting it into (3.11), we have

$$\|h\|_{L^\infty(B_r,w)} \le C^\sigma \mu^\tau w(B_r)^{\sigma(\frac{1}{pq}-\frac{1}{pQ})}\left(\varepsilon\|h\|_{L^\infty(B_r,w)}+\varepsilon^{1-q'}\|h\|_{L^p(B_r,w)}\right),$$

Take $\varepsilon=(q'-1)^{\frac{1}{q'}}\|h\|_{L^\infty(U,w)}^{-\frac{1}{q'}}\|h\|_{L^p(U,w)}^{\frac{1}{q'}}$, then

$$\begin{aligned}&\|h\|_{L^\infty(B_r,w)}\\ &\le C^\sigma \mu^\tau w(B_r)^{\sigma(\frac{1}{pq}-\frac{1}{pQ})}\left((q'-1)^{\frac{1}{q'}}+(q'-1)^{\frac{1-q'}{q'}}\right)\|h\|_{L^\infty(B_r,w)}^{1-\frac{1}{q'}}\|h\|_{L^p(B_r,w)}^{\frac{1}{q'}}.\end{aligned}$$

and by $\sigma q'(\frac{1}{pq}-\frac{1}{pQ})=-\frac{1}{p}$,

$$\|h\|_{L^\infty(B_r,w)}\le Cw(B_r)^{-\frac{1}{p}}\|h\|_{L^p(B_r,w)},$$

where $C=C^{\sigma q'}\mu^{\tau q'}\left((q'-1)^{\frac{1}{q'}}+(q'-1)^{\frac{1-q'}{q'}}\right)^{q'}$. Noting $h(x)=u^+(x)+k$ and (3.7), it follows

$$\begin{aligned}\sup_{B_r}u^+ &\le \sup_{B_r}u^+ +k=\|h\|_{L^\infty(B_r,w)}\\ &\le Cw(B_r)^{-\frac{1}{p}}\|h\|_{L^p(B_r,w)}\\ &\le Cw(B_r)^{-\frac{1}{p}}\left(\|u^+\|_{L^p(B_r,w)}+k\right).\end{aligned}$$

This completes the proof of (3.2).

Step 2. We are ready to prove (1.7). Letting

$$l=\sup_{\partial B_r}u^+,\ L=\sup_{B_r}u^+.$$

We only need to consider the case $l<\infty$, because the conclusion for $l=\infty$ holds obviously. Let us distinguish the two subcases $l=0$ and $l\neq 0$ for $l<\infty$.

If $l=0$, then $u^+=0$ on ∂B_r. Take

$$\varphi=(L+k-u^+)^{1-p}-(L+k)^{1-p},$$

then $\varphi\le k^{1-p}$, $\varphi=0$ on ∂B_r, $\varphi\in W_0^{1,p}(B_r,w)$ from $u^+\in W_0^{1,p}(B_r,w)$,

$$X\varphi=(p-1)(L+k-u^+)^{-p}Xu^+.$$

Plugging φ into (3.1) and noting (1.2),

$$\begin{aligned}(3.12)\quad &(p-1)\lambda^{-\frac{p}{2}}\int_{B_r}\frac{|Xu^+|^p}{(L+k-u^+)^p}w\mathrm{d}x\le\int_{B_r}\langle AXu,Xu\rangle^{\frac{p-2}{2}}\langle AXu,X\varphi\rangle\,\mathrm{d}x\\ &\le\int_{B_r}|g|\,|\varphi|\,\mathrm{d}x\le\int_{B_r}\frac{g}{(L+k-u^+)^{p-1}}\mathrm{d}x\le k^{1-p}\int_{B_r}\frac{|g|}{w}w\mathrm{d}x.\end{aligned}$$

Setting $v=\log\frac{L+k}{L+k-u^+}$, it yields $Xv=\frac{Xu^+}{L+k-u^+}$ and by (3.12) and (3.7),

$$\int_{B_r}|Xv|^p w\mathrm{d}x\le Ck^{1-p}w(B_r)^{1-\frac{1}{q}}\left\|\frac{g}{w}\right\|_{L^q(B_r,w)}$$
$$\le Cr^{-p}w(B_r).$$

Using (2.5) we obtain

$$\int_{B_r}|v|^p w\mathrm{d}x\le Cw(B_r). \tag{3.13}$$

Taking $\psi=\frac{\eta}{(L+k-u^+)^{p-1}}$ as a test function in (1.6), $\eta\in W_0^{1,p}(B_r,w)$, $\eta\subset\sup pu^+$, and noting

$$X\psi=\frac{X\eta}{(L+k-u^+)^{p-1}}+(p-1)\frac{\eta Xu^+}{(L+k-u^+)^p},$$

it follows

$$\begin{aligned}&\int_{B_r}\left\langle A(x)Xu^+,\ Xu^+\right\rangle^{\frac{p-2}{2}}\left\langle A(x)Xu^+,\ \frac{X\eta}{(L+k-u^+)^{p-1}}\right\rangle\mathrm{d}x\\&+\int_{B_r}\left\langle A(x)Xu^+,Xu^+\right\rangle^{\frac{p-2}{2}}\left\langle A(x)Xu^+,\ \frac{(p-1)\eta}{(L+k-u^+)^p}Xu^+\right\rangle\mathrm{d}x\\&=\int_{B_r}g(x)\frac{\eta}{(L+k-u^+)^{p-1}}\,\mathrm{d}x.\end{aligned}$$

Since the second term in the left-hand side is non-negative, we have

$$\int_{B_r}\left\langle A(x)Xv,\ Xv\right\rangle^{\frac{p-2}{2}}\left\langle A(x)Xv,\ X\eta\right\rangle\mathrm{d}x\le\int_{B_r}|g(x)|\,\frac{\eta}{k^{p-1}}\,\mathrm{d}x.$$

Thanks to $v\in W^{1,p}(B_r,w)$, $v=0$ on ∂B_r and (3.2),

$$\sup_{B_r}v\le Cw(B_r)^{-\frac{1}{p}}(\|v\|_{L^p(B_r,w)}+r^{\frac{p}{p-1}}w(B_r)^{\frac{1}{p}-\frac{1}{q(p-1)}}\frac{1}{k}\,\|g/w\|_{L^q(B_r,w)}^{\frac{1}{p-1}}),$$

then by (3.7) and (3.13),

$$\sup_{B_r}v\le Cw(B_r)^{-\frac{1}{p}}(w(B_r)^{\frac{1}{p}}+w(B_r)^{\frac{1}{p}})\le C. \tag{3.14}$$

Using $v=\log\frac{L+k}{L+k-u^+}$ into (3.14), we have

$$L\le k(e^C-1),$$

which means

$$\sup_{B_r}u^+\le(e^C-1)r^{\frac{p}{p-1}}w(B_r)^{-\frac{1}{q(p-1)}}\,\|g/w\|_{L^q(B_r,w)}^{\frac{1}{p-1}}. \tag{3.15}$$

If $l\neq 0$, then $u-l$ is also a weak solution to $L_pu=g$ and satisfies $\sup_{\partial B_r}(u-l)^+=0$. Applying (3.15) to $u-l$, it yields (1.7).

4. LOCAL BOUNDEDNESS

In this section, we use the Moser iteration technique with weight to prove the local boundedness for weak solutions to (1.1). A known result is necessary.

Lemma 4.1 ([24], Lemma 2). *Let α be a positive exponent and $0 < \alpha_i < \infty$ and $0 \le \beta_i < \alpha, i = 1, ..., N$. Suppose that z is positive satisfying*

$$z^\alpha \le \sum_{i=1}^N \alpha_i z^{\beta_i}.$$

Then

$$z \le C \sum_{i=1}^N (\alpha_i)^{\gamma_i},$$

where $\gamma_i = (\alpha - \beta_i)^{-1}$, C depends only N, α and β_i.

Lemma 4.2. *Suppose that $u \in W^{1,p}(\Omega, w)$ is a weak solution to (1.1) and $g/w \in L^q(\Omega, w)$ $(q > Q)$. Then there exist $C > 0$ and $R_0 > 0$ such that for any $0 < R \le R_0$, $B_{4R} \subset \Omega$, $B_R = B(x, R)$, we have*

$$\sup_{B_R} |u| \le C \left(\left(\frac{1}{w(B_{2R})} \int_{B_{2R}} |u|^p w \mathrm{d}x \right)^{1/p} + K(R) \right), \tag{4.1}$$

where $K(R)$ is stated in (1.9).

Proof. For $x \in \Omega$, choose $R_0 > 0$ sufficiently small such that $B_{4R} = B(x, 4R) \subset \Omega$ for any $0 < R \le R_0$. So inequalities in Lemma 2.3 hold for B_{4R}. Set $\bar{u} = |u| + K$, $K = K(R)$, then $X\bar{u} = X|u|$.

For $l > K$ and $q_0 \ge 1$, define

$$F(\bar{u}) = \begin{cases} \bar{u}^{q_0}, & K \le \bar{u} \le l, \\ q_0 l^{q_0 - 1} \bar{u} - (q_0 - 1) l^{q_0}, & \bar{u} > l, \end{cases} \tag{4.2}$$

and

$$G(u) = \operatorname{sign} u \cdot \left(F(\bar{u}) F'(\bar{u})^{p-1} - {q_0}^{p-1} K^\beta \right), -\infty < u < \infty,$$

where β satisfies $pq_0 = p + \beta - 1$. Clearly, $\beta \ge 1, F$ is continuously differential and G is piecewisely smooth. It is easy to calculate $F'' \ge 0$ and

$$FF'^{p-1} = \begin{cases} {q_0}^{p-1} \bar{u}^\beta, & K \le \bar{u} \le l, \\ (q_0 l^{q-1})^{p-1} \left(q_0 l^{q-1} \bar{u} - (q_0 - 1) l^{q_0} \right), & \bar{u} > l. \end{cases}$$

Set $v = F(\bar{u})$, then for $K \le \bar{u} \le l$,

$$FF'^{p-1} = q_0^{p-1} \bar{u}^{\beta + p - 1} \bar{u}^{1-p} \le q_0^{p-1} K^{1-p} v^p,$$

$$FF'^{p-1} \geq q_0^{p-1} K^\beta;$$

and for $\bar{u} > l$,

$$\begin{aligned} FF'^{p-1} &= (q_0 l^{q_0-1})^{p-1} \big(q_0 l^{q_0-1}\bar{u} - (q_0-1) l^{q_0}\big)^p \big(q_0 l^{q_0-1}\bar{u} - (q_0-1)l^{q_0}\big)^{1-p} \\ &\leq v^p (q_0 l^{q_0-1})^{p-1} l^{q_0(1-p)} \\ &\leq q_0^{p-1} K^{1-p} v^p, \end{aligned}$$

$$FF'^{p-1} \geq (q_0 l^{q_0-1})^{p-1} \left(q_0 l^{q_0-1} l - (q_0-1) l^{q_0}\right) = (q_0 l^{q_0-1})^{p-1} l^{q_0} \geq q_0^{p-1} K^\beta.$$

Then

$$|G| = \left|FF'^{p-1} - q_0^{p-1} K^\beta\right| \leq FF'^{p-1} \leq q_0^{p-1} K^{1-p} v^p. \tag{4.3}$$

Let us consider two cases $|u| \neq l - K$ and $|u| = l - K$, respectively. For $|u| \neq l - K$, define

$$\varphi = \eta^p G(u),$$

where $\eta \in C_0^\infty(B_{2R}), 0 \leq \eta \leq 1$, then $\varphi \in W_0^{1,p}(\Omega, w)$ and

$$X\varphi = p\eta^{p-1} G(u) X\eta + \eta^p G'(u) Xu$$

with $G'(u) \geq F'(\bar{u})^p$. Taking φ into (1.6), we obtain from (1.2) and (4.3) that

$$\begin{aligned} 0 = &\int_\Omega \langle A(x)Xu, Xu\rangle^{\frac{p-2}{2}} \langle A(x)Xu, Xu\rangle \eta^p G'(u) \mathrm{d}x \\ &+ \int_\Omega \langle A(x)Xu, Xu\rangle^{\frac{p-2}{2}} \langle A(x)Xu, X\eta\rangle p\eta^{p-1} G(u)\mathrm{d}x - \int_\Omega g(x)\eta^p G(u)\mathrm{d}x \\ \geq &\lambda^{-\frac{p}{2}} \int_{B_{2R}} \eta^p (F')^p |Xu|^p w\mathrm{d}x - p\lambda^{\frac{p}{2}} \int_{B_{2R}} \eta^{p-1} F(F')^{p-1} |Xu|^{p-1} |X\eta| w\mathrm{d}x \\ &- q_0^{p-1} K^{1-p} \int_{B_{2R}} \left|\frac{g}{w}\right| \eta^p v^p w\mathrm{d}x \\ \geq &\lambda^{-\frac{p}{2}} \int_{B_{2R}} |\eta X v|^p w\mathrm{d}x - p\lambda^{\frac{p}{2}} \int_{B_{2R}} |\eta Xv|^{p-1} |vX\eta| w\mathrm{d}x - q_0^{p-1} K^{1-p} \\ &\int_{B_{2R}} \left|\frac{g}{w}\right| (\eta v)^p w\mathrm{d}x. \end{aligned} \tag{4.4}$$

For the case $|u| = l - K$, it follows $Xu = X\bar{u} = 0$ and $X\varphi = p\eta^{p-1}G(u)X\eta$ for $\varphi = \eta^p G(u)$, therefore (4.4) also holds.

Applying (2.3), (2.5) and (1.9) into (4.4), we have

$$\begin{aligned} &q_0^{p-1} K^{1-p} \int_{B_{2R}} \left|\frac{g}{w}\right| (\eta v)^p w\mathrm{d}x \\ &\leq q_0^{p-1} K^{1-p} \left\|\frac{g}{w}\right\|_{L^q(B_{2R},w)} \|\eta v\|_{L^p(B_{2R},w)}^{p(q-Q)/q} \|\eta v\|_{L^{\frac{pQ}{Q-1}}(B_{2R},w)}^{pQ/q} \end{aligned} \tag{4.5}$$

$$\leq Cq_0^{p-1}K^{1-p}\Big\|\frac{g}{w}\Big\|_{L^q(B_{2R},w)}\|\eta v\|_{L^p(B_{2R},w)}^{p(q-Q)/q}\left(2Rw(B_{2R})^{\frac{Q-1}{pQ}-\frac{1}{p}}\right)^{pQ/q}$$
$$\|X(\eta v)\|_{L^p(B_{2R},w)}^{pQ/q}$$
$$\leq Cq_0^{p-1}\|\eta v\|_{L^p(B_{2R},w)}^{p(q-Q)/q}\left(\|\eta Xv\|_{L^p(B_{2R},w)}^{pQ/q}+\|vX\eta\|_{L^p(B_{2R},w)}^{pQ/q}\right).$$

Using (2.3) and (4.5) into (4.4),

$$\begin{aligned}\|\eta Xv\|_{L^p(B_{2R},w)}^p \leq & C\bigg(\int_{B_{2R}}|\eta Xv|^{p-1}|vX\eta|\,w\mathrm{d}x+q_0^{p-1}K^{1-p}\int_{B_{2R}}\Big|\frac{g}{w}\Big|(\eta v)^p w\mathrm{d}x\bigg)\\ \leq & C\|\eta Xv\|_{L^p(B_{2R},w)}^{p-1}\|vX\eta\|_{L^p(B_{2R},w)}\\ & +Cq_0^{p-1}\|\eta v\|_{L^p(B_{2R},w)}^{p(q-Q)/q}\Big(\|\eta Xv\|_{L^p(B_{2R},w)}^{pQ/q}+\|vX\eta\|_{L^p(B_{2R},w)}^{pQ/q}\Big).\end{aligned}$$

Noting $q_0\geq 1,\ 0<\frac{p(q-Q)}{q}<p$ and using Lemma 4.1, we have

$$\begin{aligned}\|\eta Xv\|_{L^p(B_{2R},w)} & \\ \leq & C\|vX\eta\|_{L^p(B_{2R},w)}+Cq_0^{\frac{q(p-1)}{p(q-Q)}}\|\eta v\|_{L^p(B_{2R},w)}\\ & +Cq_0^{\frac{p-1}{p}}\|\eta v\|_{L^p(B_{2R},w)}^{\frac{q-Q}{q}}\|vX\eta\|_{L^p(B_{2R},w)}^{\frac{Q}{q}}\\ \leq & C\|vX\eta\|_{L^p(B_{2R},w)}+Cq_0^{\frac{q(p-1)}{p(q-Q)}}\|\eta v\|_{L^p(B_{2R},w)}\\ & +Cq_0^{\frac{p-1}{p}}\Big(\|\eta v\|_{L^p(B_{2R},w)}+\|vX\eta\|_{L^p(B_{2R},w)}\Big)\\ \leq & C(1+q_0^{\frac{p-1}{p}})\|vX\eta\|_{L^p(B_{2R},w)}+C(q_0^{\frac{q(p-1)}{p(q-Q)}}+q_0^{\frac{p-1}{p}})\|\eta v\|_{L^p(B_{2R},w)}\\ \leq & Cq_0^{\frac{q}{q-Q}}(\|vX\eta\|_{L^p(B_{2R},w)}+\|\eta v\|_{L^p(B_{2R},w)}).\end{aligned}\tag{4.6}$$

From (2.5) and (4.6), it gets

$$\begin{aligned}&\left(\frac{1}{w(B_{2R})}\int_{B_{2R}}(\eta v)^{k_0p}w\mathrm{d}x\right)^{\frac{1}{k_0p}}\\ \leq & CR\bigg(\frac{1}{w(B_{2R})}\int_{B_{2R}}|X(\eta v)|^p w\mathrm{d}x\bigg)^{\frac{1}{p}}\\ \leq & CRw(B_R)^{-\frac{1}{p}}(\|vX\eta\|_{L^p(B_{2R},w)}+\|\eta Xv\|_{L^p(B_{2R},w)})\\ \leq & CRw(B_R)^{-\frac{1}{p}}\left(\|vX\eta\|_{L^p(B_{2R},w)}+Cq_0^{\frac{q}{q-Q}}(\|vX\eta\|_{L^p(B_{2R},w)}+\|\eta v\|_{L^p(B_{2R},w)})\right)\\ \leq & Cq_0^{\frac{q}{q-Q}}Rw(B_{2R})^{-\frac{1}{p}}(\|vX\eta\|_{L^p(B_{2R},w)}+\|\eta v\|_{L^p(B_{2R},w)}),\end{aligned}\tag{4.7}$$

where $k_0 = \frac{Q}{Q-1}$. Now select $\eta \in C_0^\infty(B_{bR})$ in (4.7) such that $\eta = 1$ in B_{aR} and $|X\eta| \le \frac{C}{(b-a)R}$, $1 \le a < b \le 2$, then

$$\begin{aligned}\left(\int_{B_{aR}} v^{k_0 p} w \mathrm{d}x\right)^{\frac{1}{k_0 p}} &\le C q_0^{\frac{q}{q-Q}} w(B_R)^{\frac{1}{k_0 p}-\frac{1}{p}} R\left(\frac{C}{(b-a)R}+1\right)\left(\int_{B_{bR}} v^p w \mathrm{d}x\right)^{\frac{1}{p}} \\ &\le C q_0^{\frac{q}{q-Q}} w(B_R)^{\frac{1}{k_0 p}-\frac{1}{p}} \frac{C+d(\Omega)}{(b-a)}\left(\int_{B_{bR}} v^p w \mathrm{d}x\right)^{\frac{1}{p}} \\ &\le C q_0^{\frac{q}{q-Q}} w(B_R)^{\frac{1}{k_0 p}-\frac{1}{p}} \frac{1}{b-a}\left(\int_{B_{bR}} v^p w \mathrm{d}x\right)^{\frac{1}{p}}.\end{aligned}$$

Letting $l \to \infty$ in (4.2), we get $v = \bar{u}^{q_0}$ and so

$$\begin{aligned}(4.8)\quad &\left(\int_{B_{aR}} \bar{u}^{k_0 p q_0} w \mathrm{d}x\right)^{\frac{1}{k_0 p q_0}} \\ &\le C^{\frac{1}{q_0}} q_0^{\frac{q}{(q-Q)q_0}} w(B_R)^{\frac{1}{q_0}\left(\frac{1}{k_0 p}-\frac{1}{p}\right)}\left(\frac{1}{b-a}\right)^{\frac{1}{q_0}}\left(\int_{B_{bR}} \bar{u}^{p q_0} w \mathrm{d}x\right)^{\frac{1}{p q_0}}.\end{aligned}$$

Taking $q_0 = k_0^i,\ \theta_i = p k_0^i,\ b = 1 + 2^{-i},\ a = 1 + 2^{-(i+1)},\ i = 0, 1, ...$, then (4.8) becomes

$$\begin{aligned}&\left(\int_{B_{\gamma_{i+1}R}} \bar{u}^{\theta_{i+1}} w \mathrm{d}x\right)^{\frac{1}{\theta_{i+1}}} \\ &\le C^{k_0^{-i}} \left(k_0^i\right)^{\frac{q}{q-Q} k_0^{-i}} w(B_R)^{k_0^{-i}\left(\frac{1}{k_0 p}-\frac{1}{p}\right)}\left(\frac{1}{b-a}\right)^{k_0^{-i}}\left(\int_{B_{\gamma_i R}} \bar{u}^{\theta_i} w \mathrm{d}x\right)^{\frac{1}{\theta_i}},\end{aligned}$$

and by iterating m times,

$$\begin{aligned}(4.9)\quad \left(\int_{B_{\gamma_{m+1}R}} \bar{u}^{\theta_{i+1}} w \mathrm{d}x\right)^{\frac{1}{\theta_{i+1}}} &\le C^{\sum\limits_{i=0}^{m} k_0^{-i}} \left(k_0^{\frac{q}{q-Q}}\right)^{\sum\limits_{i=0}^{m} i k_0^{-i}} \\ &\quad \cdot w(B_R)^{\left(\frac{1}{k_0 p}-\frac{1}{p}\right)\sum\limits_{i=0}^{m} k_0^{-i}} 2^{\sum\limits_{i=0}^{m}(i+1)k_0^{-i}}\left(\int_{B_{2R}} \bar{u}^p w \mathrm{d}x\right)^{\frac{1}{p}},\end{aligned}$$

Since $\sum\limits_{i=0}^{\infty} k_0^{-i},\ \sum\limits_{i=0}^{\infty} i k_0^{-i}$ and $\sum\limits_{i=0}^{\infty} (i+1) k_0^{-i}$ are all convergent, and $\left(\frac{1}{k_0 p} - \frac{1}{p}\right)\sum\limits_{i=0}^{\infty} k_0^{-i} = -\frac{1}{p}$, we obtain by letting $m \to \infty$ in (4.9) that

$$(4.10)\qquad \sup_{B_R} \bar{u} \le C\left(\frac{1}{w(B_{2R})}\int_{B_{2R}} \bar{u}^p w \mathrm{d}x\right)^{1/p},$$

where $\bar{u} = |u| + K$. Thus (4.1) is proved.

Lemma 4.3. *Let $u \in W^{1,p}(\Omega, w)$ be a weak solution to (1.1) and $g/w \in L^q(\Omega, w)$ $(q > Q)$, then there exist $C > 0$ and $R_0 > 0$ such that for any $0 < R \leq R_0$ and $\alpha > 0$, we have*

$$\sup_{B_R} \bar{u} \leq C\left(\frac{1}{w(B_{2R})}\int_{B_{2R}} \bar{u}^\alpha w \mathrm{d}x\right)^{1/\alpha}, \tag{4.11}$$

where $B_R = B(x,R), B_{4R} \subset \Omega$.

Proof. Since (4.11) is just (4.10) for $\alpha = p$, the remaining is to prove (4.11) for $\alpha \neq p$. We denote $q_0 = k_0^i,\ \theta_i = pk_0^i,\ k_0 = \frac{Q}{Q-1}$ in (4.8), instead $b,\ a$ with $a + (b-a)^{i+1},\ a + (b-a)^{i+2}$, and iterate m times to obtain

$$\left(\int_{B_{\chi_{m+1}R}} \bar{u}^{\theta_{i+1}} w \mathrm{d}x\right)^{\frac{1}{\theta_{i+1}}} \leq C^{\sum\limits_{i=0}^{m} k_0^{-i}} \left(k_0^{\frac{q}{q-Q}}\right)^{\sum\limits_{i=0}^{m} ik_0^{-i}} w(B_R)^{\left(\frac{1}{k_0 p} - \frac{1}{p}\right)\sum\limits_{i=0}^{m} k_0^{-i}}$$
$$\cdot (b-a)^{-\sum\limits_{i=0}^{m}(i+1)k_0^{-i}} \left(\int_{B_{bR}} \bar{u}^p w \mathrm{d}x\right)^{\frac{1}{p}}.$$

Letting $m \to \infty$ and noting $\sum\limits_{i=0}^{\infty}(i+1)k_0^{-i} = Q^2$, it has

$$\sup_{B_{aR}} \bar{u} \leq C(b-a)^{-Q^2}\left(\frac{1}{w(B_{bR})}\int_{B_{bR}} \bar{u}^p w \mathrm{d}x\right)^{1/p}. \tag{4.12}$$

Without loss of generality we assume $0 < \alpha < p$ (the case $\alpha > p$ can be treated by Hölder's inequality) and denote

$$J(s) = w(B_R)^{\frac{p}{\alpha}-1}\left(\int_{B_{2sR}} \bar{u}^p w \mathrm{d}x\right)\left(\int_{B_{2R}} \bar{u}^\alpha w \mathrm{d}x\right)^{-\frac{p}{\alpha}}.$$

Taking $a = 1,\ b = 4/3$ in (4.12) and applying (2.1), it yields

(4.13)

$$\begin{aligned}
\sup_{B_R} \bar{u}^\alpha &\leq C\left(\frac{1}{w(B_{4R/3})}\int_{B_{4R/3}} \bar{u}^p w \mathrm{d}x\right)^{\alpha/p} \\
&\leq C\left(\frac{1}{w(B_R)}\int_{B_{2R}} \bar{u}^\alpha w \mathrm{d}x\right) w(B_R)^{\frac{\alpha}{p}(\frac{p}{\alpha}-1)}\left(\int_{B_{4R/3}} \bar{u}^p w \mathrm{d}x\right)^{\frac{\alpha}{p}}\left(\int_{B_{2R}} \bar{u}^\alpha w \mathrm{d}x\right)^{-1} \\
&= \left(\frac{C}{w(B_R)}\int_{B_{2R}} \bar{u}^\alpha w \mathrm{d}x\right) J\left(\frac{2}{3}\right)^{\frac{\alpha}{p}}.
\end{aligned}$$

We claim that there exists $C > 0$ independent of R, such that

$$J\left(\frac{2}{3}\right) \leq C. \tag{4.14}$$

In fact, it is enough to prove (4.14) under the assumption $J\left(\frac{2}{3}\right) > 1$. For $\frac{1}{2} < s_1 < s_2 \le 1$, we have by using (4.12) ($a = 2s_1,\ b = 2s_2$) and (2.1) that

$$\begin{aligned} J(s_1) &\le w(B_R)^{\frac{p}{\alpha}-1} \sup_{B_{2s_1R}} \bar{u}^{p-\alpha} \left(\int_{B_{2s_1R}} \bar{u}^{\alpha} w \mathrm{d}x\right) \left(\int_{B_{2R}} \bar{u}^{\alpha} w \mathrm{d}x\right)^{-\frac{p}{\alpha}} \\ &\le w(B_R)^{\frac{p}{\alpha}-1} \frac{C}{(s_2-s_1)^{Q^2(p-\alpha)}} \left(\frac{1}{w(B_{2s_2R})} \int_{B_{2s_2R}} \bar{u}^{p} w \mathrm{d}x\right)^{\frac{p-\alpha}{p}} \\ &\quad \cdot \left(\int_{B_{2s_1R}} \bar{u}^{\alpha} w \mathrm{d}x\right) \left(\int_{B_{2R}} \bar{u}^{\alpha} w \mathrm{d}x\right)^{-\frac{p}{\alpha}} \\ &\le \frac{C}{(s_2-s_1)^{Q^2(p-\alpha)}} w(B_R)^{(\frac{p}{\alpha}-1)\frac{p-\alpha}{p}} \left(\int_{B_{2s_2R}} \bar{u}^{p} w \mathrm{d}x\right)^{\frac{p-\alpha}{p}} \left(\int_{B_{2R}} \bar{u}^{\alpha} w \mathrm{d}x\right)^{1-\frac{p}{\alpha}} \\ &= \left(C(s_2-s_1)^{-Q^2p} J(s_2)\right)^{\frac{p-\alpha}{p}}. \end{aligned}$$

Therefore

$$\log J(s_1) \le \frac{p-\alpha}{p}\left(\log C - Q^2 p \log(s_2-s_1) + \log J(s_2)\right). \tag{4.15}$$

Let $s_1 = s_2^{\theta}$ ($\theta > 1$) in (4.15) and integrate on the interval $\left[\left(\frac{2}{3}\right)^{1/\theta},\ 1\right]$ with respect to $\frac{\mathrm{d}s_2}{s_2}$, then

(4.16)

$$\frac{1}{\theta}\int_{\frac{2}{3}}^{1} \log J(\rho)\frac{\mathrm{d}\rho}{\rho} \le C + \frac{p-\alpha}{p}\int_{\left(\frac{2}{3}\right)^{1/\theta}}^{1} \log J(\rho)\frac{\mathrm{d}\rho}{\rho} \le C + \frac{p-\alpha}{p}\int_{\frac{2}{3}}^{1} \log J(\rho)\frac{\mathrm{d}\rho}{\rho},$$

where we have used $\rho = s_2^{\theta}$, $J\left(\frac{2}{3}\right) > 1$, and the fact that $\log J(\rho)$ is increasing and

$$-\int_{\left(\frac{2}{3}\right)^{\frac{1}{\theta}}}^{1} \log(s_2-s_1)\frac{\mathrm{d}s_2}{s_2} > 0.$$

Choosing $\theta \in (1, \frac{p}{p-\alpha})$, we conclude $J\left(\frac{2}{3}\right) \le C$ from (4.16).

Now (4.11) is deduced by (4.13).

5. PROOF OF THEOREM 1.2

In this section, we will use the results in Section 4 to prove the Harnack inequality.

Proof of Theorem 1.2. Without loss of generality, we assume $u > 0$ a.e. in Ω and denote $\bar{u} = u + K,\ K = K(R)$. The proof is divided into three steps.

Step 1. Let us confirm that there exist positive constants p_0 and C such that

$$\left(\frac{1}{w(B_{2R})}\int_{B_{2R}}\bar{u}^{p_0}w\mathrm{d}x\right)^{\frac{1}{p_0}}\leq C\left(\frac{1}{w(B_{2R})}\int_{B_{2R}}\bar{u}^{-p_0}w\mathrm{d}x\right)^{-\frac{1}{p_0}}. \tag{5.1}$$

In fact, take $v=\log\bar{u}$, then $Xv=Xu/\bar{u}$.

Choose $\varphi=\eta^p\bar{u}^{1-p}$ as a test function in (1.6), where $\eta\in C_0^\infty(B_{2R})$ such that $0\leq\eta\leq 1$ in B_{2R}, $\eta=1$ in B_R, and $|X\eta|\leq\frac{C}{R}$, then

(5.2)

$$\begin{aligned}
0=&\int_\Omega\langle A(x)Xu,Xu\rangle^{\frac{p-2}{2}}\langle A(x)Xu,X\eta\rangle p\eta^{p-1}\bar{u}^{1-p}\mathrm{d}x\\
&+\int_\Omega\langle A(x)Xu,Xu\rangle^{\frac{p-2}{2}}\langle A(x)Xu,Xu\rangle(1-p)\eta^p\bar{u}^{-p}\mathrm{d}x-\int_\Omega g(x)\eta^p\bar{u}^{1-p}\mathrm{d}x\\
\leq& p\lambda^{\frac{p}{2}}\int_{B_{2R}}|Xu|^{p-1}|X\eta|\eta^{p-1}\bar{u}^{1-p}w\mathrm{d}x-(p-1)\lambda^{-\frac{p}{2}}\int_{B_{2R}}\eta^p\bar{u}^{-p}|Xu|^pw\mathrm{d}x\\
&+\int_{B_{2R}}\left|\frac{g}{w}\right|\eta^p\bar{u}^{1-p}w\mathrm{d}x\\
\leq& p\lambda^{\frac{p}{2}}\int_{B_{2R}}|\eta Xv|^{p-1}|X\eta|w\mathrm{d}x-(p-1)\lambda^{-\frac{p}{2}}\int_{B_{2R}}|\eta Xv|^pw\mathrm{d}x\\
&+K^{1-p}\int_{B_{2R}}\left|\frac{g}{w}\right|\eta^pw\mathrm{d}x.
\end{aligned}$$

To the last term in the right hand side of (5.2), we apply (2.3), (2.5) and (1.9) to get

(5.3)

$$\begin{aligned}
&K^{1-p}\int_{B_{2R}}\left|\frac{g}{w}\right|\eta^pw\mathrm{d}x\\
\leq& K^{1-p}\left\|\frac{g}{w}\right\|_{L^q(B_{2R},w)}\|\eta\|^{pQ/q}_{L^{\frac{pQ}{Q-1}}(B_{2R},w)}\|\eta\|^{p(q-Q)/q}_{L^p(B_{2R},w)}\\
\leq& CK^{1-p}\left\|\frac{g}{w}\right\|_{L^q(B_{2R},w)}\|\eta\|^{p(q-Q)/q}_{L^p(B_{2R},w)}\left(2Rw(B_{2R})^{\frac{Q-1}{pQ}-\frac{1}{p}}\right)^{pQ/q}\|X\eta\|^{pQ/q}_{L^p(B_{2R},w)}\\
\leq& C\|\eta\|^{p(q-Q)/q}_{L^p(B_{2R},w)}\|X\eta\|^{pQ/q}_{L^p(B_{2R},w)}.
\end{aligned}$$

Putting (5.3) into (5.2) and using (2.3), it follows

$$\begin{aligned}
&\int_{B_{2R}}|\eta Xv|^pw\mathrm{d}x\\
\leq& C\int_{B_{2R}}|\eta Xv|^{p-1}|X\eta|w\mathrm{d}x+C\|\eta\|^{p(q-Q)/q}_{L^p(B_{2R},w)}\|X\eta\|^{pQ/q}_{L^p(B_{2R},w)}
\end{aligned}$$

$$\leq C\left(\int_{B_{2R}} |\eta Xv|^p w\mathrm{d}x\right)^{\frac{p-1}{p}}\left(\int_{B_{2R}} |X\eta|^p w\mathrm{d}x\right)^{\frac{1}{p}} + C\,\|\eta\|^{p(q-Q)/q}_{L^p(B_{2R},w)}\,\|X\eta\|^{pQ/q}_{L^p(B_{2R},w)}$$

and from Lemma 4.1,

$$\|\eta Xv\|_{L^p(B_{2R},w)} \leq C\|X\eta\|_{L^p(B_{2R},w)} + C\,\|\eta\|^{(q-Q)/p}_{L^p(B_{2R},w)}\,\|X\eta\|^{Q/q}_{L^p(B_{2R},w)}. \tag{5.4}$$

Using (2.4) and (5.4), we obtain

$$\begin{aligned}\|v - v_{B_R}\|_{L^p(B_R,w)} &\leq CR\|Xv\|_{L^p(B_R,w)}\\ &\leq R\left(\frac{C}{R}w(B_{2R})^{\frac{1}{p}} + Cw(B_{2R})^{\frac{q-Q}{pq}}\left(\frac{C}{R}\right)^{\frac{Q}{q}} w(B_{2R})^{\frac{Q}{pq}}\right)\\ &\leq C(1+\operatorname{diam}(\Omega)^{(q-Q)/q})w(B_R)^{\frac{1}{p}},\end{aligned}$$

where $v_{B_R} = \frac{1}{w(B_R)}\int_{B_R} vw\mathrm{d}x$. Hence

$$\frac{1}{w(B_R)}\int_{B_R} |v - v_{B_R}|^p w\mathrm{d}x \leq C. \tag{5.5}$$

By (5.5) and the weighted John-Nirenberg inequality (Buckley [3], Theorem 2.2), there exist positive constants p_0 and M_0 such that

$$\frac{1}{w(B_{2R})}\int_{B_{2R}} \exp\left(p_0\left|v - v_{B_{2R}}\right|\right) w\mathrm{d}x \leq M_0.$$

Since

$$\frac{1}{w(B_{2R})}\int_{B_{2R}} \exp\left(p_0\left|v - v_{B_{2R}}\right|\right) w\mathrm{d}x \geq \frac{1}{w(B_{2R})}\int_{B_{2R}} \exp\left(p_0 v - p_0 v_{B_{2R}}\right) w\mathrm{d}x,$$

$$\frac{1}{w(B_{2R})}\int_{B_{2R}} \exp\left(p_0\left|v - v_{B_{2R}}\right|\right) w\mathrm{d}x \geq \frac{1}{w(B_{2R})}\int_{B_{2R}} \exp\left(p_0 v_{B_{2R}} - p_0 v\right) w\mathrm{d}x,$$

then

$$\begin{aligned}&\frac{1}{w(B_{2R})}\int_{B_{2R}} \bar{u}^{-p_0} w\mathrm{d}x \cdot \frac{1}{w(B_{2R})}\int_{B_{2R}} \bar{u}^{p_0} w\mathrm{d}x\\ &= \frac{1}{w(B_{2R})}\int_{B_{2R}} \exp(-p_0 v) w\mathrm{d}x \cdot \frac{1}{w(B_{2R})}\int_{B_{2R}} \exp(p_0 v) w\mathrm{d}x\\ &= \frac{1}{w(B_{2R})}\int_{B_{2R}} \exp(p_0 v - p_0 v_{B_{2R}}) w\mathrm{d}x \cdot \frac{1}{w(B_{2R})}\int_{B_{2R}} \exp(p_0 v_{B_{2R}} - p_0 v) w\mathrm{d}x\\ &\leq \left(\frac{1}{w(B_{2R})}\int_{B_{2R}} \exp(p_0\left|v - v_{B_{2R}}\right|) w\mathrm{d}x\right)^2 \leq M_0^2,\end{aligned}$$

and this completes the proof of (5.1).

Step 2. We deduce that there exists a positive constant C such that

$$\inf_{B_R}\bar{u}\geq C\left(\frac{1}{w(B_{2R})}\int_{B_{2R}}\bar{u}^{-p_0}w\mathrm{d}x\right)^{-\frac{1}{p_0}}. \tag{5.6}$$

In fact, denote $\varphi=\eta^p\bar{u}^\beta$ and $h=\bar{u}^{q_1}$, where $\eta\in C_0^\infty(B_{2R})$, $0\leq\eta\leq 1$, $\beta\leq -1$ and $q_1=\frac{\beta+p-1}{p}<0$. Taking φ into (1.6) yields

$$\begin{aligned}
0=&\int_\Omega\langle A(x)Xu,Xu\rangle^{\frac{p-2}{2}}\langle A(x)Xu,X\eta\rangle p\eta^{p-1}\bar{u}^\beta\mathrm{d}x\\
&+\int_\Omega\langle A(x)Xu,Xu\rangle^{\frac{p-2}{2}}\langle A(x)Xu,Xu\rangle\beta\eta^p\bar{u}^{\beta-1}\mathrm{d}x-\int_\Omega g(x)\eta^p\bar{u}^\beta\mathrm{d}x\\
\leq&\ p\lambda^{\frac{p}{2}}\int_{B_{2R}}|Xu|^{p-1}|X\eta|\eta^{p-1}\bar{u}^\beta w\mathrm{d}x+\beta\lambda^{-\frac{p}{2}}\int_{B_{2R}}\eta^p\bar{u}^{\beta-1}|Xu|^pw\mathrm{d}x\\
&+K^{1-p}\int_{B_{2R}}\left|\frac{g}{w}\right|\eta^p\bar{u}^{\beta+p-1}w\mathrm{d}x\\
=&\ p\lambda^{\frac{p}{2}}|q_1|^{1-p}\int_{B_{2R}}|\eta Xh|^{p-1}|hX\eta|w\mathrm{d}x+\beta\lambda^{-\frac{p}{2}}|q_1|^{-p}\int_{B_{2R}}|\eta Xh|^pw\mathrm{d}x\\
&+K^{1-p}\int_{B_{2R}}\left|\frac{g}{w}\right|(\eta h)^pw\mathrm{d}x.
\end{aligned}\tag{5.7}$$

Noting (2.3), (2.5) and (1.9), we have

$$\begin{aligned}
&K^{1-p}\int_{B_{2R}}\left|\frac{g}{w}\right|(\eta h)^pw\mathrm{d}x,\\
&\leq K^{1-p}\left\|\frac{g}{w}\right\|_{L^q(B_{2R},w)}\|\eta h\|^{p(q-Q)/q}_{L^p(B_{2R},w)}\|\eta h\|^{pQ/q}_{L^{\frac{pQ}{Q-1}}(B_{2R},w)}\\
&\leq CK^{1-p}\left\|\frac{g}{w}\right\|_{L^q(B_{2R},w)}\|\eta h\|^{p(q-Q)/q}_{L^p(B_{2R},w)}\left(2Rw(B_{2R})^{\frac{Q-1}{pQ}-\frac{1}{p}}\right)^{pQ/q}\\
&\quad\cdot\left(\|\eta Xh\|^{pQ/q}_{L^p(B_{2R},w)}+\|hX\eta\|^{pQ/q}_{L^p(B_{2R},w)}\right)\\
&\leq C\|\eta h\|^{p(q-Q)/q}_{L^p(B_{2R},w)}\left(\|\eta Xh\|^{pQ/q}_{L^p(B_{2R},w)}+\|hX\eta\|^{pQ/q}_{L^p(B_{2R},w)}\right).
\end{aligned}\tag{5.8}$$

By (5.7), (5.8), (2.3) and Lemma 4.1,

$$\begin{aligned}
&\|\eta Xh\|^p_{L^p(B_{2R},w)}\leq|\beta|\,\|\eta Xh\|^p_{L^p(B_{2R},w)}\\
&\leq C|q_1|\int_{B_{2R}}|\eta Xh|^{p-1}|hX\eta|w\mathrm{d}x+C|q_1|^pK^{1-p}\int_{B_{2R}}\left|\frac{g}{w}\right|(\eta h)^pw\mathrm{d}x\\
&\leq C|q_1|\,\|\eta Xh\|^{p-1}_{L^p(B_{2R},w)}\|hX\eta\|_{L^p(B_{2R},w)}\\
&\quad+C|q_1|^p\|\eta h\|^{p(q-Q)/q}_{L^p(B_{2R},w)}\left(\|\eta Xh\|^{pQ/q}_{L^p(B_{2R},w)}+\|hX\eta\|^{pQ/q}_{L^p(B_{2R},w)}\right),
\end{aligned}\tag{5.9}$$

$$
\begin{aligned}
&\le C\,|q_1|\,\|hX\eta\|_{L^p(B_{2R},w)} + C\,|q_1|\,\|\eta h\|^{(q-Q)/q}_{L^p(B_{2R},w)}\,\|hX\eta\|^{Q/q}_{L^p(B_{2R},w)}\\
&\quad + C|q_1|^{q/(q-Q)}\|\eta h\|_{L^p(B_{2R},w)}\\
&\le C\,|q_1|\,\|hX\eta\|_{L^p(B_{2R},w)} + C\,|q_1|\left(\|\eta h\|_{L^p(B_{2R},w)}+\|hX\eta\|_{L^p(B_{2R},w)}\right)\\
&\quad + C|q_1|^{q/(q-Q)}\|\eta h\|_{L^p(B_{2R},w)}\\
&\le C\left(|q_1| + |q_1|^{q/(q-Q)}\right)\left(\|\eta h\|_{L^p(B_{2R},w)}+\|hX\eta\|_{L^p(B_{2R},w)}\right).
\end{aligned}
$$

Applying(5.9) and (2.5), then

(5.10)

$$
\begin{aligned}
&\|\eta h\|_{L^{k_0p}(B_{2R},w)} \le CRw(B_{2R})^{\frac{1}{k_0p}-\frac{1}{p}}\|X(\eta h)\|_{L^p(B_{2R},w)}\\
&\le CRw(B_{2R})^{\frac{1}{k_0p}-\frac{1}{p}}\left(\|\eta Xh\|_{L^p(B_{2R},w)} + \|hX\eta\|_{L^p(B_{2R},w)}\right)\\
&\le CRw(B_{2R})^{\frac{1}{k_0p}-\frac{1}{p}}\Big(C\left(|q_1| + |q_1|^{q/(q-Q)}\right)\\
&\quad\cdot\left(\|\eta h\|_{L^p(B_{2R},w)}+\|hX\eta\|_{L^p(B_{2R},w)}\right) + \|hX\eta\|_{L^p(B_{2R},w)}\Big)\\
&\le C\left(1+|q_1| + |q_1|^{q/(q-Q)}\right)Rw(B_{2R})^{\frac{1}{k_0p}-\frac{1}{p}}\left(\|\eta h\|_{L^p(B_{2R},w)}+\|hX\eta\|_{L^p(B_{2R},w)}\right)\\
&\le C(1+|q_1|)^{q/(q-Q)}Rw(B_{2R})^{\frac{1}{k_0p}-\frac{1}{p}}\left(\|\eta h\|_{L^p(B_{2R},w)}+\|hX\eta\|_{L^p(B_{2R},w)}\right),
\end{aligned}
$$

where $q > Q,\ k_0 = Q/(Q-1)$. Taking $h = \bar{u}^{q_1}(q_1 < 0)$ and $\eta \in C_0^\infty(B_{bR})$ in (5.10) with $\eta = 1$ in B_{aR}, $|X\eta| \le \frac{C}{(b-a)R}$, $1 \le a < b \le 2$, it follows

$$
\begin{aligned}
(5.11)\quad &\left(\int_{B_{aR}} \bar{u}^{k_0pq_1} w\mathrm{d}x\right)^{\frac{1}{k_0pq_1}}\\
&\ge C^{\frac{1}{q_1}}(1+|q_1|)^{\frac{p}{\varepsilon}\frac{1}{q_1}}w(B_{2R})^{\left(\frac{1}{k_0p}-\frac{1}{p}\right)\frac{1}{q_1}}(b-a)^{-\frac{1}{q_1}}\left(\int_{B_{bR}} \bar{u}^{pq_1} w\mathrm{d}x\right)^{\frac{1}{pq_1}}.
\end{aligned}
$$

Denoting $q_1 = -\frac{p_0}{p}k_0^i$, $\theta_i = -p_0k_0^i$, $b = \chi_i = 1+2^{-i}$, and $a = \chi_{i+1} = 1+2^{-(i+1)}$, $(i = 0, 1, ...)$ in (5.11) and iterating m times, we have

$$
\begin{aligned}
&\left(\int_{B_{\chi_{i+1}R}} \bar{u}^{\theta_{i+1}} w\mathrm{d}x\right)^{\frac{1}{\theta_{i+1}}}\\
&\ge C^{-\frac{p}{p_0}k_0^{-i}}\left(1+\frac{p_0}{p}k_0^i\right)^{-\frac{q}{q-Q}\frac{p}{p_0}k_0^{-i}} w(B_{2R})^{\left(\frac{1}{p}-\frac{1}{k_0p}\right)\frac{p}{p_0}k_0^{-i}}2^{-(i+1)\frac{p}{p_0}k_0^{-i}}\\
&\quad\left(\int_{B_{\chi_iR}} \bar{u}^{\theta_i} w\mathrm{d}x\right)^{\frac{1}{\theta_i}}
\end{aligned}
$$

$$\geq C^{-\frac{p}{p_0}\sum_{i=0}^{m}k_0^{-i}}\prod_{i=0}^{m}\left(1+\frac{p_0}{p}k_0^i\right)^{-\frac{q}{q-Q}\frac{p}{p_0}k_0^{-i}}w(B_{2R})^{\frac{1}{p_0Q}\sum_{i=0}^{m}k_0^{-i}}2^{-\frac{p}{p_0}\sum_{i=0}^{m}(i+1)k_0^{-i}}$$
$$\left(\int_{B_{2R}}\bar{u}^{-p_0}w\mathrm{d}x\right)^{-\frac{1}{p_0}}$$
$$\geq C^{-\frac{p}{p_0}\sum_{i=0}^{m}k_0^{-i}}\left(\frac{p_0}{p}k\right)^{-\frac{q}{q-Q}\frac{p}{p_0}\sum_{i=0}^{m}ik_0^{-i}}w(B_{2R})^{\frac{1}{p_0Q}\sum_{i=0}^{m}k_0^{-i}}2^{-\frac{p}{p_0}\sum_{i=0}^{m}(i+1)k_0^{-i}}$$
$$\left(\int_{B_{2R}}\bar{u}^{-p_0}w\mathrm{d}x\right)^{-\frac{1}{p_0}}.$$

Now (5.6) is proved by letting $m\to\infty$.

Step 3. From Lemma 4.3, there exists a positive constant C such that

$$\sup_{B_R}\bar{u}\leq C\left(\frac{1}{w(B_{2R})}\int_{B_{2R}}\bar{u}^{p_0}w\mathrm{d}x\right)^{\frac{1}{p_0}}. \tag{5.12}$$

Combining (4.16), (5.6) and (5.12), we conclude

$$\begin{aligned}\sup_{B_R}u\leq\sup_{B_R}\bar{u}&\leq C\left(\frac{1}{w(B_{2R})}\int_{B_{2R}}\bar{u}^{p_0}w\mathrm{d}x\right)^{\frac{1}{p_0}}\\&\leq C\left(\frac{1}{w(B_{2R})}\int_{B_{2R}}\bar{u}^{-p_0}w\mathrm{d}x\right)^{-\frac{1}{p_0}}\\&\leq C\inf_{B_R}\bar{u}=C\left(\inf_{B_R}u+K\right).\end{aligned}$$

This proves Theorem 1.2.

6. PROOF OF THEOREM 1.3

The proof of Theorem 1.3 is based on Theorem 1.2 and the following lemma.

LEMMA 6.1 ([25]). *If* ω *is a non-decreasing and non-negative function in* $[0,\tilde{R}]$*, such that for* $0<\theta_1,\theta_2<1,\ \ 0<\theta_3\leq 1,\ H\geq 0,$

$$\omega(\theta_1R)\leq\theta_2\omega(R)+HR^{\theta_3},\ \ 0<R\leq R_0,$$

then there exist $0<\theta_0\leq\theta_3$ *and* $C>0$ *such that*

$$\omega(R)\leq C\left(\frac{R}{R_0}\right)^{\theta_0}\left(\omega(R_0)+HR_0^{\theta_3}\right),$$

where θ_0 *only depends on* θ_1*,* θ_2 *and* θ_3*.*

Proof of Theorem 1.3. We have (1.8) and (1.9) from Theorem 1.2. Picking $0<\alpha<\min\left\{1,\frac{pQ}{q(p-1)}\right\}$, then $\alpha\in(0,1)$, $\frac{pQ}{q}-\alpha(p-1)>0$ and

$$K(R)=R^{\alpha}\left(R^{pQ/q-\alpha(p-1)}w(B_{2R})^{-\frac{1}{q}}\left\|\frac{g}{w}\right\|_{L^{q}(B_{2R},w)}\right)^{\frac{1}{p-1}}$$

$$\leq R^{\alpha}\left\|\frac{g}{w}\right\|^{\frac{1}{p-1}}_{L^{q,pQ-\alpha q(p-1)}(\Omega,w)}. \tag{6.1}$$

Combing (1.8) and (6.1), it yields

$$\sup_{B_R}u\leq C\left(\inf_{B_R}u+R^{\alpha}\left\|\frac{g}{w}\right\|^{\frac{1}{p-1}}_{L^{q,pQ-\alpha q(p-1)}(\Omega,w)}\right). \tag{6.2}$$

Denoting $M(R)=\sup_{B_R(x_0)}u(x)$, $m(R)=\inf_{B_R(x_0)}u(x)$ and $v_1(x)=u(x)-m(R)\geq 0$, we get from (6.2) that

$$\sup_{B_{R/4}(x_0)}v_1(x)\leq C\left(\inf_{B_{R/4}(x_0)}v_1(x)+R^{\alpha}\left\|\frac{g}{w}\right\|^{\frac{1}{p-1}}_{L^{q,pQ-\alpha q(p-1)}(\Omega,w)}\right),$$

where we have assumed $C>1$. Similarity, setting $v_2(x)=M(R)-u(x)\geq 0$, then by (6.2),

$$\sup_{B_{R/4}(x_0)}v_2(x)\leq C\left(\inf_{B_{R/4}(x_0)}v_2(x)+R^{\alpha}\left\|\frac{g}{w}\right\|^{\frac{1}{p-1}}_{L^{q,pQ-\alpha q(p-1)}(\Omega,w)}\right),$$

therefore

$$M(R/4)-m(R)\leq C\left(m(R/4)-m(R)+R^{\alpha}\left\|\frac{g}{w}\right\|^{\frac{1}{p-1}}_{L^{q,pQ-\alpha q(p-1)}(\Omega,w)}\right), \tag{6.3}$$

$$M(R)-m(R/4)\leq C\left(M(R)-M(R/4)+R^{\alpha}\left\|\frac{g}{w}\right\|^{\frac{1}{p-1}}_{L^{q,pQ-\alpha q(p-1)}(\Omega,w)}\right). \tag{6.4}$$

Using (6.3) and (6.4), we have

$$M(R/4)-m(R/4)\leq\frac{C-1}{C+1}\left(M(R)-m(R)\right)+\frac{2C}{C+1}R^{\alpha}\left\|\frac{g}{w}\right\|^{\frac{1}{p-1}}_{L^{q,pQ-\alpha q(p-1)}(\Omega,w)},$$

It shows by Lemma 6.1 that

$$M(R)-m(R)\leq CR^{\alpha}. \tag{6.5}$$

Assume $B_{9r}(x_0)\subset\Omega$ $(r\leq R_0)$ and denote $r'=d(x,y)$ for any $x,y\in B_r(x_0)$. Since $B_{8r}(x)\subset B_{9r}(x_0)\subset\Omega$, it follows by (6.5) that

$$|u(x)-u(y)|\leq\sup_{B_{r'}(x)}u-\inf_{B_{r'}(x)}u\leq Cr'^{\alpha},$$

Moreover,

$$\frac{|u(x)-u(y)|}{d(x,y)^{\alpha}} \leq C.$$

By virtue of finite covering theorem we get (1.10) for any $\Omega' \subset\subset \Omega$.

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