A NOTE ON THE BIRKHOFF-JAMES ORTHOGONALITY IN THE OPERATOR ALGEBRAS ON HILBERT SPACES

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In this paper, we present some characterizations of the Birkhoff-James orthogonality in $B(\mathcal{H})$ the algebra of bounded linear operators on Hilbert space $\mathcal{H}$ and we show that orthogonality of A and B implies orthogonality of A^n and B for positive operator A and $n \in \mathbb{N}$. We give an example to show that positivity of A is required. We also investigate some related results to this issue.

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1. INTRODUCTION

Throughout this paper $\mathcal{H}$ will be a complex Hilbert space. Denoting by $B(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$. The notion of Birkhoff-James orthogonality in an arbitrary normed linear space is studied in literature [7–9]. For $A, B \in B(\mathcal{H})$, the operator A is said to be Birkhoff-James orthogonal (shortly orthogonal) to B, denoted by $A \perp B$, if $\|A+\lambda B\| \geq \|A\|$ for all complex numbers of λ.

In inner product spaces (such Hilbert spaces), this orthogonality is equivalent to the usual notion of orthogonality. Obviously, Birkhoff-James orthogonality is nondegenerate, thus $A \perp A \iff A = 0$. It is homogenous, thus $A \perp B \iff \lambda A \perp \mu B$ for $\lambda, \mu \in \mathbb{C}$, and not symmetric, thus $A \perp B$ need not imply $B \perp A$. However, Turnšek in [11] proved that $A \perp B$ always implies $B \perp A$ if and only if B is a scalar multiple of an isometry or coisometry. If $\mathcal{H}$ is a Hilbert C^*-module, some characterizations of the Birkhoff-James orthogonality were given by Arambašič and Rajič in [1, 2]. Characterizations of the Birkhoff-James orthogonality in C^*-algebras and $B(\mathcal{H})$ were obtained in [5]. Bhatia and Šemrl [4] obtained one of the most important results of the Birkhoff-James orthogonality in the C^*-algebra $B(\mathcal{H})$. The following result is the content of Theorem 1.1 and Remark 3.1 of [4].

Theorem 1. *Let $A, B \in B(\mathcal{H})$. The following statements hold.*

(1) *If dim($\mathcal{H}$)$< \infty$, then $A \perp B$ if and only if there exists a unit vector $\xi \in \mathcal{H}$ such that $\|A\xi\| = \|A\|$ and $\langle A\xi, B\xi\rangle = 0$.*

(2) *If dim($\mathcal{H}$)$= \infty$, then $A \perp B$ if and only if there exists a sequence of unit vectors $(\xi_n) \subset \mathcal{H}$ such that $\lim_{n\to\infty} \|A\xi_n\| = \|A\|$ and $\lim_{n\to\infty}\langle A\xi_n, B\xi_n\rangle = 0$.*

The authors in [3] showed a similar theorem for the algebra of bounded linear operators on a real finite dimention normed space $\mathcal{X}$ with the norm induced by an inner product. Motivated by these, the purpose of this paper is to improve these characterizations for elements of $B(\mathcal{H})$. In addition, we show that orthogonality of A and B implies orthogonality of A^n and B for positive operator A and $n \in \mathbb{N}$. We give an example to show that positivity of A is required. In addition, by giving an example we show that if $A \perp B$ then $A + I$ is not necessarily orthogonal to B. Also, some related results are discussed for elements of $B(\mathcal{H})$.

We recall some basic facts about the C^*-algebra $B(\mathcal{H})$, before stating our results. For $A \in B(\mathcal{H})$ the symbol $\|A\|$ denotes the operator norm of A satisfying $\|A^*A\| = \|A\|^2$. Denoting by I the identity operator on $\mathcal{H}$. An operator $A \in B(\mathcal{H})$ is said to be positive if it is self-adjoint whose spectrum is contained in $[0, \infty)$, or equivalently, $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$. We write $A \geq 0$, for a positive element $A \in B(\mathcal{H})$. If A, B are self-adjoint elements of $B(\mathcal{H})$ such that $A - B \geq 0$, we write $A \geq B$. For every $A \geq 0$, there exists a unique positive $B \in B(\mathcal{H})$ such that $A = B^2$, such an element B denoted by $A^{\frac{1}{2}}$. Our reference for the theory of C^*-algebras and $B(\mathcal{H})$ is [10].

2. MAIN RESULTS

We start this section with a lemma.

Lemma 1. *If $A, B \in B(\mathcal{H})$ and A is a nonzero positive operator. The following statements hold.*

(1) *If dim($\mathcal{H}$)$< \infty$, then $A \perp B$ if and only if there exists a unit vector $\xi \in \mathcal{H}$ such that $A\xi = \|A\|\xi$ and $\langle \xi, B\xi\rangle = 0$.*

(2) *If dim($\mathcal{H}$)$= \infty$, then $A \perp B$ if and only if there exists a sequence of unit vectors $(\xi_n) \subset \mathcal{H}$ such that $\lim_{n\to\infty} A\xi_n - \|A\|\xi_n = 0$ and $\lim_{n\to\infty}\langle \xi_n, B\xi_n\rangle = 0$.*

Proof. Let dim($\mathcal{H}$)$= \infty$ and let $A \perp B$. By Theorem 1 ((2)), there exists a sequence of unit vectors $(\xi_n) \subset \mathcal{H}$ such that $\lim_{n\to\infty} \|A\xi_n\| = \|A\|$ and $\lim_{n\to\infty}\langle A\xi_n, B\xi_n\rangle = 0$. By Lemma 2.1 in [11], which shows that if (ξ_n) is a sequence of unit vectors in $\mathcal{H}$ such that $\|A\xi_n\| \longrightarrow \|A\|$ then $A\xi_n - \|A\|\xi_n \longrightarrow 0$, we obtain $\lim_{n\to\infty} A\xi_n - \|A\|\xi_n = 0$. Also, we have $0 = \lim_{n\to\infty}\langle A\xi_n, B\xi_n\rangle = \lim_{n\to\infty}\langle \|A\|\xi_n, B\xi_n\rangle$. So $\lim_{n\to\infty}\langle \xi_n, B\xi_n\rangle = 0$.

Conversely, the proof is obvious.

Using Theorem 1 (1), we can similarly prove the statement (1). □

Remark 1. If $A \in B(\mathcal{H})$ is a nonzero positive operator then A^n is a nonzero positive operator for each $n \in \mathbb{N}$. Moreover, we have $\|A^n\| = \|A\|^n$.

THEOREM 2. *Let $A, B \in B(\mathcal{H})$ and let A be a positive operator. If $A \perp B$ then $A^m \perp B$ for all $m \in \mathbb{N}$. Moreover, if $A^m \perp B$ for some $m \in \mathbb{N}$ then $A \perp B$.*

Proof. Let $A > 0$ and $A \perp B$. Then by Lemma 1, there exists a sequence of unit vectors $(\xi_n) \subset \mathcal{H}$ such that $A\xi_n - \|A\|\xi_n \longrightarrow 0$. Now we have:

$$\begin{aligned}
\lim_{n\to\infty} A^m\xi_n &= \lim_{n\to\infty} A^{m-1}(A\xi_n) \\
&= \lim_{n\to\infty} A^{m-1}(\|A\|\xi_n) = \lim_{n\to\infty} \|A\|A^{m-1}(\xi_n) \\
&= \lim_{n\to\infty} \|A\|A^{m-2}(A\xi_n) \\
&= \lim_{n\to\infty} \|A\|^2A^{m-2}(\xi_n)
\end{aligned}$$

By continuing this process we obtain that $\lim_{n\to\infty} A^m\xi_n = \lim_{n\to\infty} \|A\|^m\xi_n$. Hence

$$\lim_{n\to\infty} A^m\xi_n - \|A^m\|\xi_n = \lim_{n\to\infty} \|A\|^m\xi_n - \|A^m\|\xi_n = 0.$$

Since $\langle \xi_n, B\xi_n\rangle \longrightarrow 0$ we get $A^m \perp B$ for all $m \in \mathbb{N}$.

Moreover, if $A^m \perp B$ for some $m \in \mathbb{N}$ then there is a sequence of unit vectors $(\xi_n) \subset \mathcal{H}$ such that $\lim_{n\to\infty} \|A^m(\xi_n)\| = \|A\|^m$ and $\lim_{n\to\infty} A^m\xi_n - \|A\|^m\xi_n = 0$ and $\lim_{n\to\infty}\langle A^m\xi_n, B\xi_n\rangle = 0$. Put $\eta_n := \frac{A^{m-1}\xi_n}{\|A\|^{m-1}}$ which is a sequence of unit vectors. So we obtain that:

$$\lim_{n\to\infty} \|A(\eta_n)\| = \lim_{n\to\infty} \|A(\frac{A^{m-1}\xi_n}{\|A\|^{m-1}})\| = \|A\|.$$

Thus $\lim_{n\to\infty} A\eta_n - \|A\|\eta_n = 0$. Also:

$$\begin{aligned}
\lim_{n\to\infty}\langle A\eta_n, B\eta_n\rangle &= \lim_{n\to\infty}\langle \frac{1}{\|A\|^{m-1}}A^m\xi_n, \frac{1}{\|A\|}B(\|A\|\eta_n)\rangle \\
&= \lim_{n\to\infty}\langle \frac{1}{\|A\|^{m-1}}A^m\xi_n, \frac{1}{\|A\|^m}B(A^m\xi_n)\rangle \\
&= \lim_{n\to\infty}\langle \frac{1}{\|A\|^{m-1}}A^m\xi_n, B\xi_n\rangle = 0
\end{aligned}$$

Hence $A \perp B$. □

The following example shows that the positivity condition is required in the last theorem.

Example 1. Let $A : l^2 \longrightarrow l^2$ such that $A(x_1, x_2, x_3, ...) = (\lambda x_2, 0, \lambda_1 x_3, \lambda_2 x_4, ...)$ that $0 < \lambda_1 < \lambda_2 < ... < \lambda$ and $\lambda_j \to \lambda$. Clearly, A is not positive. Also, we have

$$\begin{aligned}
\|Ax\|^2 &= \sum_{i=1}^{\infty} |(Ax)_i|^2 \\
&= |\lambda|^2 |x_2|^2 + \sum_{i=1}^{\infty} |\lambda_i x_{i+2}|^2 \\
&< |\lambda|^2 \sum_{i=1}^{\infty} |x_i|^2.
\end{aligned}$$

Thus $\|A\| \leq \lambda$. Since $Ae_2 = (\lambda, 0, 0, ...)$, we have $\|Ae_2\| = \lambda$. So $\|A\| = \lambda$. Hence $\|Ae_2\| = \lambda = \|A\|$. If $B(x_1, x_2, x_3, ...) = (\lambda x_1, 0, \lambda_1 x_3, \lambda_2 x_4, ...)$ then $B(e_2) = 0$. Therefore $A \perp B$.

On the other hand, $A^2(x_1, x_2, x_3, ...) = (0, 0, \lambda_1^2 x_3, \lambda_2^2 x_4, ...)$ and we have

$$\|A^2 x\|^2 = \sum_{i=1}^{\infty} |\lambda_i^2 x_{i+2}|^2 < \lambda^4 \sum_{i=1}^{\infty} |x_i|^2.$$

So $\|A^2\| \leq \lambda^2$. Also, $\sup_j \|A^2 e_j\| \leq \|A^2\|$ and $\|A^2 e_j\| = \lambda_j^2$ for $j \geq 3$. Thus $\|A^2\| \geq \lim_j \lambda_j^2 = \lambda^2$. Hence $\|A^2\| = \lambda^2$. But for every $x \in \mathcal{H}$, we obtain that $A^2 x \neq \|A^2\| x = \lambda^2 x$. It means that A^2 is not orthogonal to B.

PROPOSITION 1. *Let $A, B \in B(\mathcal{H})$ and A be a positive operator. The following statements hold.*

(1) *If $dim(\mathcal{H}) < \infty$, then $A \perp B$ if and only if there is a unit vector $\xi \in \mathcal{H}$ such that $\|A\| = \langle A\xi, \xi \rangle$ and $\langle \xi, B\xi \rangle = 0$.*
(2) *If $dim(\mathcal{H}) = \infty$, then $A \perp B$ if and only if there is a sequence of unit vectors $(\xi_n) \in \mathcal{H}$ such that $\|A\| = \lim_{n\to\infty} \langle A\xi_n, \xi_n \rangle$ and $\lim_{n\to\infty} \langle \xi_n, B\xi_n \rangle = 0$.*

Proof.

(1) Let $\|A\| = \langle A\xi, \xi \rangle$ for some $\xi \in \mathcal{H}$. Put $\tilde{A} := \|A\| I - A$. We have $\langle A\xi, \xi \rangle + \langle \tilde{A}\xi, \xi \rangle = \|A\|$. So $\langle \tilde{A}\xi, \xi \rangle = 0$. Since $\tilde{A}$ is positive we obtain that $\tilde{A}\xi = 0$, so $\|A\|\xi = A\xi$. Hence $A \perp B$.
Conversely, the proof is obvious.
(2) The proof is similar to the last part. □

In the next example, we want to show that if $A \perp B$ then $A + I$ is not necessarily orthogonal to B.

Example 2. Let $\mathcal{H}$ be a Hilbert space with an orthonormal basis $(e_j)_j$. We define the operator $A : \mathcal{H} \longrightarrow \mathcal{H}$ by $Ae_j = \lambda_j e_j$ and $B : \mathcal{H} \longrightarrow \mathcal{H}$ by $Be_j = \lambda_j e_{j+1}$ with $\lambda_j = \alpha_j + i\beta_j$ which $\alpha_j^2 + \beta_j^2 = 1$ and $0 < \alpha_1 < \alpha_2 < ... < \alpha$ with $\alpha_j \longrightarrow \alpha$. We can see $\|Ax\| = \|x\|$ for all $x \in \mathcal{H}$. Since $\|Ae_1\| = \|e_1\| = 1 = \|A\|$ and $Be_1 = 0$, so $A \perp B$.

Now we want to show $A+I$ is not orthogonal to B. Since for every $x \in \mathcal{H}$ we have $x = \sum_j \langle x, e_j\rangle e_j$, we get $Ax + x = \sum_j (1+\lambda_j)\langle x, e_j\rangle e_j$. So

$$\begin{aligned}
\|Ax + x\|^2 &= \sum_j |1+\lambda_j|^2 |\langle x, e_j\rangle|^2 \\
&= \sum_j (1+\lambda_j)(1+\bar{\lambda_j})|\langle x, e_j\rangle|^2 \\
&= \sum_j 2(1+\alpha_j)|\langle x, e_j\rangle|^2 \\
&< 2(1+\alpha)\|x\|^2.
\end{aligned}$$

On the other hand, $\|(A+I)e_j\| = \sqrt{2(1+\alpha_j)}$ and $\|A+I\| \geq \lim_j \|(A+I)e_j\| = \sqrt{2(1+\alpha)}$. Thus $\|A+I\| = \sqrt{2(1+\alpha)}$. But for every $x \in \mathcal{H}$ we have $\|(A+I)x\| \lneqq \|A+I\|$. Hence $A+I$ is not orthogonal to B.

Now we define $[A] := \inf_{\|x\|=1} \|Ax\|$ and show that if $A \perp B$ then $(A^n - [A]^n I) \perp B$. At first, we need the following lemma.

Lemma 2. *Let $\mathcal{H}$ be a Hilbert space and $A \in B(\mathcal{H})$ be a positive operator then $[A]\|x\|^2 \leq \langle Ax, x\rangle$.*

Proof. If $Ker(A) \neq 0$ then $[A] = 0$. Let $Ker(A) = 0$. We have $|\langle Ax, y\rangle|^2 \leq \langle Ax, x\rangle\langle Ay, y\rangle$, for $x, y \in \mathcal{H}$. Put $y := \frac{Ax}{\|Ax\|}$. So we obtain

$$\|Ax\|^2 \leq \langle Ax, x\rangle \langle \frac{1}{\|Ax\|}A(Ax), \frac{1}{\|Ax\|}Ax\rangle. \tag{2.1}$$

On the other hand, put $B := A - [A]I$. So

$$\begin{aligned}
0 \leq \|Bx\|^2 &= \langle Ax, Ax\rangle - \langle Ax, [A]x\rangle + \langle [A]x, [A]x\rangle - \langle [A]x, Ax\rangle \\
&= \|Ax\|^2 - 2[A]\langle Ax, x\rangle + [A]^2\|x\|^2 \\
&\leq 2\|Ax\|^2 - 2[A]\langle Ax, x\rangle.
\end{aligned}$$

Hence $[A]\langle Ax, x\rangle \leq \|Ax\|^2$. By combining this with 2.1 we have $[A] \leq \langle \frac{1}{\|Ax\|}A(Ax), \frac{1}{\|Ax\|}Ax\rangle$. If we write $\mathcal{H} = Ker(A) \oplus (Ker(A))^{\perp}$ then we get $\mathcal{H} = \overline{A(\mathcal{H})}$. Hence $[A]\|x\|^2 \leq \langle Ax, x\rangle$ for every $x \in \mathcal{H}$. $\square$

Theorem 3. *Let A be a positive operator in $B(\mathcal{H})$ and $A \perp B$. Then $(A^m - [A]^m I) \perp B$ for each $m \in \mathbb{N}$.*

Proof. If $\dim(\mathcal{H}) = \infty$ and $A \perp B$ then there is a sequence of unit vectors $(\xi_n) \in \mathcal{H}$ such that $\|A\|^m = \lim_{n\to\infty}\langle A^m\xi_n, \xi_n\rangle$ and $\lim_{n\to\infty}\langle \xi_n, B\xi_n\rangle = 0$. Since by Lemma 2, $\langle (A^m - [A]^m I)x, x\rangle = \langle A^m x, x\rangle - [A]^m\|x\|^2 \geq 0$ for every $x \in \mathcal{H}$, so $(A^m - [A]^m I)$ is a positive operator. Hence

$$\begin{aligned} \|A^m - [A]^m I\| &= \sup_{\|x\|=1} \langle (A^m - [A]^m I)x, x\rangle \\ &= \|A^m\| - [A]^m = \lim_{n\to\infty}\langle A^m\xi_n, \xi_n\rangle - [A]^m \\ &= \lim_{n\to\infty}\langle (A^m - [A]^m I)\xi_n, \xi_n\rangle. \end{aligned}$$

So by Proposition 1 we get $(A^m - [A]^m I) \perp B$.

If $\dim(\mathcal{H}) < \infty$, the proof is similar. □

Next we obtain some characterizations of the Birkhoff-James orthogonality for elements of $B(\mathcal{H})$.

THEOREM 4. *Let $\dim(\mathcal{H}) = \infty$ and $A \in B(\mathcal{H})$ then $A \perp B$ if and only if $\lim_{n\to\infty}(A^*A\xi_n - \|A\|^2\xi_n) = 0$ and $\lim_{n\to\infty}\langle A\xi_n, B\xi_n\rangle = 0$. If $\dim(\mathcal{H}) < \infty$ we use a unit vector $\xi \in \mathcal{H}$ instead of a sequence of unit vectors.*

Proof. Let $A \perp B$, then there exists a sequence of unit vectors such that $\lim_{n\to\infty}\|A\xi_n\| = \|A\|$. Since we have

$$\lim_{n\to\infty}\|(A^*A)^{\frac{1}{2}}\xi_n\|^2 = \lim_{n\to\infty}\|A\xi_n\|^2 = \|A\|^2 = \|(A^*A)^{\frac{1}{2}}\|^2.$$

So $\lim_{n\to\infty}(A^*A)^{\frac{1}{2}}\xi_n - \|A\|\xi_n = 0$. Hence we obtain that $\lim_{n\to\infty}(A^*A)\xi_n - \|A\|^2\xi_n = 0$.

The converse is obvious. □

THEOREM 5. *Let A, B be two operators with $A \perp ABA$. Then $A^* \perp B$.*

Proof. Let $dim\mathcal{H} = \infty$. Since $A \perp ABA$, there exists a sequence of unit vectors such that $\lim_{n\to\infty}(A^*A\xi_n - \|A\|^2\xi_n) = 0$ and $\lim_{n\to\infty}\langle A\xi_n, ABA\xi_n\rangle = 0$. So $\lim_{n\to\infty}\langle A^*A\xi_n, BA\xi_n\rangle = 0$. Also, we have $\lim_{n\to\infty}\|A^*(A(\frac{1}{\|A\|}\xi_n)\| = \|A\| = \|A^*\|$. Put $\eta_n := (A(\frac{1}{\|A\|}\xi_n)$. Obviously, the sequence (η_n) is a sequence of unit vectors that $\|A^*\eta_n\| \to \|A^*\|$ and $\langle A^*\eta_n, B\eta_n\rangle \to 0$. Hence by Theorem 1 $A^* \perp B$.

In the case of $dim\mathcal{H} < \infty$, the proof is similar. □

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