INFINITELY MANY SOLUTIONS FOR A CLASS OF PERTURBED DAMPED VIBRATION PROBLEMS

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In the present paper, the existence of infinitely many solutions for a class of perturbed damped vibration boundary value problems is established. Our approach is based on variational methods.

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1. INTRODUCTION

The aim of this paper is to study the following perturbed damped vibration problem

(1.1)
$$\begin{cases} -\ddot{u}(t) - q(t)\dot{u}(t) + A(t)u(t) = \lambda\nabla F(t, u(t)) + \mu\nabla G(t, u(t)) \quad a.e.\ t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0 \end{cases}$$

where $T > 0$, $q \in \mathrm{L}^1(0, T; \mathbb{R})$, $Q(t) = \int_0^t q(s)\mathrm{d}s$ for all $t \in [0, T)$, $Q(T) = 0$, $A : [0, T] \to \mathbb{R}^{N\times N}$ is a continuous map from the interval $[0, T]$ to the set of N-order symmetric matrices, $\lambda > 0$, $\mu \geq 0$, and $F, G : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ are measurable with respect to t, for all $u \in \mathbb{R}^N$, continuously differentiable in u, for almost every $t \in [0, T]$, satisfies the following standard summability condition:

$$\sup_{|\xi|\leq a} \max\{|F(\cdot, \xi)|, |G(\cdot, \xi)|,\ |\nabla F(\cdot, \xi)|\ , |\nabla G(\cdot, \xi)|\} \in \mathrm{L}^1([0, T]) \tag{1.2}$$

for any $a > 0$.

Assume that $\nabla F, \nabla G : [0, T] \times \mathbb{R}^N \to \mathbb{R}^N$ are continuous, then the condition (1.2) is satisfied.

Inspired by the monographs [15,17], the existence and multiplicity of periodic solutions for Hamiltonian systems have been investigated in many papers (see [1, 3, 4, 10–13, 16, 20–23, 29, 30] and the references therein) *via* variational methods. For example, in [3] Bonanno and Livrea ensured the existence of infinitely many periodic solutions for a class of second-order Hamiltonian systems under an appropriate oscillating behavior of the nonlinear term. Moreover, they obtained the multiplicity of periodic solutions for the system with a coercive potential and also in the noncoercive case. In [22] the authors obtained existence theorems for periodic solutions of a class of unbounded nonautonomous nonconvex subquadratic second order Hamiltonian systems by using the minimax methods in critical point theory. In [10] Cordaro established a multiplicity result to an eigenvalue problem related to second-order Hamiltonian systems, and proved the existence of an open interval of positive eigenvalues in which the problem admits three distinct periodic solutions. In [12] Faraci studied the multiplicity of solutions of a second order nonautonomous system. In [16] the authors obtained an existence theorem of homoclinic solution for a class of the nonautonomous second order Hamiltonian systems, by the minimax methods in the critical point theory, specially, the generalized mountain pass theorem. In [21] the existence and multiplicity of periodic solutions are obtained for nonautonomous second order systems with sublinear nonlinearity by using the least action principle and the minimax methods. In [23] the author presented two new existence results of periodic solutions with saddle point character and one new multiplicity result for Hamiltonian systems by using the critical point reduction method. In [29] the existence of homoclinic orbits for the second-order Hamiltonian systems without periodicity was studied and infinitely many homoclinic orbits for both superlinear and asymptotically linear cases were obtained. In [30] the author considered two classes of the second-order Hamiltonian systems with symmetry. In fact, if the systems are asymptotically linear with resonance, infinitely many small-energy solutions by minimax technique was obtained. If the systems possess sign-changing potential, an existence theorem of infinitely many solutions by Morse theory was established.

We also refer to [8, 14, 19] in which based on variational methods and critical point theory the existence of multiple solutions for second-order impulsive Hamiltonian systems was established.

Very recently, some researchers have paid attention to the existence and multiplicity of solutions for damped vibration problems, for instance, see [6,7,9, 24–27] and references therein. For example, Chen in [6,7] studied a class of nonperiodic damped vibration systems with subquadratic terms and with asymptotically quadratic terms, respectively, and obtained infinitely many nontrivial homoclinic orbits by a variant fountain theorem developed recently by Zou [28]. Wu and Chen in [26] based on a variational principle gave three existence theorems for periodic solutions of a class of damped vibration problems. In particular, the authors in [25] based on variational methods and critical point theory studied the existence of one solution and multiple solutions for damped vibration problems.

In the present paper, motivated by [25], employing a smooth version of [5, Theorem 2.1] which is a more precise version of Ricceri's Variational Principle [18, Theorem 2.5] under some hypotheses on the behavior of the nonlinear terms at infinity, under conditions on F and G we prove the existence of a definite interval about λ and μ in which the problem (1.1) admits a sequence of solutions which is unbounded in the space E which will be introduced later (Theorem 3.1).

We also refer the reader to [2] in which the existence of infinitely many solutions to a fourth-order boundary value problem has been studied.

2. PRELIMINARIES

Our main tool to investigate the existence of infinitely many periodic solutions for the problem (1.1) is a smooth version of Theorem 2.1 of [5] which is a more precise version of Ricceri's Variational Principle [18, Theorem 2.5] that we now recall here.

THEOREM 2.1. *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous, and coercive and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let us put*

$$\varphi(r) := \inf_{u \in \Phi^{-1}(-\infty, r)} \frac{\sup_{v \in \Phi^{-1}(-\infty, r)} \Psi(v) - \Psi(u)}{r - \Phi(u)}$$

and

$$\gamma := \liminf_{r \to +\infty} \varphi(r), \quad \delta := \liminf_{r \to (\inf_X \Phi)^+} \varphi(r).$$

Then, one has

(a) for every $r > \inf_X \Phi$ and every $\lambda \in]0, \frac{1}{\varphi(r)}[$, the restriction of the functional $I_\lambda = \Phi - \lambda \Psi$ to $\Phi^{-1}(]-\infty, r[)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.

(b) If $\gamma < +\infty$ then, for each $\lambda \in]0, \frac{1}{\gamma}[$, the following alternative holds: either

(b_1) I_λ possesses a global minimum,

or

(b_2) there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that

$$\lim_{n \to +\infty} \Phi(u_n) = +\infty.$$

(c) *If* $\delta < +\infty$ *then, for each* $\lambda \in]0, \frac{1}{\delta}[$, *the following alternative holds: either*

(c_1) *there is a global minimum of* Φ *which is a local minimum of* I_λ, *or*

(c_2) *there is a sequence of pairwise distinct critical points (local minima) of* I_λ *which weakly converges to a global minimum of* Φ.

We assume that A satisfies the following conditions:

(A1) $A(t) = (a_{ij}(t))$ is a symmetric matrix with $a_{ij} \in \mathrm{L}^\infty[0,T]$ for any $t \in [0,T]$, $i,j = 1, \cdots, N$;

(A2) there exists $\kappa > 0$ such that $(A(t)x, x) \geq \kappa |x|^2$ for any $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$, where $(\cdot, \cdot)$ denotes the inner product in $\mathbb{R}^N$.

Let us recall some basic concepts. Denote

$$\mathrm{E} = \{u : [0,T] \to \mathbb{R}^N |\ u \text{ is absolutely continuous}, \ \ u(0) = u(T), \\ \dot{u} \in \mathrm{L}^2([0,T], \mathbb{R}^N)\}$$

with the inner product

$$\prec u, v \succ_E = \int_0^T [(\dot{u}(t), \dot{v}(t)) + (u(t), v(t))]\mathrm{d}t.$$

The corresponding norm is defined by

$$\|u\|_E = \int_0^T (|\dot{u}(t)|^2 + |u(t)|^2)\mathrm{d}t, \quad \text{for all}\, u \in \mathrm{E}.$$

For every $u, v \in E$, we define

$$\prec u, v \succ = \int_0^T e^{Q(t)}[(\dot{u}(t), \dot{v}(t)) + (A(t)u(t), v(t))]\mathrm{d}t,$$

and we observe that, by assumptions (A1) and (A2), it defines an inner product in E. Then E is a separable and reflexive Banach space with the norm

$$\|u\| = \prec u, u \succ^{\frac{1}{2}}, \quad \text{for all}\, u \in \mathrm{E}.$$

Obviously, E is an uniformly convex Banach space.

Clearly, the norm $\|\cdot\|$ is equivalent to the norm $\|\cdot\|_{\mathrm{E}}$ (see [13]).

Since $(\mathrm{E}, \|\cdot\|)$ is compactly embedded in $\mathrm{C}([0,T], \mathbb{R}^N)$ (see [15]), there exists a positive constant c such that

$$\|u\|_\infty \leq c\|u\|, \tag{2.1}$$

where $\|u\|_\infty = \max_{t\in[0,T]} |u(t)|$. We mean by a weak solution of the problem (1.1), any $u \in \mathrm{E}$ such that

$$\int_0^T e^{Q(t)}\Big[(\dot{u}(t), \dot{v}(t)) + (A(t)u(t), v(t))\Big]\mathrm{d}t - \lambda \int_0^T e^{Q(t)}\big(\nabla F(t, u(t)), v(t)\big)\mathrm{d}t$$

$$- \mu \int_0^T e^{Q(t)} \big(\nabla G(t, u(t)), v(t)\big) \mathrm{d}t = 0$$

for every $v \in$ E.

A special case of our main result is the following theorem.

THEOREM 2.2. *Assume that Assumptions (A1) and (A2) hold. Let* $F : \mathbb{R}^N \to \mathbb{R}$ *be continuously differentiable function such that*

$$\liminf_{\zeta \to +\infty} \frac{\sup_{|x| \le \zeta} F(x)}{\zeta^2} = 0 \quad \textit{and} \quad \limsup_{\xi \to +\infty} \frac{F(\xi)}{\sum_{i=1}^N \sum_{j=1}^N \xi_i \, \xi_j} = +\infty$$

where $\xi = (\xi_1, \dots, \xi_n)$ *and* $\xi \to +\infty$ *means that* $(\xi_1, \dots, \xi_n) \to (+\infty, \dots, +\infty)$. *Then, the problem*

$$\begin{cases} -\ddot{u}(t) - q(t)\dot{u}(t) + A(t)u(t) = \nabla F(u(t)) & \textit{a.e. } t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0 \end{cases}$$

has an unbounded sequence of solutions.

3. MAIN RESULTS

In this section, we formulate our main results and prove them. For this purpose we put

$$\mathcal{B} := \max_{i,j=1}^{N} \|a_{ij}\|_\infty.$$

THEOREM 3.1. *Assume that Assumptions (A1) and (A2) hold and*

$$(a_1) \quad \liminf_{\zeta \to +\infty} \frac{\int_0^T e^{Q(t)} \sup_{|x| \le \zeta} F(t, x) \mathrm{d}t}{\zeta^2} < \frac{1}{c^2 \mathcal{B} \int_0^T e^{Q(t)} \mathrm{d}t} \limsup_{\xi \to +\infty} \frac{\int_0^T e^{Q(t)} F(t, \xi) \mathrm{d}t}{\sum_{i=1}^N \sum_{j=1}^N \xi_i \xi_j}$$

where $\xi \in \mathbb{R}^N$, $\xi = (\xi_1, \dots, \xi_n)$ *and* $\xi \to +\infty$ *means that* $(\xi_1, \dots, \xi_n) \to (+\infty, \dots, +\infty)$.

Then, for each $\lambda \in]\lambda_1, \lambda_2[$ *where*

$$\lambda_1 := \frac{\mathcal{B} \int_0^T e^{Q(t)} \mathrm{d}t}{2 \limsup_{\xi \to +\infty} \frac{\int_0^T e^{Q(t)} F(t,\xi) \mathrm{d}t}{\sum_{i=1}^N \sum_{j=1}^N \xi_i \xi_j}}$$

where $\xi = (\xi_1, \dots, \xi_n)$ *and* $\xi \to +\infty$ *means that* $(\xi_1, \dots, \xi_n) \to (+\infty, \dots, +\infty)$, *and*

$$\lambda_2 := \frac{1}{2c^2 \liminf_{\zeta \to +\infty} \frac{\int_0^T e^{Q(t)} \sup_{|x| \le \zeta} F(t,x) \mathrm{d}t}{\zeta^2}},$$

for every arbitrary non-negative function $G : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ which is measurable with respect to t, for all $x \in \mathbb{R}^N$, continuously differentiable in x, for almost every $t \in [0,T]$, satisfying the condition

$$G_\infty := 2c^2 \lim_{\zeta\to+\infty} \frac{\int_0^T e^{Q(t)} \sup_{|x|\le\zeta} G(t,x)\mathrm{d}t}{\zeta^2} < +\infty, \tag{3.1}$$

and for every $\mu \in [0, \mu_{G,\lambda}[$ where $\mu_{G,\lambda} := \frac{1}{G_\infty}\big(1 - \frac{\lambda}{\lambda_2}\big)$, the problem (1.1) has an unbounded sequence of solutions.

Proof. Fix $\overline{\lambda} \in]\lambda_1, \lambda_2[$ and let G be a function satisfying the condition (3.1). Since, $\overline{\lambda} < \lambda_2$, one has $\mu_{G,\overline{\lambda}} > 0$. Fix $\overline{\mu} \in [0, \mu_{G,\overline{\lambda}}[$ and set $\nu_1 := \lambda_1$ and $\nu_2 := \frac{\lambda_2}{1+\frac{\overline{\mu}}{\overline{\lambda}}\lambda_2 G_\infty}$. If $G_\infty = 0$, clearly, $\nu_1 = \lambda_1$, $\nu_2 = \lambda_2$ and $\overline{\lambda} \in]\nu_1, \nu_2[$. If $G_\infty \neq 0$, since $\overline{\mu} < \mu_{g,\overline{\lambda}}$, we obtain $\frac{\overline{\lambda}}{\lambda_2} + \overline{\mu} G_\infty < 1$, and so $\frac{\lambda_2}{1+\frac{\overline{\mu}}{\overline{\lambda}}\lambda_2 G_\infty} > \overline{\lambda}$, namely, $\overline{\lambda} < \nu_2$. Hence, since $\overline{\lambda} > \lambda_1 = \nu_1$, one has $\overline{\lambda} \in]\nu_1, \nu_2[$. Now, put $H(t,\xi) = F(t,\xi) + \frac{\overline{\mu}}{\overline{\lambda}} G(t,\xi)$ for all $(t,\xi) \in [0,T] \times \mathbb{R}^N$. Take $X = \mathrm{E}$ and consider the functionals $\Phi,\ \Psi : X \to \mathbb{R}$ defined as follows

$$\Phi(u) = \frac{1}{2}\|u\|^2$$

and

$$\Psi(u) = \int_0^T H(t, u(t))\mathrm{d}t$$

for all $u \in X$. It is well known that Ψ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $u \in X$ is the functional $\Psi'(u) \in X^*$, given by

$$\Psi'(u)v = \int_0^T e^{Q(t)} \Big(\nabla F(t,u(t)) + \frac{\overline{\mu}}{\overline{\lambda}} \nabla G(t,u(t)), v(t)\Big)\mathrm{d}t$$

for every $v \in X$. Moreover, Φ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $u \in X$ is the functional $\Phi'(u) \in X^*$, given by

$$\Phi'(u)v = \int_0^T e^{Q(t)} \Big[(\dot{u}(t), \dot{v}(t)) + (A(t)u(t), v(t))\Big]\mathrm{d}t$$

for every $v \in X$. Furthermore, Φ is sequentially weakly lower semicontinuous and coercive. From the definition of Φ, since $(X, \|\cdot\|)$ is compactly embedded in $\mathrm{C}([0,T], \mathbb{R}^N)$, we observe that Φ is strongly continuous. Put $I_{\overline{\lambda}} := \Phi - \overline{\lambda}\Psi$. We observe that the weak solutions of the problem (1.1) are exactly the solutions of the equation $I'_{\overline{\lambda}}(u) = 0$. Now, we want to show that $\gamma < +\infty$, where γ

is defined in Theorem 2.1. Let $\{\zeta_n\}$ be a real sequence such that $n \in \mathbb{N}$ and $\zeta_n \to +\infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{\int_0^T e^{Q(t)} \sup_{|x|\le\zeta_n} H(t,x)\mathrm{d}t}{\zeta_n^2} = \liminf_{\zeta\to+\infty} \frac{\int_0^T e^{Q(t)} \sup_{|x|\le\zeta} H(t,x)\mathrm{d}t}{\zeta^2}.$$

Put $r_n = \frac{1}{2}(\frac{\zeta_n}{c})^2$ for all $n \in \mathbb{N}$. Taking (2.1) into account that, we have

$$\Phi^{-1}(-\infty, r_n) \subseteq \{u \in X; ||u||_\infty \le \zeta_n\}.$$

Hence, one has

$$\begin{aligned}
\varphi(r_n) &\le 2c^2 \frac{\int_0^T e^{Q(t)} \sup_{|x|\le\zeta_n} H(t,x)\mathrm{d}t}{\zeta_n^2} \\
&= 2c^2 \frac{\int_0^T e^{Q(t)} \sup_{|x|\le\zeta_n} \left[F(t,x) + \frac{\overline{\mu}}{\lambda} G(t,x)\right]\mathrm{d}t}{\zeta_n^2} \\
&\le 2c^2 \frac{\int_0^T e^{Q(t)} \sup_{|x|\le\zeta_n} F(t,x)\mathrm{d}t}{\zeta_n^2} + \frac{\overline{\mu}}{\lambda} 2c^2 \frac{\int_0^T e^{Q(t)} \sup_{|x|\le\zeta_n} G(t,x)\mathrm{d}t}{\zeta_n^2}.
\end{aligned}$$

Moreover, it follows from Assumption (a_1) that

$$\liminf_{\zeta\to+\infty} \frac{\int_0^T e^{Q(t)} \sup_{|x|\le\zeta} F(t,x)\mathrm{d}t}{\zeta^2} < +\infty,$$

so we obtain

$$\lim_{n\to\infty} \frac{\int_0^T e^{Q(t)} \sup_{|x|\le\zeta_n} F(t,x)\mathrm{d}t}{\zeta_n^2} < +\infty. \tag{3.2}$$

Then, in view of (3.1) and (3.2), we have

$$\lim_{n\to\infty} \frac{\int_0^T e^{Q(t)} \sup_{|x|\le\zeta_n} F(t,x)\mathrm{d}t}{\zeta_n^2} + \lim_{n\to\infty} \frac{\overline{\mu}}{\lambda} \frac{\int_0^T e^{Q(t)} \sup_{|x|\le\zeta_n} G(t,x)\mathrm{d}t}{\zeta_n^2} < +\infty,$$

which follows

$$\lim_{n\to\infty} \frac{\int_0^T e^{Q(t)} \sup_{|x|\le\zeta_n} \left[F(t,x) + \frac{\overline{\mu}}{\lambda} G(t,x)\right]\mathrm{d}t}{\zeta_n^2} < +\infty.$$

Therefore,

$$\gamma \le \liminf_{n\to+\infty} \varphi(r_n) \le 2c^2 \lim_{n\to\infty} \frac{\int_0^T e^{Q(t)} \sup_{|x|\le\zeta_n} \left[F(t,x) + \frac{\overline{\mu}}{\lambda} G(t,x)\right]\mathrm{d}t}{\zeta_n^2} < +\infty. \tag{3.3}$$

Since

$$\frac{\int_0^T e^{Q(t)}\sup_{|x|\le\zeta_n}H(t,x)\mathrm{d}t}{\zeta_n^2}\le\frac{\int_0^T e^{Q(t)}\sup_{|x|\le\zeta_n}F(t,x)\mathrm{d}t}{\zeta_n^2}+\frac{\overline{\mu}}{\overline{\overline{\lambda}}}\frac{\int_0^T e^{Q(t)}\sup_{|x|\le\zeta_n}G(t,x)\mathrm{d}t}{\zeta_n^2},$$

taking (3.1) into account, one has

$$2c^2\liminf_{\zeta\to+\infty}\frac{\int_0^T e^{Q(t)}\sup_{|x|\le\zeta}H(t,x)\mathrm{d}t}{\zeta^2}\le 2c^2\liminf_{\zeta\to+\infty}\frac{\int_0^T e^{Q(t)}\sup_{|x|\le\zeta}F(t,x)\mathrm{d}t}{\zeta^2}+\frac{\overline{\mu}}{\overline{\overline{\lambda}}}G_\infty. \tag{3.4}$$

Moreover, since G is nonnegative, we have

$$\limsup_{\xi\to+\infty}\frac{\int_0^T e^{Q(t)}H(t,\xi)\mathrm{d}t}{\sum_{i=1}^N\sum_{j=1}^N\xi_i\,\xi_j}\ge\limsup_{\xi\to+\infty}\frac{\int_0^T e^{Q(t)}F(t,\xi)\mathrm{d}t}{\sum_{i=1}^N\sum_{j=1}^N\xi_i\,\xi_j} \tag{3.5}$$

where $\xi=(\xi_1,\dots,\xi_n)$ and $\xi\to+\infty$ means that $(\xi_1,\dots,\xi_n)\to(+\infty,\dots,+\infty)$.

Therefore, from (3.4) and (3.5), and from Assumption (a_1) and (3.3) we observe

$$\overline{\lambda}\in(\nu_1,\nu_2)\subseteq\Bigg(\frac{\mathcal{B}\int_0^T e^{Q(t)}\mathrm{d}t}{2\limsup_{\xi\to+\infty}\frac{\int_0^T e^{Q(t)}H(t,\xi)\mathrm{d}t}{\sum_{i=1}^N\sum_{j=1}^N\xi_i\,\xi_j}},\frac{1}{2c^2\liminf_{\zeta\to+\infty}\frac{\int_0^T e^{Q(t)}\sup_{|x|\le\zeta}H(t,x)\mathrm{d}t}{\zeta^2}}\Bigg)\subseteq\left(0,\frac{1}{\gamma}\right)$$

where $\xi=(\xi_1,\dots,\xi_n)$ and $\xi\to+\infty$ means that $(\xi_1,\dots,\xi_n)\to(+\infty,\dots,+\infty)$.

For the fixed $\overline{\lambda}$, the inequality (3.3) concludes that the condition (b) of Theorem 2.1 can be applied and either $I_{\overline{\lambda}}$ has a global minimum or there exists a sequence $\{u_n\}$ of solutions of the problem (1.1) such that $\lim_{n\to\infty}\|u\|=+\infty$.

The other step is to show that for the fixed $\overline{\lambda}$ the functional $I_{\overline{\lambda}}$ has no global minimum. Let us verify that the functional $I_{\overline{\lambda}}$ is unbounded from below. Since

$$\frac{1}{\overline{\overline{\lambda}}}<\frac{2}{\mathcal{B}\int_0^T e^{Q(t)}\mathrm{d}t}\limsup_{\xi\to+\infty}\frac{\int_0^T F(t,\xi)\mathrm{d}t}{\sum_{i=1}^N\sum_{j=1}^N\xi_i\,\xi_j}$$

where $\xi=(\xi_1,\dots,\xi_n)$ and $\xi\to+\infty$ means that $(\xi_1,\dots,\xi_n)\to(+\infty,\dots,+\infty)$, we can consider a sequence $\{d_n=(d_{n1},\dots,d_{nN})\}$ and a positive constant τ

such that $d_{ni} \to +\infty$ as $n \to +\infty$ for $i = 1, \ldots, N$ and

$$\frac{1}{\overline{\lambda}} < \tau < \frac{2}{\mathcal{B}\int_0^T e^{Q(t)}\mathrm{d}t} \limsup_{n\to+\infty} \frac{\int_0^T F(t, d_n)\mathrm{d}t}{\sum_{i=1}^N \sum_{j=1}^N d_{ni}\, d_{nj}} \tag{3.6}$$

for each $n \in \mathbb{N}$ large enough. Let $\{w_n\}$ be a sequence in X defined by

$$w_n(t) = d_n, \qquad t \in [0, T]. \tag{3.7}$$

For any fixed $n \in \mathbb{N}$, $w_n \in X$ and

$$\Phi(w_n) \leq \frac{\mathcal{B}\int_0^T e^{Q(t)}\mathrm{d}t}{2} \sum_{i=1}^N \sum_{j=1}^N d_{ni}\, d_{nj}. \tag{3.8}$$

On the other hand, since G is nonnegative, from the definition of Ψ, we infer

$$\Psi(w_n) \geq \int_0^T F(t, d_n)\mathrm{d}t. \tag{3.9}$$

So, according to (3.6), (3.8) and (3.9) we obtain

$$\begin{aligned} I_{\overline{\lambda}}(w_n) &\leq \frac{1}{2}\sum_{i=1}^N \sum_{j=1}^N \|a_{ij}\|_\infty d_{ni}\, d_{nj} \int_0^T e^{Q(t)}\mathrm{d}t - \overline{\lambda}\int_0^T F(t, d_n)\mathrm{d}t \\ &\leq \frac{\mathcal{B}(1-\overline{\lambda}\tau)\int_0^T e^{Q(t)}\mathrm{d}t}{2} \sum_{i=1}^N \sum_{j=1}^N d_{ni}\, d_{nj} \end{aligned}$$

for every $n \in \mathbb{N}$ large enough. Hence, the functional $I_{\overline{\lambda}}$ is unbounded from below, and it follows that $I_{\overline{\lambda}}$ has no global minimum. Therefore, applying Theorem 2.1 we deduce that there is a sequence $\{u_n\} \subset X$ of critical points of $I_{\overline{\lambda}}$ such that $\lim_{n\to\infty} \Phi(u_n) = +\infty$, which from the definition of Φ follows that $\lim_{n\to\infty} \|u_n\| = +\infty$. Hence, since the critical points of $I_{\overline{\lambda}}$ are actually the solutions of the problem (1.1) (see [25, Theorem 2.2]) we have conclusion. □

We now present the following examples in which the hypotheses of Theorem 3.1 are satisfied, whose constructions are motivated by Examples 3.11 and 3.12 of [5], respectively.

Example 3.1. Let $N = 1$, $T = 1$ and put

$$a_n := \frac{2n!(n+2)! - 1}{4(n+1)!}, \qquad b_n := \frac{2n!(n+2)! + 1}{4(n+1)!}$$

for every $n \in \mathbb{N}$. Consider the problem

$$\begin{cases} -u''(t) - \cos(\pi t)\, u'(t) + u(t) = \lambda f(t, u(t)) + \mu g(t, u(t)) & \text{a.e. } t \in [0, 1], \\ u(0) - u(1) = u'(0) - u'(1) = 0 \end{cases} \tag{3.10}$$

where

$$f(t,x)=\begin{cases}\dfrac{32\cos(\pi t)(n+1)!^2[(n+1)!^2-n!^2]}{\pi} \\ \quad\times\sqrt{\dfrac{1}{16(n+1)!^2}-(x-\dfrac{n!(n+2)}{2})^2} & \text{if } (t,x)\in[0,1]\times\cup_{n\in\mathbb{N}}[a_n,b_n], \\ 0 & \text{elsewhere,}\end{cases}$$

and $g(t,x)=e^{-\frac{\sin(\pi t)}{\pi}-x^+}(x^+)(2-x^+)$ where $x^+=\max\{x,0\}$, for all $t\in[0,1]$ and $x\in\mathbb{R}$.

One has $\int_{n!}^{(n+1)!} f(1,x)dx=(n+1)!^2-n!^2$ for every $n\in\mathbb{N}$. Then, one has $\lim_{n\longrightarrow+\infty}\frac{F(1,b_n)}{b_n^2}=4$ and $\lim_{n\longrightarrow+\infty}\frac{F(1,a_n)}{a_n^2}=0$. Therefore, simple computations show that

$$\liminf_{\xi\longrightarrow+\infty}\frac{F(1,\zeta)}{\zeta^2}=0 \quad\text{and}\quad \limsup_{\xi\longrightarrow+\infty}\frac{F(1,\zeta)}{\zeta^2}=4.$$

Hence

$$\liminf_{\zeta\longrightarrow+\infty}\frac{\int_0^1 e^{\frac{\sin(\pi t)}{\pi}}\sup_{|x|\le\zeta}F(t,x)\mathrm{d}t}{\zeta^2} = \liminf_{\zeta\longrightarrow+\infty}\frac{\int_0^1\cos(\pi t)e^{\frac{\sin(\pi t)}{\pi}}\sup_{|x|\le\zeta}F(1,x)\mathrm{d}t}{\zeta^2}=0$$

and

$$\frac{1}{c^2\mathcal{B}\int_0^1 e^{\frac{\sin(\pi t)}{\pi}}\mathrm{d}t}\limsup_{\xi\longrightarrow+\infty}\frac{\int_0^1 e^{\frac{\sin(\pi t)}{\pi}}F(t,\xi)\mathrm{d}t}{\xi^2} = \frac{\int_0^1\cos(\pi t)e^{\frac{\sin(\pi t)}{\pi}}\mathrm{d}t}{c^2\int_0^1 e^{\frac{\sin(\pi t)}{\pi}}\mathrm{d}t}\limsup_{\xi\longrightarrow+\infty}\frac{F(1,\xi)}{\xi^2}=0.$$

Hence, using Theorem 3.1, since

$$G_\infty=2c^2\lim_{\zeta\to+\infty}\frac{\int_0^1 e^{\frac{\sin(\pi t)}{\pi}}\sup_{|x|\le\zeta}e^{-\frac{\sin(\pi t)}{\pi}-x^+}(x^+)^2\mathrm{d}t}{\zeta^2}=2c^2\lim_{\zeta\to+\infty}e^{-\zeta}=0,$$

the problem (3.10) for every $\lambda>0$ and $\mu\ge 0$ has an unbounded sequence of solutions.

Example 3.2. Let $N=2$, $T=1$, where $A:[0,1]\to\mathbb{R}^{2\times 2}$ is an unit matrix. Consider the problem

(3.11)
$$\begin{cases}-u_1''(t)-\cos(\pi t)\,u_1'(t)+u_1(t)=\lambda f_1(t,u(t))+\mu g_1(t,u(t)) & a.e.\ \ t\in[0,1],\\ -u_2''(t)-\cos(\pi t)\,u_2'(t)+u_2(t)=\lambda f_2(t,u(t))+\mu g_2(t,u(t)) & a.e.\ \ t\in[0,1],\\ u_1(0)-u_1(1)=u_1'(0)-u_1'(1)=0,\\ u_2(0)-u_2(1)=u_2'(0)-u_2'(1)=0\end{cases}$$

where

$$f_1(t,x) = 4\cos(\pi t)x(1-\sin x) + 2\cos(\pi t)x^2\cos x \text{ for all } (t,x) \in [0,1]\times\mathbb{R}$$

and

$$f_2(t,y) = \begin{cases} \cos(\pi t)(a_{n+1})^3 e^{\frac{1}{(y-(a_{n+1}-1))(y+(a_{n+1}+1))+1}} \\ \quad \times \dfrac{2(a_{n+1}-y)}{(y-(a_{n+1}-1))^2(y-(a_{n+1}+1))^2} & \text{if } (t,y)\in[0,1]\times S \\ 0 & \text{otherwise} \end{cases}$$

where

$$a_1 := 2, \qquad a_{n+1} := (a_n)^{\frac{3}{2}}$$

for every $n \in \mathbb{N}$ and $S := \bigcup_{n\geq 2}]a_{n+1}-1, a_{n+1}+1[$, and $g_1(t,\xi) = g_2(t,\xi) = 2(\xi^+)e^{-t^2}$ where $\xi^+ = \max\{\xi, 0\}$, for all $t \in [0,1]$ and $\xi \in \mathbb{R}$. Thus, setting $(f_1(t,x), f_2(t,y)) = \nabla F(t,x,y)$ and $(g_1(t,x), g_2(t,y)) = \nabla G(t,x,y)$, one has

$$F(t,x,y) =: \begin{cases} 2\cos(\pi t)(1+\sin x)x^2 \\ \quad +t(a_{n+1})^3 e^{\frac{1}{(y-(a_{n+1}-1))(y+(a_{n+1}+1))+1}} & \text{if } (t,x,y)\in[0,1]\times\mathbb{R}\times S \\ 2\cos(\pi t)(1+\sin x)x^2 & \text{otherwise} \end{cases}$$

and

$$G(t,x,y) = [(x^+)^2 + (y^+)^2]e^{-\frac{\sin \pi t}{\pi}} \quad \text{for all } (t,x,y)\in[0,1]\times\mathbb{R}\times S.$$

Simple calculations show that

$$\liminf_{\zeta\longrightarrow+\infty} \frac{\int_0^1 e^{\frac{\sin\pi t}{\pi}} \sup_{\sqrt{x^2+y^2}\leq\zeta} F(t,x,y)\mathrm{d}t}{\zeta^2} = \int_0^1 \cos(\pi t)e^{\frac{\sin\pi t}{\pi}}\mathrm{d}t = 0$$

and

$$\limsup_{(\xi_1,\xi_2)\longrightarrow(+\infty,+\infty)} \frac{\int_0^1 e^{\frac{\sin\pi t}{\pi}} F(t,\xi_1,\xi_2)\mathrm{d}t}{\sum_{i=1}^2\sum_{j=1}^2 \xi_i\,\xi_j} = +\infty.$$

Hence, since all assumptions of Theorem 3.1 are satisfied, taking into account that

$$G_\infty = 2c^2 \lim_{\zeta\to+\infty} \frac{\int_0^1 e^{\frac{\sin\pi t}{\pi}} \sup_{\sqrt{x^2+y^2}\leq\zeta} e^{-\frac{\sin\pi t}{\pi}}[(x^+)^2+(y^+)^2]\mathrm{d}t}{\zeta^2} = 2c^2 < \infty.$$

Thus the problem (3.11) for every $\lambda > 0$ and $\mu \in \left[0, \frac{1}{2c^2}\right)$ has an unbounded sequence of solutions.

Remark 3.1. Under the conditions

$$\liminf_{\zeta\to+\infty}\frac{\int_0^T e^{Q(t)}\sup_{|x|\leq\zeta}F(t,x)\mathrm{d}t}{\zeta^2}=0 \quad\text{and}\quad \limsup_{\xi\to+\infty}\frac{\int_0^T e^{Q(t)}F(t,\xi)\mathrm{d}t}{\sum_{i=1}^N\sum_{j=1}^N\xi_i\,\xi_j}=+\infty$$

where $\xi=(\xi_1,\ldots,\xi_n)$ and $\xi\to+\infty$ means that $(\xi_1,\ldots,\xi_n)\to(+\infty,\ldots,+\infty)$, Theorem 3.1 concludes that for every $\lambda>0$ and for each $\mu\in[0,\frac{1}{G_\infty}[$ the problem (1.1) admits infinitely many solutions. Moreover, if $G_\infty=0$, the result holds for every $\lambda>0$ and $\mu\geq 0$.

Here, we point out a simple consequence of Theorem 3.1.

COROLLARY 3.2. *Assume that*

(a_2) $\liminf_{\zeta\to+\infty}\frac{\int_0^T e^{Q(t)}\sup_{|x|\leq\zeta}F(t,x)\mathrm{d}t}{\zeta^2}<\frac{1}{2c^2}$;

(a_3) $\limsup_{\xi\to+\infty}\frac{\int_0^T e^{Q(t)}F(t,\xi)\mathrm{d}t}{\sum_{i=1}^N\sum_{j=1}^N\xi_i\,\xi_j}>\frac{\mathcal{B}}{2}\int_0^T e^{Q(t)}\mathrm{d}t$, *where* $\xi=(\xi_1,\ldots,\xi_n)$ *and* $\xi\to+\infty$ *means that* $(\xi_1,\ldots,\xi_n)\to(+\infty,\ldots,+\infty)$.

Then, for every arbitrary non-negative function $G:[0,T]\times\mathbb{R}^N\to\mathbb{R}$ *which is measurable with respect to* t*, for all* $x\in\mathbb{R}^N$*, continuously differentiable in* x*, for almost every* $t\in[0,T]$*, satisfying the condition* (3.1)*, and for every* $\mu\in[0,\mu_{G,\lambda}[$ *where* $\mu_{G,\lambda}:=\frac{1}{G_\infty}\left(1-\frac{1}{\lambda_2}\right)$*, the problem*

$$\begin{cases} -\ddot{u}(t)-q(t)\dot{u}(t)+A(t)u(t)=\nabla F(t,u(t))+\mu\nabla G(t,u(t)) & a.e.\ \ t\in[0,T],\\ u(0)-u(T)=\dot{u}(0)-\dot{u}(T)=0. \end{cases}$$

has an unbounded sequence of solutions.

Remark 3.2. Theorem 2.2 is an immediate consequence of Corollary 3.2 when $\mu=0$.

Remark 3.3. We observe in Theorem 3.1 we can replace $\xi\to+\infty$ with $\xi\to 0^+$, that by the same arguing as in the proof of Theorem 3.1 but using conclusion (c) of Theorem 2.1 instead of (b), the problem (1.1) has a sequence of pairwise distinct solutions, which strongly converges to 0 in E.

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