A NEW CHARACTERIZATION OF $L_2(127)$

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Let G be a group and $\pi_e(G)$ be the set of element orders of G. Assume that $k \in \pi_e(G)$ and let m_k be the number of elements of order k in G. Set $nse(G) := \{m_k \mid k \in \pi_e(G)\}$. In this paper, we give a new characterization of the simple group $L_2(127)$ by the set $nse(L_2(127))$.

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1. INTRODUCTION

Throughout this paper, all groups are finite and G is always a group. Denote by $\pi(G)$ the set of prime divisors of $|G|$, and by $\pi_e(G)$ the set of element orders of G. If r is a prime divisor of the order of G, then P_r denotes a Sylow r-subgroup of G and $n_r(G)$ denotes the number of Sylow r-subgroups of G. Let n be an integer. We denote by $\varphi(n)$ the Euler function of n. We call that G is a simple K_n-group if G is simple with $|\pi(G)| = n$.

Recall that the prime graph $\mathrm{GK}(G)$ of a group G is defined as a graph with vertex set $\pi(G)$ and two distinct primes $p, q \in \pi(G)$ are adjacent if G contains an element of order pq. Further, the connected components of $\mathrm{GK}(G)$ are denoted by $\pi_i, 1 \leq i \leq t(G)$, where $t(G)$ is the number of connected components of G. In particular, we denote by π_1 the component containing the prime 2 for a group of even order.

The motivation of this article is to investigate Thompson's Problem related to algebraic number fields as follows (see [7, Problem 12.37]).

Let $k \in \pi_e(G)$ and m_k be the number of elements of order k in G. Set $nse(G) := \{m_k \mid k \in \pi_e(G)\}$. Write $M_t(G) := \{g \in G \mid g^t = 1\}$. G_1 and G_2 are called of the same order type if and only if $|M_t(G_1)| = |M_t(G_2)|$, $t = 1, 2, \ldots$.

Thompson's Problem. Suppose that G_1 and G_2 are of the same order type. If G_1 is solvable, is it true that G_2 is also necessarily solvable?

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So far, no one could prove it completely, or even give a counterexample. However, if groups G_1 and G_2 are of the same order type, we see clearly that $nse(G_1) = nse(G_2)$. So it is natural to try to investigate the Thompson's Problem by $|G|$ and $nse(G)$.

Note that not all groups can be characterized by $nse(G)$ and $|G|$. For instance, in 1987, Thompson gave an example as follows: Let $G_1 = (C_2 \times C_2 \times C_2 \times C_2) \rtimes A_7$ and $G_2 = L_3(4) \rtimes C_2$ be two maximal subgroups of M_{23}. Then $nse(G_1) = nse(G_2)$ and $|G_1| = |G_2|$, but $G_1 \ncong G_2$.

Then authors of [10] proved that all simple K_4-groups can be uniquely determined by $nse(G)$ and $|G|$. Moreover, in [11], it is proved that $L_2(3) \cong A_4$, $L_2(4) \cong L_2(5) \cong A_5$ and $L_2(9) \cong A_6$ are uniquely determined by $nse(G)$. M. Khatami, B. Khosravi and Z. Akhlaghi [6] deduced that simple groups $L_2(p)$ are characterizable uniquely by the set $nse(L_2(p))$ if $p \in \{7, 8, 11, 13\}$. Recently, S.T. Liu [5] proved that $L_5(2)$ is uniquely determined by $nse(G)$.

In this paper, by using prime graph properties and Lemma 2.2, we prove that $L_2(127)$ is characterizable by its associated set $nse(L_2(127))$. Our result is:

Theorem A. *Let G be a finite group with*

$$nse(G) = \{1, 16256, 48768, 97536, 292608, 8001, 16002, 32004, \\ 64008, 128016, 256032, 16128\} = nse(L_2(127)),$$

then $G \cong L_2(127)$.

In this paper, we denote $n_r(G)$ by n_r and $m_k(G)$ by m_k if there is no confusion. Further unexplained notation is standard, readers may refer to [2].

2. PRELIMINARIES

In this section, we give the lemmas which will be used in the sequel. We begin with a classical by Frobenius.

Lemma 2.1 ([3]). *Let G be a group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G | g^m = 1\}$, then $m \mid |L_m(G)|$.*

Although the following lemma is simple, it is powerful in characterizing simple groups G by $nse(G)$.

Lemma 2.2 ([9, Lemma 2.2]). *Let G be a group and P be a cyclic Sylow p-subgroup of G. Assume further that $|P| = p^a$ and r is an integer such that $p^a r \in \pi_e(G)$. Then $m_{p^a r} = m_r(C_G(P)) m_{p^a}$. In particular, $\varphi(r) m_{p^a} \mid m_{p^a r}$.*

LEMMA 2.3 ([8]). *Let G be a group and $p \in \pi(G)$ be odd. Suppose that P is a Sylow p-subgroup of G and $n = p^s m$, where $(p, m) = 1$. If P is not cyclic and $s > 1$, then the number of elements of order n is always a multiple of p^s.*

Recall that G is a 2-Frobenius group, if G has a normal series $G \trianglerighteq K \trianglerighteq H \trianglerighteq 1$ such that G/H and K are Frobenius groups with K/H and H as Frobenius kernel.

LEMMA 2.4 ([13, Theorem]). *Let G be a group such that $t(G) \geq 2$. Then G has one of the following structures:*

(a) *G is a Frobenius or 2-Frobenius group.*

(b) *G has a normal series $1 \trianglelefteq N \trianglelefteq G_1 \trianglelefteq G$ such that $\pi(N) \cup \pi(G/G_1) \subseteq \pi_1$ and G_1/N is a nonabelian simple group.*

LEMMA 2.5 ([1, Theorem 2]). *If G is a 2-Frobenius group of even order, then $t(G) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(K/H) = \pi_2$, $\pi(H) \bigcup \pi(G/K) = \pi_1$, $|G/K| \mid |Aut\ (K/H)|$, G/K and K/H are cyclic. In particular, $|G/K| < |K/H|$ and G is solvable.*

LEMMA 2.6 ([10, Lemma 2.5]). *Let G be a group with a normal series: $K \trianglelefteq L \trianglelefteq G$. Suppose that $P \in Syl_p(G)$, where $p \in \pi(G)$. If $P \leq L$ and $p \nmid |K|$, then the following statements hold:*

(1) *$|G : N_G(P)| = |L : N_L(P)|$, that is, $n_p(G) = n_p(L)$;*

(2) *$|L/K : N_{L/K}(PK/K)|t = |G : N_G(P)| = |L : N_L(P)|$, that is, $n_p(L/K)t = n_p(G) = n_p(L)$ for some positive integer t. Furthermore, $|N_K(P)|t = |K|$.*

LEMMA 2.7 ([4, Theorem 2]). *If G is a simple K_3-group, then G is isomorphic to one of the following groups: $A_5, A_6, L_2(7), L_2(8), L_2(17), L_3(3)$, $U_3(3)$ or $U_4(2)$.*

LEMMA 2.8. *Let G be a simple K_4-group. If $\pi(G) = \{2, 3, 7, 127\}$, then $G \cong L_2(127)$.*

Proof. This follows from [12. Corollary 3]. □

3. PROOF OF THEOREM A

The necessity is obvious by [2]. We only prove the sufficiency. Let $n \in \pi_e(G)$ and k be the number of cyclic subgroups of G of order n. Then $m_n = k \cdot \varphi(n)$. In particular, $\varphi(n) \mid m_n$. Further, along with Lemma 2.1 we obtain that for any positive integer $n \mid |G|$, the following hold:

$$(1) \qquad \begin{cases} \varphi(n) \mid m_n \\ n \mid \sum_{d \mid n} m_d \end{cases}$$

For convenience, we denote by Ω_i the set of all elements of order i in G.

Proof. Note that $nse(G) = \{1, 16256, 48768, 97536, 292608, 8001, 16002, 32004, 64008, 128016, 256032, 16128\} = nse(L_2(127))$. By (1), it follows that $\pi(G) \subseteq \{2, 3, 5, 7, 127\}$, $2 \in \pi(G)$ and $m_2 = 8001$. Moreover, if $3, 5, 7, 127 \in \pi(G)$, then $m_3 = 16256$, $m_5 = 32004$, $m_7 = 48768$, $m_{127} = 16128$. Assume that $\exp(P_2) = 2^s$. Since $\varphi(2^s) \mid m_{2^s} \in nse(G)$, we obtain $s \leq 9$. Moreover, (1) indicates that $|P_2| \mid (1 + m_2 + \cdots + m_{2^s})$, and thus $|P_2| \leq 2^{12}$. Similarly, if $5, 7, 127 \in \pi(G)$, then $\exp(P_{127}) = 127$, $|P_{127}| \leq 127^2$, $|P_5| = 5$ and $|P_7| = 7$.

We prove that $\pi(G) = \{2, 3, 7, 127\}$. If $5 \in \pi(G)$, then $n_5(G) = m_5/4 = 3^2 \cdot 7 \cdot 127$, which implies that $127 \in \pi(G)$. On the other hand, Lemma 2.2 indicates that $5 \cdot 127 \notin \pi_e(G)$. Hence, P_5 acts fixed-pointed-freely on Ω_{127}. That is, $5 \mid m_{127}$, a contradiction. Hence $\pi(G) \subseteq \{2, 3, 7, 127\}$.

If $3 \in \pi(G)$, assume that $\exp(P_3) = 3^s$. Then $s \leq 3$ since $\varphi(3^s) \mid m_{3^s} \in nse(G)$. If $s = 1$, then $|P_3| \mid 1 + m_3$ by (1), implying P_3 is cyclic. If $s \geq 2$, again by applying (1), $m_9 = 48768$ and $m_{27} \in \{16002, 64008, 256032\}$. Along with Lemma 2.3, we see that P_3 is always cyclic. Hence $n_3 = m_{3^s}/\varphi(3^s)$, leading to $127 \in \pi(G)$. Therefore $\{2, 3, 127\} \subseteq \pi(G)$. Similarly, if $7 \in \pi(G)$, then $n_7(G) = m_7/6 = 2^6 \cdot 127$, also implying that $127 \in \pi(G)$. Consequently, we only need to consider one of the following cases $\{2\}$, $\{2, 127\}$, $\{2, 3, 127\}$ and $\{2, 7, 127\}$.

Easily, G is not a 2-group since $|nse(G)| = 12$. Assume that $\pi(G) = \{2, 127\}$. It follows $|G| \leq 2^{12} \cdot 127 = 520192 < \sum_{i \in nse(G)} i = 975360$ that $|P_{127}| = 127^2$. Further, $|G| = 2^a \cdot 127^2 \geq 975360$ implies that $a \geq 6$.

Note that $\exp(P_2) \leq 2^9$ and $\exp(P_{127}) = 127$. Hence $\pi_e(G) \subseteq \{1, 2, 2^2, \cdots, 2^9\} \cup \{127, 2 \cdot 127, \cdots, 2^8 \cdot 127\}$, which indicates that $|G| = 2^a 127^2 = 975360 + 16256k_1 + 48768k_2 + 97536k_3 + 292608k_4 + 16002k_5 + 32004k_6 + 64008k_7 + 128016k_8 + 256032k_9 + 16128k_{10}$ and $\sum_{i=1}^{10} k_i \leq 9$. Moreover, $a \leq 7$. If $|P_2| = 2^7$, then P_2 is not cyclic since otherwise, $m_{2^7} = 16128$ and thus $n_2 = m_{2^7}/\varphi(2^7) = 2^2 \cdot 3^2 \cdot 7$, a contradiction. Hence $\exp(P_2) \leq 2^6$ and $\sum_{i=1}^{10} k_i \leq 2$. Easily, $a \leq 6$, leading to $a = 6$. However, the equation $2^6 \cdot 127^2 = 975360 + 16256k_1 + 48768k_2 + 97536k_3 + 292608k_4 + 16002k_5 + 32004k_6 + 64008k_7 + 128016k_8 + 256032k_9 + 16128k_{10} = 2^a 3^b \leq 4779264$ and thus $2^6 \cdot 127 = 7680 + 2^7 k_1 + 2^7 \cdot 3k_2 + 2 \cdot 3^2 \cdot 7k_5 + 2^2 \cdot 3^2 \cdot 7k_6$ with $\sum_{i=1}^{6} k_i \leq 2$ has no solutions.

Assume then $\pi(G) = \{2, 7, 127\}$. Further, $127^2 \notin \pi_e(G)$. Assume that $|P_{127}| = 127^2$, then $127 \mid |C_G(P_7)|$ since $|G : N_G(P_7)| = 2^6 \cdot 127$. By Lemma 2.2, $126m_7 \mid m_{7 \cdot 127}$, a contradiction. Hence $|P_{127}| = 127$. By Lemma 2.2 and (1), we have that $127r \notin \pi_e(G)$, where $r = 2, 7$ and $14 \notin \pi_e(G)$. It follows that $t(G) \geq 2$. By Lemma 2.4, G is not solvable. Note that there is no simple

K_3-group of order divisible by 127, also a contradiction. Hence the remaining case is $\pi(G) = \{2, 3, 127\}$.

We show that $3^2 \cdot 127 \notin \pi_e(G)$. If not, (1) indicates that $\varphi(3^2 \cdot 127) \mid m_{3^2 \cdot 127} \in nse(G)$, a contradiction. Suppose that $|P_{127}| = 127$. If $\exp(P_3) \geq 3^2$, then P_{127} acts fixed-point-freely on Ω_{3^2}, which implies that $127^2 \mid m_9$, a contradiction. Hence $\exp(P_3) = 3$. Recall that $|P_3| = 3$, leading to $n_3 = m_3/\varphi(3) = 2^6 \cdot 127$. As a result, $127 \mid |N_G(P_3)|$. Let $N \in \mathrm{Syl}_{127}(N_G(P_3))$. Then $N \trianglelefteq P_3N$ by Sylow's Theorem and thus $3 \cdot 127 \in \pi_e(G)$. Lemma 2.2 gives that $126m_3 \mid m_{3 \cdot 127}$, this is impossible and as a result, $|P_{127}| = 127$. Again by Lemma 2.2, it follows that $3 \cdot 127 \notin \pi_e(G)$ and $2 \cdot 127 \notin \pi_e(G)$. Hence $t(G) \geq 2$. As there is no simple K_3-group whose order is divisible by 127, G is solvable. Further, we see from Lemma 2.5 that G is either a Frobenius group or a 2-Frobenius group. If the former holds, write $G = K \rtimes H$, then either $|H| = 127$ or $|K| = 127$. Suppose that $|H| = 127$. Then $m_2 \leq |K_2| - 1 \leq 4095$, a contradiction. If $|K| = 127$, then $m_{127} = 126$, also a contradiction. Hence G is a 2-Frobenius group and then G has the following normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ with $|K/H| = 127$ and $|G/K| \mid 126$.

Therefore, $\pi(G) = \{2, 3, 7, 127\}$. By the same reasoning above we deduce that $|P_{127}| = 127$ and $|P_7| = 7$. Further, $|G : N_G(P_7)| = 2^6 \cdot 127$. Let $N \in \mathrm{Syl}_2(C_G(P_7))$. Then $P_7 \rtimes N \leq G$. Since $14 \notin \pi_e(G)$, P_7N is a Frobenius group with Frobenius kernel P_7. Hence $|N| \mid 2$, leading to $|P_2| = 2^6$ or 2^7. If G is solvable, then G is either a Frobenius group or a 2-Frobenius group. Write $G = K \rtimes H$, where K and H are Frobenius kernel and Frobenius complement, respectively. Then either $|H| = 127$ or $|K| = 127$. Assuming first that $|H| = 127$, then $m_3 \leq |K_3| - 1 \leq 3^3 - 1 = 26$, a contradiction. On the other hand, if $|K| = 127$, then $m_{127} \leq |K| - 1 \leq 3^3 - 1 = 126$, also a contradiction. As a result, G is unsolvable and has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $|K/H| = 127$, $|G/K| \mid |Aut\ K/H| = 126$ and G/K is cyclic. Since K is a Frobenius group with kernel H, we have that $127 \mid |H_3| - 1$. However, $\exp(P_3) \leq 3$ and P_3 is cyclic, we have that H is a 2-group. By the same reason, we have that $127 \mid (|H| - 1)$. So we have that $|H| = 2^8$. Since $|G/K| \mid 126$, we have that $|P_3| \leq 3^2$ and $|P_2| \leq 2^9$. Hence $|G| \leq 2^9 \cdot 3^2 \cdot 127 = 585216 < \Sigma_{i \in nse(G)} i = 975360$, a contradiction. Hence G is unsolvable.

Further, Lemma 2.4 shows that G has the following normal series:

$$1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$$

with K/H a simple K_4-group and $127 \nmid |H|$. By Lemma 2.8, we have that $K/H \cong L_2(127)$. By Lemma 2.6, $|N_H(P_{127})|t = |H|$ and $n_{127}(K/H)t = n_{127}(K)$. Since $K/H \cong L_2(127)$, we have that $m_{127}(K/H) = m_{127}$. Hence $n_{127}(K/H) = n_{127}(K)$ and thus $t = 1$. Furthermore, we have that $H \leq$

$N_G(P_{127})$. It follows that $H \times P_{127} \leq G$. Note that $127r \notin \pi_e(G)$, we have that $H = 1$. So we have that $K \cong L_2(127)$ and $G/K \leq C_2$. If $G = K.2$, then $m_2 = 16129 \notin nse(G)$, a contradiction. Hence $G = K \cong L_2(127)$. This completes the proof.

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