A NOTE ON THE BIRKHOFF-JAMES ORTHOGONALITY IN THE OPERATOR ALGEBRAS ON HILBERT SPACES

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In this paper, we present some characterizations of the Birkhoff-James orthogonality in \( B(\mathcal{H}) \) the algebra of bounded linear operators on Hilbert space \( \mathcal{H} \) and we show that orthogonality of \( A \) and \( B \) implies orthogonality of \( A^n \) and \( B \) for positive operator \( A \) and \( n \in \mathbb{N} \). We give an example to show that positivity of \( A \) is required. We also investigate some related results to this issue.

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1. INTRODUCTION

Throughout this paper \( \mathcal{H} \) will be a complex Hilbert space. Denoting by \( B(\mathcal{H}) \) the algebra of bounded linear operators on \( \mathcal{H} \). The notion of Birkhoff-James orthogonality in an arbitrary normed linear space is studied in literature [7–9]. For \( A, B \in B(\mathcal{H}) \), the operator \( A \) is said to be Birkhoff-James orthogonal (shortly orthogonal) to \( B \), denoted by \( A \perp B \), if \( \|A + \lambda B\| \geq \|A\| \) for all complex numbers of \( \lambda \).

In inner product spaces (such Hilbert spaces), this orthogonality is equivalent to the usual notion of orthogonality. Obviously, Birkhoff-James orthogonality is nondegenerate, thus \( A \perp A \iff A = 0 \). It is homogenous, thus \( A \perp B \iff \lambda A \perp \mu B \) for \( \lambda, \mu \in \mathbb{C} \), and not symmetric, thus \( A \perp B \) need not imply \( B \perp A \). However, Turnšek in [11] proved that \( A \perp B \) always implies \( B \perp A \) if and only if \( B \) is a scalar multiple of an isometry or coisometry. If \( \mathcal{H} \) is a Hilbert \( C^* \)-module, some characterizations of the Birkhoff-James orthogonality were given by Arambašić and Rajič in [1,2]. Characterizations of the Birkhoff-James orthogonality in \( C^* \)-algebras and \( B(\mathcal{H}) \) were obtained in [5]. Bhatia and Šemrl [4] obtained one of the most important results of the Birkhoff-James orthogonality in the \( C^* \)-algebra \( B(\mathcal{H}) \). The following result is the content of Theorem 1.1 and Remark 3.1 of [4].

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Theorem 1. Let $A, B \in B(H)$. The following statements hold.

(1) If $\dim(H) < \infty$, then $A \perp B$ if and only if there exists a unit vector $\xi \in H$ such that $\|A\xi\| = \|A\|$ and $\langle A\xi, B\xi \rangle = 0$.

(2) If $\dim(H) = \infty$, then $A \perp B$ if and only if there exists a sequence of unit vectors $(\xi_n) \subset H$ such that $\lim_{n \to \infty} \|A\xi_n\| = \|A\|$ and $\lim_{n \to \infty} \langle A\xi_n, B\xi_n \rangle = 0$.

The authors in [3] showed a similar theorem for the algebra of bounded linear operators on a real finite dimension normed space $X$ with the norm induced by an inner product. Motivated by these, the purpose of this paper is to improve these characterizations for elements of $B(H)$. In addition, we show that orthogonality of $A$ and $B$ implies orthogonality of $A^n$ and $B$ for positive operator $A$ and $n \in \mathbb{N}$. We give an example to show that positivity of $A$ is required. In addition, by giving an example we show that if $A \perp B$ then $A + I$ is not necessarily orthogonal to $B$. Also, some related results are discussed for elements of $B(H)$.

We recall some basic facts about the $C^*$-algebra $B(H)$, before stating our results. For $A \in B(H)$ the symbol $\|A\|$ denotes the operator norm of $A$ satisfying $\|A^*A\| = \|A\|^2$. Denoting by $I$ the identity operator on $H$. An operator $A \in B(H)$ is said to be positive if it is self-adjoint whose spectrum is contained in $[0, \infty)$, or equivalently, $\langle Ax, x \rangle \geq 0$ for all $x \in H$. We write $A \geq 0$, for a positive element $A \in B(H)$. If $A, B$ are self-adjoint elements of $B(H)$ such that $A - B \geq 0$, we write $A \geq B$. For every $A \geq 0$, there exists a unique positive $B \in B(H)$ such that $A = B^2$, such an element $B$ denoted by $A^{1/2}$. Our reference for the theory of $C^*$-algebras and $B(H)$ is [10].

2. MAIN RESULTS

We start this section with a lemma.

Lemma 1. If $A, B \in B(H)$ and $A$ is a nonzero positive operator. The following statements hold.

(1) If $\dim(H) < \infty$, then $A \perp B$ if and only if there exists a unit vector $\xi \in H$ such that $A\xi = \|A\|\xi$ and $\langle \xi, B\xi \rangle = 0$.

(2) If $\dim(H) = \infty$, then $A \perp B$ if and only if there exists a sequence of unit vectors $(\xi_n) \subset H$ such that $\lim_{n \to \infty} A\xi_n - \|A\|\xi_n = 0$ and $\lim_{n \to \infty} \langle \xi_n, B\xi_n \rangle = 0$.

Proof. Let $\dim(H) = \infty$ and let $A \perp B$. By Theorem 1 ((2)), there exists a sequence of unit vectors $(\xi_n) \subset H$ such that $\lim_{n \to \infty} \|A\xi_n\| = \|A\|$ and $\lim_{n \to \infty} \langle A\xi_n, B\xi_n \rangle = 0$. By Lemma 2.1 in [11], which shows that if $(\xi_n)$ is a
sequence of unit vectors in $\mathcal{H}$ such that $\|A\xi_n\| \to A$ then $A\xi_n - A\|\xi_n\| \to 0$, we obtain $\lim_{n \to \infty} A\xi_n - A\|\xi_n\| = 0$. Also, we have $0 = \lim_{n \to \infty} \langle A\xi_n, B\xi_n \rangle = \lim_{n \to \infty} \langle\|A\|\xi_n, B\xi_n \rangle$. So $\lim_{n \to \infty} \langle\xi_n, B\xi_n \rangle = 0$.

Conversely, the proof is obvious.

Using Theorem 1 (1), we can similarly prove the statement (1). □

Remark 1. If $A \in B(\mathcal{H})$ is a nonzero positive operator then $A^n$ is a nonzero positive operator for each $n \in \mathbb{N}$. Moreover, we have $\|A^n\| = \|A\|^n$.

Theorem 2. Let $A, B \in B(\mathcal{H})$ and let $A$ be a positive operator. If $A \perp B$ then $A^m \perp B$ for all $m \in \mathbb{N}$. Moreover, if $A^m \perp B$ for some $m \in \mathbb{N}$ then $A \perp B$.

Proof. Let $A > 0$ and $A \perp B$. Then by Lemma 1, there exists a sequence of unit vectors $(\xi_n) \subset \mathcal{H}$ such that $A\xi_n - A\|\xi_n\| \to 0$. Now we have:

$$
\lim_{n \to \infty} A^m\xi_n = \lim_{n \to \infty} A^{m-1}(A\xi_n) = \lim_{n \to \infty} A^{m-1}(\|A\|\xi_n) = \lim_{n \to \infty} \|A\|A^{m-1}(\xi_n) = \lim_{n \to \infty} \|A\|^2 A^{m-2}(\xi_n)
$$

By continuing this process we obtain that $\lim_{n \to \infty} A^m\xi_n = \lim_{n \to \infty} \|A\|^m\xi_n$. Hence

$$
\lim_{n \to \infty} A^m\xi_n - \|A\|^m\xi_n = \lim_{n \to \infty} \|A\|^m\xi_n - \|A\|^m\xi_n = 0.
$$

Since $\langle\xi_n, B\xi_n\rangle \to 0$ we get $A^m \perp B$ for all $m \in \mathbb{N}$.

Moreover, if $A^m \perp B$ for some $m \in \mathbb{N}$ then there is a sequence of unit vectors $(\xi_n) \subset \mathcal{H}$ such that $\lim_{n \to \infty} \|A^m(\xi_n)\| = \|A\|^m$ and $\lim_{n \to \infty} A^m\xi_n - A\|A\|^m\xi_n = 0$ and $\lim_{n \to \infty} \langle A^m\xi_n, B\xi_n \rangle = 0$. Put $\eta_n := \frac{A^{m-1}\xi_n}{\|A\|^m} = \frac{A^{m-1}\xi_n}{\|A\|^m}$ which is a sequence of unit vectors. So we obtain that:

$$
\lim_{n \to \infty} \|A(\eta_n)\| = \lim_{n \to \infty} \|A\|\frac{A^{m-1}\xi_n}{\|A\|^m} = \|A\|.
$$

Thus $\lim_{n \to \infty} A\eta_n - A\|\eta_n\| = 0$. Also:

$$
\lim_{n \to \infty} \langle A\eta_n, B\eta_n \rangle = \lim_{n \to \infty} \left\langle \frac{1}{\|A\|^{m-1}} A^m\xi_n, \frac{1}{\|A\|^m} B(\|A\|\eta_n) \right\rangle = \lim_{n \to \infty} \left\langle \frac{1}{\|A\|^{m-1}} A^m\xi_n, \frac{1}{\|A\|^m} B(A^m\xi_n) \right\rangle = \lim_{n \to \infty} \langle A^{m-1}\xi_n, B\xi_n \rangle = 0
$$

Hence $A \perp B$. □
The following example shows that the positivity condition is required in the last theorem.

**Example 1.** Let $A : l^2 \rightarrow l^2$ such that $A(x_1, x_2, x_3, ...) = (\lambda x_2, 0, \lambda_1 x_3, \lambda_2 x_4, ...) \text{ that } 0 < \lambda_1 < \lambda_2 < ... < \lambda$ and $\lambda_j \rightarrow \lambda$. Clearly, $A$ is not positive. Also, we have

$$\|Ax\|^2 = \sum_{i=1}^{\infty} |(Ax)_i|^2 = |\lambda|^2 |x_2|^2 + \sum_{i=1}^{\infty} |\lambda_i x_{i+2}|^2 < |\lambda|^2 \sum_{i=1}^{\infty} |x_i|^2.$$  

Thus $\|A\| \leq \lambda$. Since $Ae_2 = (\lambda, 0, 0, ..., )$, we have $\|Ae_2\| = \lambda$. So $\|A\| = \lambda$. Hence $\|Ae_2\| = \lambda = \|A\|$. If $B(x_1, x_2, x_3, ...) = (\lambda x_1, 0, \lambda_1 x_3, \lambda_2 x_4, ...) \text{ then } B(e_2) = 0$. Therefore $A \perp B$.

On the other hand, $A^2(x_1, x_2, x_3, ...) = (0, 0, \lambda_1^2 x_3, \lambda_2^2 x_4, ...) \text{ and we have }$

$$\|A^2x\|^2 = \sum_{i=1}^{\infty} |\lambda_i^2 x_{i+2}|^2 < \lambda^4 \sum_{i=1}^{\infty} |x_i|^2.$$  

So $\|A^2\| \leq \lambda^2$. Also, $\sup_j \|A^2 e_j\| \leq \|A^2\|$ and $\|A^2 e_j\| = \lambda_j^2 \text{ for } j \geq 3$. Thus $\|A^2\| \geq \lim_j \lambda_j^2 = \lambda^2$. Hence $\|A^2\| = \lambda^2$. But for every $x \in \mathcal{H}$, we obtain that $A^2 x \neq \|A^2\| x = \lambda^2 x$. It means that $A^2$ is not orthogonal to $B$.

**Proposition 1.** Let $A, B \in B(\mathcal{H})$ and $A$ be a positive operator. The following statements hold.

1. If $\dim(\mathcal{H}) < \infty$, then $A \perp B$ if and only if there is a unit vector $\xi \in \mathcal{H}$ such that $\|A\| = \langle A\xi, \xi \rangle$ and $\langle \xi, B\xi \rangle = 0$.

2. If $\dim(\mathcal{H}) = \infty$, then $A \perp B$ if and only if there is a sequence of unit vectors $(\xi_n) \in \mathcal{H}$ such that $\|A\| = \lim_{n \rightarrow \infty} \langle A\xi_n, \xi_n \rangle$ and $\lim_{n \rightarrow \infty} \langle \xi_n, B\xi_n \rangle = 0$.

**Proof.**

(1) Let $\|A\| = \langle A\xi, \xi \rangle$ for some $\xi \in \mathcal{H}$. Put $\tilde{A} := \|A\| I - A$. We have $\langle A\xi, \xi \rangle + \langle \tilde{A}\xi, \xi \rangle = \|A\|$. So $\langle \tilde{A}\xi, \xi \rangle = 0$. Since $\tilde{A}$ is positive we obtain that $\tilde{A}\xi = 0$, so $\|A\| \xi = A\xi$. Hence $A \perp B$.

Conversely, the proof is obvious.

(2) The proof is similar to the last part. □

In the next example, we want to show that if $A \perp B$ then $A + I$ is not necessarily orthogonal to $B$. 

Example 2. Let $\mathcal{H}$ be a Hilbert space with an orthonormal basis $(e_j)_j$. We define the operator $A : \mathcal{H} \to \mathcal{H}$ by $Ae_j = \lambda_j e_j$ and $B : \mathcal{H} \to \mathcal{H}$ by $Be_j = \lambda_j e_{j+1}$ with $\lambda_j = \alpha_j + i\beta_j$ which $\alpha_j^2 + \beta_j^2 = 1$ and $0 < \alpha_1 < \alpha_2 < \ldots < \alpha$ with $\alpha_j \to \alpha$. We can see $\|Ax\| = \|x\|$ for all $x \in \mathcal{H}$. Since $\|Ae_1\| = \|e_1\| = 1 = \|A\|$ and $Be_1 = 0$, so $A \perp B$.

Now we want to show $A + I$ is not orthogonal to $B$. Since for every $x \in \mathcal{H}$ we have $x = \sum_j \langle x, e_j \rangle e_j$, we get $Ax + x = \sum_j (1 + \lambda_j) \langle x, e_j \rangle e_j$. So

$$\|Ax + x\|^2 = \sum_j |1 + \lambda_j|^2 |\langle x, e_j \rangle|^2$$

$$= \sum_j (1 + \lambda_j)(1 + \bar{\lambda}_j) |\langle x, e_j \rangle|^2$$

$$= \sum_j 2(1 + \alpha_j) |\langle x, e_j \rangle|^2$$

$$< 2(1 + \alpha) \|x\|^2.$$ 

On the other hand, $\|(A + I)e_j\| = \sqrt{2(1 + \alpha_j)}$ and $\|A + I\| \geq \lim_j \|(A + I)e_j\| = \sqrt{2(1 + \alpha)}$. Thus $\|A + I\| = \sqrt{2(1 + \alpha)}$. But for every $x \in \mathcal{H}$ we have $\|(A + I)x\| \leq \|A + I\|$. Hence $A + I$ is not orthogonal to $B$.

Now we define $[A] := \inf_{\|x\|=1} \|Ax\|$ and show that if $A \perp B$ then $(A^n - [A]^n I) \perp B$. At first, we need the following lemma.

Lemma 2. Let $\mathcal{H}$ be a Hilbert space and $A \in B(\mathcal{H})$ be a positive operator then $[A]\|x\|^2 \leq \langle Ax, x \rangle$.

Proof. If $\text{Ker}(A) \neq 0$ then $[A] = 0$. Let $\text{Ker}(A) = 0$. We have $\|Ax, y\|^2 \leq \langle Ax, x \rangle \langle Ay, y \rangle$, for $x, y \in \mathcal{H}$. Put $y := \frac{Ax}{\|Ax\|}$. So we obtain

$$\langle Ax, x \rangle \langle \frac{1}{\|Ax\|} A(Ax), \frac{1}{\|Ax\|} Ax \rangle.$$ 

On the other hand, put $B := A - [A]I$. So

$$0 \leq \|Bx\|^2 = \langle Ax, Ax \rangle - \langle Ax, [A]x \rangle + \langle [A]x, [A]x \rangle - \langle [A]x, Ax \rangle$$

$$= \|Ax\|^2 - 2[A] \langle Ax, x \rangle + [A]^2 \|x\|^2$$

$$\leq 2\|Ax\|^2 - 2[A] \langle Ax, x \rangle.$$ 

Hence $[A] \langle Ax, x \rangle \leq \|Ax\|^2$. By combining this with 2.1 we have $[A] \leq \langle \frac{1}{\|Ax\|} A(Ax), \frac{1}{\|Ax\|} Ax \rangle$. If we write $\mathcal{H} = \text{Ker}(A) \oplus (\text{Ker}(A))^\perp$ then we get $\mathcal{H} = A(\mathcal{H})$. Hence $[A]\|x\|^2 \leq \langle Ax, x \rangle$ for every $x \in \mathcal{H}$. □

Theorem 3. Let $A$ be a positive operator in $B(\mathcal{H})$ and $A \perp B$. Then $(A^m - [A]^m I) \perp B$ for each $m \in \mathbb{N}$. 

Proof. If \( \dim(\mathcal{H}) = \infty \) and \( A \perp B \) then there is a sequence of unit vectors \( (\xi_n) \in \mathcal{H} \) such that \( \|A\|^m = \lim_{n \to \infty} \langle A^m \xi_n, \xi_n \rangle \) and \( \lim_{n \to \infty} \langle \xi_n, B \xi_n \rangle = 0 \). Since by Lemma 2, \( (A^m - [A]^m) x, x \rangle = \langle A^m x, x \rangle - [A]^m \|x\|^2 \geq 0 \) for every \( x \in \mathcal{H} \), so \( (A^m - [A]^m) \) is a positive operator. Hence
\[
\|A^m - [A]^m I\| = \sup_{\|x\| = 1} \langle (A^m - [A]^m) x, x \rangle = \|A^m\| - [A]^m = \lim_{n \to \infty} \langle A^m \xi_n, \xi_n \rangle - [A]^m = \lim_{n \to \infty} \langle (A^m - [A]^m) I \xi_n, \xi_n \rangle.
\]
So by Proposition 1 we get \( (A^m - [A]^m I) \perp B \).
If \( \dim(\mathcal{H}) < \infty \), the proof is similar. \( \square \)

Next we obtain some characterizations of the Birkhoff-James orthogonality for elements of \( B(\mathcal{H}) \).

**Theorem 4.** Let \( \dim(\mathcal{H}) = \infty \) and \( A \in B(\mathcal{H}) \) then \( A \perp B \) if and only if \( \lim_{n \to \infty} (A^* A \xi_n - \|A\|^2 \xi_n) = 0 \) and \( \lim_{n \to \infty} \langle A \xi_n, B \xi_n \rangle = 0 \). If \( \dim(\mathcal{H}) < \infty \) we use a unit vector \( \xi \in \mathcal{H} \) instead of a sequence of unit vectors.

**Proof.** Let \( A \perp B \), then there exists a sequence of unit vectors such that \( \lim_{n \to \infty} \|A \xi_n\| = \|A\| \). Since we have
\[
\lim_{n \to \infty} \|(A^* A)^{\frac{1}{2}} \xi_n\|^2 = \lim_{n \to \infty} \|A \xi_n\|^2 = \|A\|^2 = \|(A^* A)^{\frac{1}{2}}\|^2.
\]
So \( \lim_{n \to \infty} (A^* A)^{\frac{1}{2}} \xi_n - \|A\| \xi_n = 0 \). Hence we obtain that \( \lim_{n \to \infty} (A^* A) \xi_n - \|A\|^2 \xi_n = 0 \).

The converse is obvious. \( \square \)

**Theorem 5.** Let \( A, B \) be two operators with \( A \perp ABA \). Then \( A^* \perp B \).

**Proof.** Let \( \dim \mathcal{H} = \infty \). Since \( A \perp ABA \), there exists a sequence of unit vectors such that \( \lim_{n \to \infty} (A^* A \xi_n - \|A\|^2 \xi_n) = 0 \) and \( \lim_{n \to \infty} \langle A \xi_n, ABA \xi_n \rangle = 0 \). So \( \lim_{n \to \infty} \langle A^* A \xi_n, B A \xi_n \rangle = 0 \). Also, we have \( \lim_{n \to \infty} \|A^* (A \frac{1}{\|A\|} \xi_n)\| = \|A\| = \|A^*\| \). Put \( \eta_n := (A \frac{1}{\|A\|} \xi_n) \). Obviously, the sequence \( (\eta_n) \) is a sequence of unit vectors that \( \|A^* \eta_n\| \to \|A^*\| \) and \( \langle A^* \eta_n, B \eta_n \rangle \to 0 \). Hence by Theorem 1 \( A^* \perp B \).

In the case of \( \dim \mathcal{H} < \infty \), the proof is similar. \( \square \)

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